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Proper Holomorphic Self-maps of smooth bounded Reinhardt domains in $\mathbb{C}^2$

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Abstract It is proved that every proper holomorphic self-map of a smooth bounded Reinhardt domain in $\mathbb{C}^2$ is an automorphism.

Keywords: Reinhardt domain proper holomorphic maps

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1 Introduction

In [1], Berteloot proved the following theorem regarding proper holomorphic self-maps.

Theorem 1.1. *Let Ω be a complete Reinhardt domain with C^2 smooth boundary. Then every proper holomorphic self-map of Ω is an automorphism.*

The novelty here is that neither pseudoconvexity nor finite type of the domain is assumed, only completeness. This result provides an important case of domains for which the following classical theorem of H. Alexander [2] proved in 1977 is true.

Theorem 1.2. *Every proper holomorphic self-map of the unit ball in $\mathbb{C}^n$ $(n > 1)$ is an automorphism.*

Motivated by this theorem, the following problem seems to be well-known.

Open Problem 1.3. *Is every proper holomorphic self-map of a smooth bounded domain in $\mathbb{C}^n$ $(n > 1)$ an automorphism.*

In this paper, we will consider non-complete Reinhardt domains and prove the following theorems.

Theorem 1.4. *Let Ω be a smooth bounded Reinhardt domain in $\mathbb{C}^2$. Then every proper holomorphic self-map of Ω is an automorphism.*

In higher dimensions, we have a weaker result which is a simple consequence of Berteloot [1], and used in the proof of Theorem 1.4.

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Theorem 1.5. *Let Ω be a C^2 smooth bounded Reinhardt domain in $\mathbb{C}^n$ $(n > 1)$ that contains the origin. Then every proper holomorphic self-map of Ω is an automorphism.*

Since Alexander's theorem, the problem has been solved for many cases. In 1978, Pinchuk [3] proved the case for strictly pseudoconvex domains. Pinchuk's approach was to reduce the case to that of the unit ball by his scaling method. Bedford and Bell [4] solved the case for pseudoconvex domains with real analytic boundary. They were able to use a stratification of weakly pseudoconvex points to control the branching behavior of the maps. In 1996, Huang and Pan [5] modified this method to solve the case for any domain with real analytic boundary provided the map is C^∞ smooth up to the boundary. Using this result and holomorphic extension results of Diederich and Pinchuk [6], one concludes that every proper holomorphic self-map of a real analytic domain in $\mathbb{C}^2$ is an automorphism.

Despite these results, the problem remains open, even for the case of pseudoconvex domains of finite type. However, many positive results have been obtained for domains with symmetries. We [7] verified the case of smooth bounded pseudoconvex Reinhardt domains of finite type in $\mathbb{C}^n$. Berteloot [1] solved the case of complete Reinhardt domains with C^2 smooth boundary (without pseudoconvexity assumption) in $\mathbb{C}^n$. His method is based on the study of the Lie Algebra of holomorphic tangent vector fields of a strictly pseudoconvex Reinhardt hypersurface. In [8],[9] Coupet, Sukhov and Pan were able to solve the problem for pseudoconvex circular and Hartogs domains of finite type in $\mathbb{C}^2$. They used the scaling method to reduce to the situation where theory of complex dynamics in $\mathbb{C}^2$ can be employed.

We want to point out that the smoothness assumption of domains is very important. Besides obvious examples, Berteloot and Loeb [10] proved, based on complex dynamics in $\mathbb{C}^2$, that there exists a complete circular domain in $\mathbb{C}^2$ with real analytic strictly pseudoconvex boundary outside of the union of three circles (where the boundary is not smooth) such that there is a proper holomorphic self-map of the domain which is not biholomorphic. However for some other non-smooth domains, certain generalized Hartogs triangles, Chen and Xu [11] proved that self-maps are biholomorphic. We also point out that proper self-maps of smooth bounded domains in a complex manifold may not be automorphisms by examples of Burns and Schnider. However, Zhou proved that proper self-maps of certain symmetric domains in a Lie Group indeed are automorphisms, see details in [12].

2 Branching behavior of proper holomorphic maps

Lemma 2.1. *If Ω is a pseudoconvex Reinhardt domain in $\mathbb{C}^n$ that contains the origin, then Ω is complete.*

Proof The result follows from a classical result of Reinhardt: If Ω is pseudoconvex and $\Omega \cap \{z_j = 0\} \neq \emptyset$, then $(z_1, ..., z_{j-1}, \lambda z_j, z_{j+1}, ..., z_n) \in \Omega$ whenever $z \in \Omega$ and $|\lambda| \leqslant 1$.

The following lemma is important to us and is proved in [13] (Lemma 4).

Lemma 2.2. *Let $f : \Omega_1 \to \Omega_2$ be a proper holomorphic map between domains in $\mathbb{C}^n$. Assume that the hulls of holomorphy $\hat{\Omega}_1$,$\hat{\Omega}_2$ of Ω_1, Ω_2 are domains in $\mathbb{C}^n$. Then f extends to some proper holomorphic map $\hat{f} : \hat{\Omega}_1 \to \hat{\Omega}_2$.*

Lemma 2.3. *Let f be a proper holomorphic self-map of a bounded Reinhardt domain Ω with C^2 smooth boundary in $\mathbb{C}^n$. Then f extends holomorphically to a neighborhood of $\overline{\Omega}$.*

Proof First we observe that $0 \notin \partial\Omega$ since Ω is of smoothness C^2. Indeed, if we write $\partial\Omega$ as $r = r(|z|^2, |w|^2) = 0$, then $\nabla r = r_z \bar{z} + r_w \bar{w}$. If $(0,0) \in \partial\Omega$, we will have $\nabla r(0,0) = 0$, a contradiction to smoothness at 0. So if $0 \notin \Omega$, then by a theorem of Barrett [14] that f extends holomorphically to a neighborhood of $\overline{\Omega}$. Now we only have to consider the case that $0 \in \Omega$. Consider the hull of holomorphy $\hat{\Omega}$ of Ω. Since Ω is Reinhardt, then $\hat{\Omega} \subset \mathbb{C}^n$. By Lemma 2.2, we conclude that f extends to a proper holomorphic self-map of $\hat{\Omega}$. Since $\hat{\Omega}$ is pseudoconvex, Reinhardt and $0 \in \hat{\Omega}$, it follows from Lemma 2.1 that $\hat{\Omega}$ is complete, and therefore, by a result of Bell [15], that f extends holomorphically to a neighborhood of $\bar{\hat{\Omega}}$, in particular, $\bar{\Omega}$. The proof is complete.

Using the same idea, we can give a proof of Theorem 1.5 as follows.

Proof First we extend f to $\hat{f} : \hat{\Omega} \to \hat{\Omega}$, by Lemma 2.2. Since Ω contains the origin, so does $\hat{\Omega}$. By Lemma 2.1, we see $\hat{\Omega}$ is complete. Since Ω is C^2 smooth and bounded, it contains a strictly pseudoconvex point on its boundary, and therefore $\hat{\Omega}$ contains the same point. Thus a result of Berteloot ([1], Theorem 1.3) implies that f is biholomorphic.

Now we are ready to apply the results of Berteloot [1] using lemmas above to get the following properties of self-maps on the branch locus. The following is a special case of Proposition 3.1 of [1], and is the key to control the branch locus.

Lemma 2.4. *Let Ω be a bounded Reinhardt domain Ω of C^2 smoothness in $\mathbb{C}^{n+1}$. Then there exists a finite family of multi-indices $\mathcal{I} \subset \mathbb{Z}^n \times \mathbb{Z}$ and an associated space of rational functions*

$$\mathcal{G} =: \bigoplus_{(K,l) \in \mathcal{I}} \mathbb{C} \cdot z^K w^l$$

such that for any proper holomorphic self-map of Ω, there exists an $(n+1, n+1)$ matrix $Q_f =: [(Q_f)_{k,p}]$ with entries in $\mathcal{G}$ which satisfies the following identity:

$$\left[\frac{\partial f_k}{\partial z_p}\right] [(Q_f)_{k,p}] = i[\delta_{k,p} f_k].$$

Proof We only have to verify that the condition of Proposition 3.1 in [1] is met. Namely, we have to prove the existence of some point $\eta \in \partial\Omega \cap (\mathbb{C} \setminus \{0\})^n$ such that $\partial\Omega$ is C^2 strictly pseudoconvex at η and f does not branch on the torus $T_\eta = \{|z_1| = |\eta_1|, ..., |z_n| = |\eta_n|\}$. The existence of strictly pseudoconvex points is ensured by the global smoothness. If η is strictly pseudoconvex, then T_η is a strictly pseudoconvex torus, and therefore f cannot branch at any point of T_η because f maps T_η to smooth boundary points by a well-known result [1]. Therefore the lemma is proved.

By the same reasoning, we have the following, which is Proposition 3.3 of [1].

Lemma 2.5. *Let f be a proper holomorphic self-map of a bounded Reinhardt domain Ω of C^2 smoothness in $\mathbb{C}^n$. If $V_f \neq \emptyset$, then (after some linear transformation permuting the coordinate axes) there exists an integer $m \leqslant n$ so that*

$$V_f = \cup_{j=1}^m \mathcal{H}_j,$$

$$f(\mathcal{H}_j) = \mathcal{H}_j,$$

$$f^{-1}(\mathcal{H}_j) = \mathcal{H}_j,$$

where $\mathcal{H}_j = \Omega \cap \{z_j = 0\}$ *and* $j = 1, ..., m$.

3 The Proof of Theorem 1.4

First we point out the following simple but useful results.

Lemma 3.1. *Let* f *be a proper holomorphic self-map of* $\{z \in \mathbb{C} : r < |z| < R\}$ *where* $r > 0$. *Then either* $f = e^{i\theta} z$ *or* $e^{i\theta} \frac{rR}{z}$.

Proof Following the proof of Lemma 3 in [9], we give the simple proof here. It is well-known that f extends continuously to the boundary. Consider the harmonic function $\phi(z) = ln|f(z)|$. Then ϕ is continuous up to the boundary since $r > 0$. If f maps $|z| = r$ to $|z| = r$, then f maps $|z| = R$ to $|z| = R$. Therefore both $\phi(z)$ and $ln|z|$ solve the same Dirichlet problem, which implies $\phi(z) = ln|z|$ or $f(z)| = |z|$. It follows $f(z) = e^{i\theta} z$. If f maps $|z| = r$ to $|z| = R$, then f maps $|z| = R$ to $|z| = r$. We can apply the function $\frac{rR}{f}$ to the above case to conclude that $f = e^{i\theta} \frac{rR}{z}$.

The useful result of this lemma is the following.

Lemma 3.2. *Suppose* Ω *is a bounded smooth Reinhardt domain in* $\mathbb{C}$ *with finite connectivity but not simply connected. Then every proper holomorphic self-map of* Ω *is biholomorphic. In particular* f *is* $e^{i\theta} z$ *or* $\Omega = \{r < |z| < R\}$ *and* $f = e^{i\theta} rR/z$.

Proof Since the domain is assumed smooth then the origin is not the boundary of Ω. Now we assume $\partial\Omega$ is given by $|z| = r_j, j = 1, ..., k$ where r_j is increasing and $r_i > 0$. Iterating if necessary, we may assume f maps $\{r_1 < |z| < r_2\}$ to itself. Hence Lemma 3.1 applies to conclude that either $f = e^{i\theta} z$ or $\Omega = \{r_1 < |z| < r_2\}$ and $f = e^{i\theta} r_1 r_2 / z$. Indeed, if Ω has more than one component, $\Omega = \{r_1 < |z| < r_2\} \cup \{r_3 < |z| < r_4\}$, then we would also have $f = e^{i\theta} r_3 r_4 / z$, a contradiction since $r_1 r_2 \neq r_3 r_4$.

We will prove Theorem 1.1 using results in the above section. Here we assume that Ω is a bounded Reinhardt domain with C^∞ boundary in $\mathbb{C}^2$. In order to prove Theorem 1.2, it suffices to prove $V_F = \emptyset$ by a well-known result of Pinchuk. By Lemma 2.5, we have that if $V_f \neq \emptyset$, then either $V_f = \Omega \cap \{z = 0\}$ or $V_f = \Omega \cap \{z = 0\} \cup \Omega \cap \{w = 0\}$.

Lemma 3.3. *Let* $F = (f, g)$ *be a proper holomorphic self-map of a bounded Reinhardt domain* Ω *of* C^∞ *smoothness in* $\mathbb{C}^2$. *If* V_f *contains* $\Omega \cap \{z = 0\}$, *then* g *is independent of* z.

Proof First, we have $F : \Omega \cap \{z = 0\} \to \Omega \cap \{z = 0\}$ by Lemma 2.5. It follows that $f(0, w) = 0$, and $g(0, w)$ is a proper map from $\Omega \cap \{z = 0\}$ to $\Omega \cap \{z = 0\}$. Take a point $p = (0, w_0) \in \partial\Omega$, and we may assume $F(p) = p$ since Ω is Reinhardt. In a neighborhood U of p in $\mathbb{C}^2$, we can define $\partial\Omega$ near p by the equation

$$|w|^2 + \phi(|z|^2) = 0.$$

In fact, we can always have $\partial\Omega$ defined by a defining function of the form $r(|z|^2, |w|^2) = 0$. Since $\Omega \cap \{z = 0\}$ is smooth, so we can solve $|w|^2$ by the implicit function theorem near p. Consider the complex tangential derivative

$$L = \bar{w} \frac{\partial}{\partial z} - \phi_r(|z|^2) \bar{z} \frac{\partial}{\partial w}.$$

By properness, it follows, on $U \cap \partial\Omega$, that

$$|g(z, w)|^2 + \phi(|f(z, w)|^2) = 0.$$

Operating L on both sides of above equation , we have, on $U \cap \partial\Omega$

$$\bar{w}g_z\bar{g} - \phi_r(|z|^2)\bar{z}g_w\bar{g} + \bar{w}\phi_r(|f|^2)f_z\bar{f} - \phi_r(|z|^2)\bar{z}\phi_r(|f|^2)f_w\bar{f} = 0.$$

Letting $z = 0$,we have on $|w|^2 = -\phi(0)$ near p

$$\bar{w}g_z g = 0,$$

which implies that $g_z(0, w) = 0$ on $|w|^2 = -\phi(0)$. Hence $g_z(0, w) = 0$ for $(0, w) \in \Omega$ by the uniqueness of holomorphic functions. Consider $L^k = L...L$. Rewrite the above equation as

$$\bar{w}g_z\bar{g} + A\bar{z} + B\bar{f} = 0,$$

where A, B are defined by the above equation. Taking L^k, we have

$$\bar{w}L^k(g_z)\bar{g} + (L^k A)\bar{z} + (L^k B)\bar{f} = 0.$$

Letting $z = 0$, we have

$$\bar{w}(g_z)^{(k)}\bar{g} = 0$$

which implies that $g_z^{(k)}(0, w) = 0$ for all k. The Taylor expansion of g at $z = 0$ for a fixed w tells that g is independent of z. This completes the proof.

Lemma 3.4. *Let* $g(w) = e^{i\theta}\frac{w-a}{1-\bar{a}w}$. *There exists* $0 < b < 1$ *such that for* $|w_0| \neq 1$,

$$|1 - |g^n(w_0)|^2| \geqslant Cb^n$$

for some constant $C > 0$ *and* $n = 1, 2, ...$

Proof We consider the case $|w_0| < 1$. If $\{g^n(w)\}$ is a compact family, this is obvious. If $\{g^n(w)\}$ is not compact, by the Wolff-Denjoy theorem, $\{g^n(w)\}$ converges to 1 (say). Let $b_n = g^n(w_0)$. Then

$$b_{n+1} = \frac{b_n - a}{1 - \bar{a}b_n}$$

We observe that a is real since $b_n \to 1$. We have

$$b_{n+1} - 1 = (b_n - 1)\frac{1 + a}{1 - ab_n},$$

$$|b_{n+1} - 1| = |(b_n - 1)|\left|\frac{1 + a}{1 - ab_n}\right|$$

$$\geqslant \frac{1 - |a|}{1 + |a|}|b_n - 1| \geqslant b^n\left|\frac{1 - b_1}{b}\right|$$

where $b = \frac{1-|a|}{1+|a|}$. By iteration theory, one has

$$(1 - |b_n|^2) \geqslant C|1 - b_n|^2.$$

Therefore

$$1 - |b_n|^2 \geqslant C(b^2)^n\left|\frac{1 - b_1}{b}\right|^2 \approx Cb^n.$$

For the case $|w_0| > 1$, we rewrite for $|w| > 1$, $g(w) = \frac{1}{\tilde{g}(1/w)}$, where $\tilde{g}(\zeta) = \frac{\zeta - \bar{a}}{1 - a\zeta}$ for $|\zeta| < 1$ and $\zeta = 1/w$. It is easy to check that

$$g^n(w) = \frac{1}{\tilde{g}^n(1/w)}.$$

For $|w_0| > 1$, we have

$$|g^n(w_0)|^2 - 1 = \frac{1 - |\tilde{g}^n(1/w_0)|^2}{|\tilde{g}^n(1/w_0)|^2} \geqslant 1 - |\tilde{g}^n(1/w_0)|^2 \geqslant Cb^n.$$

The lemma is proved.

Lemma 3.5. *Let $F = (f, g)$ be a proper holomorphic self-map of a bounded Reinhardt domain Ω of C^∞ smoothness in $\mathbb{C}^2$. If $V_f = \Omega \cap \{z = 0\}$, then $\partial\Omega$ is Levi flat near $\{z = 0\} \cap \partial\Omega$. More precisely, $\partial\Omega$ is defined by $|w|^2 = const$ for z near 0.*

Proof Let $p = (0, a) \in \partial\Omega \cap \{z = 0\}$. We may assume $F(p) = p$. Choose a neighborhood U of p so that

$$\partial\Omega \cap U = \{|w|^2 + \phi(|z|^2) = 0\} \cap U$$

$$\Omega \cap U = \{|w|^2 + \phi(|z|^2) < 0\} \cap U$$

where we assume $\phi(0) = -1$. Since F maps $\{z = 0\}$ to $\{z = 0\}$ by Lemma 2,5 and $V_f \supset \{z = 0\}$, we see that $f(0, w) = f_z(0, w) = 0$. By considering iteration of F and the chain rule, we can further assume $f_{zz}(0, w) = 0$. So we conclude that there exists a $0 < \delta < 1$ such that

$$|f(z, w)| < |z|^2$$

for $|z| < \delta$ and $(z, w) \in \bar{\Omega} \cap U$.

By Lemma 3.3, g is independent of z, and we have $\det JF = f_z(z, w) g_w(w)$. Since $V_f = \Omega \cap \{z = 0\}$ by assumption, we have $g_w(w) \neq 0$. That is $g(w)$ is a biholomorphic map from $\Omega \cap \{z = 0\}$ to $\Omega \cap \{z = 0\}$. When $\Omega \cap \{z = 0\}$ is simply connected, then it is the unit disk, otherwise it is an annulus with finitely many boundary components (with the origin not on the boundary) and Lemma 3.2 applies. Therefore we see that either $g(w) = e^{i\theta} \frac{w - a}{1 - \bar{a}w}$, or $g(w) = e^{i\theta} w$ or $g(w) = e^{i\theta} r/w$ and the annulus is given by $\{r < |w| < 1\}$

We claim that $\phi(|z|^2) = -1$ for $|z| < \delta$. Indeed, if there is z_0 such that $|z_0| < \delta$ and $\phi(|z_0|^2) > -1$. We define w_0 such that $|w_0|^2 = -\phi(|z_0|^2)$. It follows that $|w_0| < 1$ and $(z_0, w_0) \in \partial\Omega$. By properness of F we have

$$|g^n(w_0)|^2 + \phi(|f^n(z_0, w_0)|^2) = 0.$$

It follows that

$$1 - |g^n(w_0)|^2 = 1 + \phi(|f^n(z_0, w_0)|^2) \leqslant C|f^n(z_0, w_0)|^2 \leqslant C|z_0|^{2^{n+1}} \leqslant C\delta^{2^{n+1}},$$

where we have used the fact $\phi(0) = -1$ and $\phi(r) = -1 + O(r)$ when $r \approx 0$. On the other hand, we have by Lemma 3.4

$$1 - |g^n(w_0)|^2 \geqslant Cb^n$$

for some b: $0 < b < 1$. This is a contradiction.

If there is z_0 such that $|z_0| < \delta$ and $\phi(|z_0|^2) < -1$. We define w_0 such that $|w_0|^2 = -\phi(|z_0|^2)$. It follows that $|w_0| > 1$ and $(z_0, w_0) \in \partial\Omega$. By the same argument above and Lemma 3.4, we will reach a contradiction again. We would like to point out when $g(w) = e^{i\theta} r/w$, we may have to consider an iteration $F^{2n}(z, w)$ instead, then $g^{(2n)}(w) = e^{i\theta} w$. This completes the proof.

We consider the natural projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ defined by $\pi(z, w) = z$.

Now we are ready to prove Theorem 1.1. We shall prove it by two cases. According to Theorem 1.5, we can assume now that Ω does not contain the origin.

Proof Case I: $V_F = \Omega \cap \{z = 0\} \cup \Omega \cap \{w = 0\}$. In this case by Lemma 2.5, we have

$$F(\Omega \cap \{z = 0\}) = \Omega \cap \{z = 0\},$$

$$F(\Omega \cap \{w = 0\}) = \Omega \cap \{w = 0\},$$
$$F^{-1}(\Omega \cap \{z = 0\}) = \Omega \cap \{z = 0\},$$
$$F^{-1}(\Omega \cap \{w = 0\}) = \Omega \cap \{w = 0\}.$$

Since $0 \notin \Omega$, we can also assume that $0 \notin \hat{\Omega}$, otherwise similar to the proof (using Theorem 1.3 of [1]) of Theorem 1.5, we can prove that f is biholomorphic. Therefore by the proof of Lemma 2.1, we see that $\hat{\Omega} \cap \{z = 0\}, \hat{\Omega} \cap \{w = 0\}$ are not simply connected but of finite connectivity due to smoothness of the boundary of $\partial\Omega$. By Lemma 3.3, f is independent of w, and g independent of z, and therefore by Lemma 3.2 we have $f(z) = az, g(w) = bw$ where a, b are nonzero complex numbers. In particular, $F = (f, g)$ is biholomorphic and this is a contradiction.

Case II: $V_F = \Omega \cap \{z = 0\}$. Since $0 \notin \Omega$, we see that $w \neq 0$ on $\bar{\Omega}$. Therefore V_f is a Reinhardt domain in $\mathbb{C}$ with finite connectivity (but not simply connected) and therefore, by Lemma 3.2 g is aw or a/w with a nonzero constant. We assume $a = 1$ by iteration if necessary. Consider $E = \pi(\Omega)$. By Lemma 3.5 Ω is Levi flat near $z = 0$. Let E_0 be a open set in E such that $O = \pi^{-1}(E_0) \cap \partial\Omega$ is the largest connected Levi flat piece on $\partial\Omega$ containing $\partial\Omega \cap \{z = 0\}$. Then we can see that $O = E_0 \times \{|w| = 1\}$. We can choose a small neighborhood U of O so that $\partial\Omega$ is defined by $|w|^2 + \phi(|z|^2) = 0$ on $U \cap \partial\Omega$ so that $\phi(|z|^2) = -1$ when $z \in E_0$, and $\phi(|z|^2) \neq -1$ when $z \notin \bar{E}_0$ but $z \in \pi(U)$. Now we claim that

$$F(O) = O, F^{-1}(O) = O,$$

$$F(\partial O) = \partial O, F^{-1}(\partial O) = \partial O.$$

This can be seen as follows. Consider $r = |w|^2 + \phi(|z|^2)$. We observe that $\rho = r \circ F$ is a defining function near O since $g_w(w) \neq 0$ and the complex normal direction remains to be in w near ∂O. Therefore we have the following identity

$$\Lambda_r(p) = |JF|^2 \Lambda_\rho(F(p)),$$

which easily implies the above claim. Here $\Lambda_r(p)$ is the Levi-determinant. Choose a point $(z_0, w_0) \in \partial\Omega \cap U$ such that $\phi(|z_0|^2) \neq -1$. We see that $z_0 \notin \overline{E}_0$. By properness of F, and $g = w$, we conclude that $f(z, w_0)$ is a proper map from $\{|z| < |z_0|\} \setminus \bar{E}$ to itself. By Lemma 3.2, we see that $f(z, w_0)$ is biholomorphic in z. But this is impossible, since $f(0, w) = 0, f_z(0, w) = 0$ by the assumption that $V_F = \Omega \cap \{z = 0\}$.

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