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UNIQUE CONTINUATION FOR SCHRODINGER OPERATORS WITH SINGULAR POTENTIALS

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ABSTRACT: It is shown in this paper that the Schrodinger operator with a potential satisfying $|V(x)| \leq M/|x|^2$ *a.e.* in the unit ball B has the strong unique continuation property in $H^{2,2}(B)$ for $n \geq 2$.

1. Introduction

Consider the Laplace operator $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ on R^n and a function $V(x)$ on the unit ball $B = B(0,1)$ of R^n. We say the Schrodinger operator

$$\Delta u + Vu = 0$$

satisfies the strong unique continuation property in Sobolev space $H^{2,q}(B)$, if whenever $u \in H^{2,q}(B)$ is a solution of the operator and vanishes to infinite order at a point, u is identically zero. Here, a function $f \in L^2_{loc}$ is said to vanish to infinite order at 0 if

$$\int_{|x|<r} f^2 dx = O(r^k)$$

for every k as $r \to 0$. When $n = 2$ and V is bounded, Carleman proved a unique continuation theorem, and all subsequent work follows his basic idea. There is large literature of applications of Carleman-type inequalities to this and other uniqueness questions.

Over the past several years, considerable attention has been paid to the case $V \in L^{\omega}(B)$ for $\omega < \infty$; this is largely because the unique continuation property can be used to prove that the Schrodinger operator on $L^2(R^n)$ has no positive eigenvalues provided $V \in L^{\omega}_{loc}(R^n)$ and V has suitable behavior at infinity. Recently, Jerison and Kenig [JK] proved that if $n > 2$, the Schrodinger operator has the strong unique continuation property for $V \in L^{n/2}$ in the Sobolev space $H^{2,q}(B)$ with $q = 2n/(n+2)$. This improves all previous results (see the references in [JK]) and is best possible in the context of L^p spaces. However Jerison and Kenig's theorem does not cover the case $V(x) = C/|x|^2$ since it does not belong to $L^{n/2}(B)$. We take up this issue and prove the following.

Theorem 1. *Suppose $n \geq 2$. The Schrodinger operator with the potential $|V(x)| \leq M/|x|^2$ a.e. in B has the strong unique continuation property in $H^{2,2}(B)$.*

Remarks. 1) Theorem 1 establishes the unique continuation result in an important borderline case not coverd by the general result of Jerison and Kenig [JK] since $1/|x|^2 \notin L^{n/2}$. Moreover, since we do not require any local restriction on the size of V, Theorem 1 is not contained either in E. Stein's subsequent improvement of Jerison and Kenig's result concerning $V \in L^{n/2,\infty}_{loc}(R^n)$, the Lorentz space of weak $n/2$ type, see [JK]. It is worth mentioning that T. Wolff has constructed an example showing that in the result of Stein, a smallness condition on the norm of the potential is needed [W]. On the other hand, easy examples show that the potential $C/|x|^2$ is of strongest singularity for unique continuation to hold. For example, as in [JK], if $\epsilon > 0, f(x) = e^{-(\log 1/|x|^2)^{1+\epsilon}}$ for $|x| < 1$, then f vanishes at 0 of infinite order, and $(\Delta f/f) = V$ satisfies $V \approx (\log 1/|x|)^{2\epsilon}(1/|x|^2)$.

2) The Schrodinger operators

$$H = -\Delta + V \tag{1}$$

were also considered in [GL] and some related results were obtained. It was shown by Garofalo and F.H. Lin in [GL] that if $V = \frac{f(\omega)}{|x|^2}$ where f is a bounded function homogeneous of degree zero, i.e., if $\omega = x/|x|, x \neq 0$, we have

$$f(\omega) = f(\frac{x}{|x|}) \quad \text{and} \quad |f(\omega)| \leq C$$

the operator H has the unique continuation property. So this result may be considered as a special case of Theorem 1. In particular it was also shown in [GL] that if $V(x) = V^+(x) - V^-(x)$ satisfies

$$0 \le V^+(x) \le \frac{C}{|x|^2}$$

and

$$0 \le V^-(x) \le \frac{C}{|x|^2}$$

for $x \in B$, then any solution u of (1) in $H^{1,2}$ satisfying

$$\int_{|x|<r} u^2 dx = O(\exp -Ar^\alpha) \quad \text{for some} \quad A, \alpha > 0$$

as $r \to 0$ must be identically zero in B.

In this paper, we also consider the partial differential equation

$$Lu = -\Delta + b(x)\nabla u + V(x)u = 0. \tag{2}$$

in B. Assume that there exists $r_0 > 0$, $f : (0, r_0) \to R^+$ and $c > 0$ such that f is increasing and satisfies

$$\int_0^{r_0} \frac{f(r)}{r} dr < \infty. \tag{3}$$

and for $x \in B$,

$$|b(x)| \le \frac{Cf(|x|)}{|x|}, \quad |V(x)| \le \frac{Cf(|x|)}{|x|^2}. \tag{4}$$

In [GL], Garofalo and Lin proved in particular that the operator L satisfying the conditions (2), (3) and (4) has the strong unique continuation property in the sense that the only $H^{1,2}_{loc}(\Omega)$ solution of $Lu = 0$ which vanishes of infinite order at 0 is $u = 0$. Concerning the function f we note from (3) that $f(x)$ tends to zero as x tends to 0. So it would be interesting to see if one can weaken the condition (4) by replacing f by 1 in (4) to prove the unique continuation property.

We prove the unique continuation result for smooth solutions of (2) with some restrictions on the coefficents.

Theorem 2. *Let $u \in C^\infty(B)$ be a solution of*

$$-\Delta u + a(|x|)\partial_r u + V(|x|)u = 0 \tag{5}$$

where $r = |x|$ and a and V are radial functioons satisfying

$$|a(|x|)| \leq \frac{C}{r}, \quad |V(|x|)| \leq \frac{C}{r^2}.$$

If u vanishes of infinite order at 0, then $u = 0$ in B.

In R^2, we prove the following, the proof of which is reduced to the Cauchy-Riemman operator.

Theorem 3. *Let $u \in H^{1,2}(B)$ where $B \subset R^2$, be a solution of*

$$-\Delta u + a(x)\nabla u = 0$$

where the vector function $a(x)$ satisfies

$$|a(x)| \leq \frac{C}{|x|}.$$

If u vanishes of infinite order at 0, then $u = 0$ in B.

Remark. A special case of the main theorem in R^2 in [K] can be stated as follows: an operator of the form

$$-\Delta + a(x)\nabla u$$

in R^2 has the strong unique continuation property if $a(x)$ satisfies

$$|a(x)| \leq \frac{C}{r(\log \frac{2}{r})}$$

in B. So Theorem 3 generalizes this special case by allowing the best possible singularity for unique continuation to hold in this case. A counterexample of this kind is given in the end of the paper.

2. Proof of Theorem 1

Our proof of Theorem 1 is based on the Carleman-type inequality of Amerin et al. in [ABG]. We state a special case of it for our need. Let us denote by $H_c^{2,q}(\Omega)$ the subspace of $H^{2,q}(\Omega)$ of functions having compact support in Ω.

Theorem 4. **(Amerin et al.)** *The inequality*

$$|||x|^\tau f||_{L^2(R^n)} \leq C(\tau)|||x|^{\tau+2}\Delta f||_{L^2(R^n)} \tag{6}$$

holds for any $\tau \in R$ *and for all* $f \in H_c^{2,2}$ $(R^n - 0)$. *The constant* $C(\tau)$ *is finite provided that*

$$(\tau - l + 2 - n/2)(\tau + l + n/2) \neq 0$$

for $l = 0, 1, 2, 3, ...$ *and it is given by*

$$C(\tau) = Sup_{l\geq 0}|(\tau - l + 2 - n/2)(\tau + l + n/2)|^{-1}.$$

Remark. With $\tau = -(n/2 + 1/2 + m), m = 1, 2, ...,$ $C(\tau)$ is not only uniformally bounded but also goes to zreo as $m \to \infty$. This is one of the key facts that will be used in the proof of Theorem 1. Indeed, if $\tau_m = -(n/2 + 1/2 + m)$, then

$$(\tau - l + 2 - n/2)(\tau + l + n/2) = -(n + m + l - 3/2)(l - m - 1/2)$$

Since l, m are nonegative integers, we have $|l - m - 1/2| \geq 1/2$. It follows that

$$\begin{aligned} C(\tau_m) &= Sup_{l\geq 0}|(\tau_m - l + 2 - n/2)(\tau_m + l + n/2)|^{-1} \\ &= Sup_{l\geq 0}((n + m + l - 3/2)|l - m - 1/2|)^{-1} \\ &\leq \frac{2}{n + m - 3/2}, \end{aligned}$$

which tends to 0 as $m \to \infty$.

To prove Theorem 1, we need to check if the vanishing to infinite order of a solution of the Shrodinger operator implies to that of its gradient. To this end, we begin with the following lemma.

Lemma 5. *Let* f *be a* $H^{2,2}(B)$ *solution of (1) with* $|V(x)| \leq M/|x|^2$ *a.e. such that it vanishes to infinite order at the origin. Then* ∇f *also vanishes to infinite order at 0.*

Proof. We will use the well-known L^2 inequality,

$$\int_{|x|<\frac{a}{2}} |\nabla f|^2 dx \le C\{\frac{1}{a^2}\int_{|x|<a} f^2 dx + a^2 \int_{|x|<a} (\nabla f)^2\}.$$

Now we prove that if f vanishes to infinite order, then so does $|x|^{-2}f$. Given n, choose M so that

$$\int_{|x|<a} f^2 dx \le M a^{n+4}.$$

Then we have

$$\begin{aligned}\int_{|x|<a} (|x|^{-2}f)^2 dx &= \sum_{j=0}^{\infty} \int_{2^{-(j+1)}a<|x|<2^{-j}a} (|x|^{-2}f)^2 \\ &\le \sum_{j=0}^{\infty} (2^{-j+1}a)^{-4} \int_{|x|<2^{-j}a} f^2 dx \\ &\le \sum_{j=0}^{\infty} (2^{-j+1}a)^{-4}(2^{-j}a)^{n+4} M \\ &= 32 M a^n.\end{aligned}$$

So $|x|^{-2}f$ vanishes to infinite order. Since $|\Delta f| \le M|x|^{-2}|f|$, both terms on the right side of the L^2 inequality are $O(a^n)$ for every n and the lemma follows.

Proof of Theorem 1.

Let $f \in H^{2,2}(B)$ be a solution of the Schrodinger operator. Let $\phi \in C_0^\infty(B)$ be a cutoff function such that $0 \le \phi \le 1, \phi(x) = 1$ for $|x| < ./2$. Let $g = \phi f$. Choose $0 \le \Psi_j \le \Psi_{j+1} \le 1, \Psi_j(x) = 1$ for $|x| > 2/j, \Psi_j(x) = 0$ for $|x| < 1/j, |\nabla\Psi_j(x)| \le C'j, |\Delta\Psi_j(x)| \le C'j^2$. Let $g_j(x) = \Psi_j(x)g(x)$.

Now it is easy to that $g_j \in H_c^{2,2}(R^n - 0)$ since $f \in H^{2,2}(B)$. By Theorem 4, we see that (6) holds for g_j. From now on we set $\tau = \tau^m = -(n/2 + 1/2 + m), m = 1, 2, 3, ...,$ for which it is shown early that the constants $C(\tau)$ appearing in Theorem 5 satisfy $C(\tau_m) = O(1/m)$. Now choose $0 < \rho < 1/2$ and m sufficiently large such that $C(\tau_m)M < 1/2$. we shall show that $f = 0$ in $|x| < \rho$. We have by (6)

$$\begin{aligned}|||x|^\tau \Psi_j f||_{L^2(|x|<\rho)} &\le |||x|^\tau g_j||_{L^2(dx)} \le C(\tau)|||x|^{\tau+2}\Delta g_j||_{L^2(dx)} \\ &\le C(\tau)|||x|^{\tau+2}\Delta(\Psi_j f)||_{L^2(|x|<\rho)} \\ &+ C(\tau)|||x|^{\tau+2}\Delta g||_{L^2(|x|>\rho)}. \qquad (9)\end{aligned}$$

Leibniz's rule shows that $\Delta(\Psi_j f) = \Psi_j \Delta f + f\Delta\Psi_j + 2\nabla\Psi_j\nabla f$. Thus, the right hand side of (9) is bounded (for j large)by

$$C(\tau)|||x|^{\tau+2}\Psi_j\Delta f||_{L^2(|x|<\rho)} + C'C(\tau)j^{-\tau}||f||_{L^2(|x|<2/j)}$$

$$+C'C(\tau)j^{-\tau-1}||\nabla f||_{L^2(|x|<2/j)} + C(\tau)\rho^{\tau+2}||\Delta g||_{L^2(|x|>\rho)}.$$

Since $|\Delta f| \leq |V(x)||f(x)|$, we have, using $|x|^2|V| \leq M$,

$$\begin{aligned} C(\tau)|||x|^{\tau+2}\Psi_j\Delta f||_{L^2(|x|<\rho)} &\leq C(\tau)|||x|^{\tau+2}\Psi_j V f||_{L^2(|x|<\rho)} \\ &= C(\tau)|||x|^{\tau}\Psi_j f|x|^2 V||_{L^2(|x|<\rho)} \\ &\leq C(\tau)M|||x|^{\tau}\Psi_j f||_{L^2(|x|<\rho)} \\ &\leq 1/2|||x|^{\tau}\Psi_j f||_{L^2(|x|<\rho)} < +\infty. \qquad (10) \end{aligned}$$

The last term is finite since $f \in H^{2,2}(B)$ Thus from (9) and (10)

$$1/2|||x|^{\tau}\Psi_j f||_{L^2(|x|<\rho)} \leq C'C(\tau)j^{-\tau}||f||_{L^2(|x|<2/j)}$$

$$+C'C(\tau)j^{-\tau-1}||\nabla f||_{L^2(|x|<2/j)} + C(\tau)\rho^{\tau+2}||\Delta g||_{L^2(|x|>\rho)}. \qquad (11)$$

We need to show the first two terms go to zero as $j \to \infty$. But by the definition and Lemma 5, we have

$$j^{-\tau}|||f||_{L^2(|x|<2/j)} \to 0$$

$$j^{-\tau-1}||\nabla f||_{L^2(|x|<2/j)} \to 0$$

as $j \to \infty$. It is worth mentioning that $\tau = \tau_m$ are negative. So by the monotone convergence theorem, it follows from (11)

$$||(|x|/\rho)^{\tau} f||_{L^2(|x|<\rho)} \leq 2C(\tau)\rho^2||\Delta g||_{L^2(|x|>\rho)}$$

Letting $m \to \infty$ with $\tau = \tau_m$,we see that $f = 0$ on $|x| < \rho$. The proof of Theorem is then complete.

3. Proofs of Theorem 2 and 3

In this section we shall see that the proof of Theorem 2 is reduced to prove the following lemma.

Lemma 6. *Let $u \in C^\infty((0,R])$ and satisfy*

$$|u''(r)| \le C(\frac{1}{r^2}|u(r)| + \frac{1}{r}|u'(r)|). \tag{12}$$

If $u = O(r^N)$ for every N as $r \to 0$, then $u = 0$ in $[0,R]$.

Assuming the truth of Lemma 6, we begin with

Proof of Theorem 2. For $x \in R^n$, we set $r = |x|$ and $x = r\omega$ where ω varies over the uint sphere S_{n-1}. The Laplacian Δ can be written in the spherical coordinates

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega$$

where Δ_ω is the Laplace-Beltrami operator on S_{n-1}. In $L^2(S_{n-1})$ we choose an orthonormal basis of spherical harmonics $P_{l,\alpha}$ with $l \in N$ and $1 \le \alpha \le \alpha(l)$ where

$$\alpha(l) = \frac{(2l+n-2)(n+l-3)!}{(n-2)!l!}$$

for $n \ge 2$. In particular, we have

$$\Delta_\omega(P_{l,\alpha}(\omega)) = -l(l+n-2)P_{l,\alpha}(\omega). \tag{13}$$

For $R' < R, u \in C^\infty(\overline{B}'_R)$, and we have

$$u(x) = \sum_{l=0}^{\infty}\sum_{l=0}^{\alpha(l)} \tilde{u}_{l,\alpha}(r)P_{l,\alpha}(\omega)$$

where

$$\tilde{u}_{l,\alpha}(r) = \int_{S_{n-1}} u(r\omega)\overline{P_{l,\alpha}}(\omega)d\omega. \tag{14}$$

The series above converges uniformaly in $B_{R'}$ and $\tilde{u}_{l,\alpha} \in C^2$.

We will show that $\tilde{u}_{l,\alpha} = 0$ in $B_{R'}$. Indeed, subsituting Laplacian into the equation (5) in Theorem 2, multiplying the resulting equation by $\overline{P}_{l,\alpha}$, integrating with respect to $d\omega$ over S_{n-1}, and using the self-adjontness of Δ_ω and the equations (13) and (14) we obtain the equation satisfied by $\tilde{u}_{l,\alpha}$

$$\frac{\partial^2}{\partial r^2}\tilde{u}_{l,\alpha} + \frac{n-1}{r}\frac{\partial}{\partial r}\tilde{u}_{l,\alpha} + a(r)\frac{\partial}{\partial r}\tilde{u}_{l,\alpha} + \left\{V(r) - \frac{l(l+n-2)}{r^2}\right\}\tilde{u}_{l,\alpha} = 0. \tag{15}$$

From (15), there exists a constant C depending on a, b such that

$$|\tilde{u}''_{l,\alpha}(r)| \leq C(\frac{1}{r}|\tilde{u}'_{l,\alpha}(r)| + \frac{1}{r^2}|\tilde{u}_{l,\alpha}(r)|). \tag{16}$$

Then by Lemma 6 and (16), $\tilde{u}_{l,\alpha} = 0$, so $u = 0$.

Now we are in a position to prove Lemma 6 and Theorem 3.

Proof of Lemma 6. First we observe that if $u = O(r^N)$, then $u' = O(r^N)$ and $u'' = O(r^N)$ for every N. Therefore the functions $r^{-\beta}u, r^{-\beta}u'$ and $r^{-\beta}u''$ are integrable on $(0, R]$. We claim that if $u = O(r^N)$ for every N, then the following holds: there exists a constant C such that for $\beta \geq 2$

$$\int_0^R (r^{-4}|u(r)|^2 + r^{-2}|u'(r)|^2)r^{-\beta}dr \leq \frac{C}{(\beta+1)^2}\int_0^R |u''(r)|^2 r^{-\beta}dr. \tag{17}$$

Indeed, consider the identity

$$(u'^2)' = 2u'u''. \tag{18}$$

Mutltiplying $r^{-\beta-1}$ on (18) gives

$$r^{-\beta-1}(u'^2)' = r^{-\beta-1}2u'u''. \tag{19}$$

We get from (19)

$$(r^{-\beta-1}u'^2)' + (\beta+1)r^{-\beta-2}u'^2 = 2r^{-\beta-1}u'u''. \tag{20}$$

Integrating (20) over $[0, R]$ gives, using $u = O(r^N)$,

$$R^{-\beta-1}u'(R)^2 + (\beta+1)\int_0^R r^{-\beta-2}u'^2 dr = 2\int_0^R r^{-\beta-1}u'u'' dr.$$

Then we have the inequality

$$(\beta+1)\int_0^R r^{-\beta-2}u'^2 dr \leq 2\int_0^R r^{-\beta-1}u'u'' dr. \tag{21}$$

On the other hand we have

$$\int_0^R r^{-\beta-1}u'u'' dr = \int_0^R (\frac{\beta+1}{4})^{1/2} r^{-\beta-2/2}u'(\frac{4}{\beta+1})^{1/2} r^{-\beta/2}u'' dr$$

$$\leq \frac{\beta+1}{4}\int_0^R u'^2 r^{-\beta-2}dr + \frac{4}{\beta+1}\int_0^R u''^2 r^{-\beta}dr. \tag{22}$$

From (21) and (22) it follows that

$$\int_0^R r^{-\beta-2}u'^2 dr \leq \frac{4}{(\beta+1)^2}\int_0^R u''^2 r^{-\beta}dr. \tag{23}$$

Now we consider

$$\begin{aligned}\int_0^R u^2 r^{-\beta-4}dr &= -\frac{R^{-\beta-3}}{\beta+3}u^2(R) + \frac{1}{\beta+3}\int_0^R 2uu'r^{-\beta-3}dr \\ &\leq \frac{2}{\beta+3}\int_0^R uu'r^{-\beta-3}dr \\ &= \frac{2}{\beta+3}\left\{(\frac{\beta+3}{4})^{1/2}ur^{-\beta-4/2}(\frac{4}{\beta+3})^{1/2}u'r^{\beta-2/2}dr\right\} \\ &\leq \frac{1}{2}\int_0^R u^2 r^{-\beta-4}dr + \frac{8}{(\beta+3)^2}\int_0^R u'^2 r^{-\beta-2}dr. \end{aligned} \tag{24}$$

Therefore we get from (23) and (24)

$$\int_0^R u^2 r^{-\beta-4}dr \leq \frac{16}{(\beta+3)^2}\int_0^R u''^2 r^{-\beta}dr \tag{25}$$

Combining (23) and (25) yields the proof of (12). Now we are back to prove Lemma 6. Indeed if u is not identically zero then we come out with a controdication by letting $\beta \to \infty$ in (17).

To prove Theorem 3 it suffices to prove the following lemma which can be considered as a generalization of a lemma in [BL].

Lemma 7. *Suppose v is a function on the unit disk such that $|v_{\bar{z}}| \leq \frac{C}{|z|}|v|$ for some positive constant C. If v vanishes of infinite order at the origin, then v is identically zero.*

Proof. Define a function λ on the unit disk to be equal to $v_{\bar{z}}/v$ when v is not zero and equal to zero when v is zero. Note that λ is well-defined except at $z = 0$. But by assumption , λ is L^1 integrable on the unit disk. So $\lambda(z)dz \wedge d\bar{z}$ is a Borel measure with compact support U , the unit disk. Define a function u on U via

$$u(z) = \frac{1}{2\pi i}\int_U \frac{\lambda(\zeta)}{\zeta - z}d\zeta \wedge d\bar{\zeta}. \tag{26}$$

We claim that $u(z) = O(\log \frac{1}{|z|})$ when $z \neq 0$,i.e, we have to prove by (26)

$$\int_U \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta||\zeta - z|} = O(\log \frac{1}{|z|}). \tag{27}$$

Assuming the truth of the claim, we prove the lemma. Note that u is continuous on U except at 0 and has the property that it is C^∞ where v is not zero and $u_{\bar{z}} = v_{\bar{z}}/v$ when $v \neq 0, z \neq 0$. Now consider the function $h = ve^{-u}$. So h is continuous on $U - 0$ and is holomorphic when it is not zero. Thus, Rado's Theorem implies that h is holomorphic on $U - 0$. Now we want to show that 0 is also a removable singularity of h. Indeed, let $O = \{\Re u > 0\}$. Then on O

$$|e^{-u}| = e^{-\Re u} \leq 1.$$

On $U - O$, by the claim,

$$-\Re u \leq |u| \leq C \log \frac{1}{|z|}$$

for some positive constant C. Therefore. on $U - O$

$$|e^{-u}| = e^{-\Re u} \leq \frac{1}{|z|^C}.$$

Since v vanishes of infinite order at 0, there exists a $M > 0$ such that

$$|v(z)| \leq M|z|^C.$$

Then

$$|h(z)| = |v(z)||e^{-u}| \leq M$$

for $z \in U$, i.e., 0 is a removable singularity of h.

Now it is clear that h vanishes of infinite order at 0 so it must be zero identically, and hence, so must v. The proof of Lemma 7 is comlpete provided we prove the claim. To this end, let us break U into three pieces as follows:

$$I = \{|\zeta| < a/2\}$$

where $0 < a = |z| < 1/2$,

$$II = \{a/2 < |\zeta| < 2a\} \quad \text{and} III = \{2a < |\zeta| < 1\}$$

Then we have

$$\int_I \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta||\zeta - a|} \le \frac{C}{a} \int_I \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta|} = O(1), \tag{28}$$

and

$$\begin{aligned}\int_{II} \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta||\zeta - a|} &\le \frac{C}{a} \int_I \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta - a|} \\ &\le \frac{C}{a} \int_0^{2\pi} \int_0^{2a} d\rho d\theta = O(1),\end{aligned} \tag{29}$$

where we have used the polar coordinates centered at a. Finally,

$$\begin{aligned}\int_{III} \frac{d\zeta \wedge d\bar{\zeta}}{|\zeta||\zeta - a|} &= \int_{III} \frac{d\rho d\theta}{|\rho e^{i\theta} - a|} \\ &= \int_0^{2\pi} \int_a^1 \frac{d\rho d\theta}{((\rho - a)^2 + 2\rho a \sin^2 \theta/2)^{1/2}} \\ &\le 2\pi \int_{2a}^1 \frac{d\rho}{(\rho - a)} = \log \frac{1}{a} + C.\end{aligned} \tag{30}$$

Combining all estmates (28), (29) and (30) togather yields the proof of (27)

Proof of Theorem 3. It is a consequence of Lemma 7 by observing that $\Delta = \frac{1}{4} \frac{\partial^2}{\partial z \partial \bar{z}}$ and the condition

$$|\Delta u| \le \frac{C}{|x|} |\nabla u|$$

is equivelent to

$$|\bar{\partial} v| \le \frac{C}{|z|} |v|$$

where $v = \frac{\partial}{\partial z} u$.

We conclude the section by giving a simple example showing that the singularity in Lemma 7 cannot be replaced by $1/|z|^{1+\epsilon}$. Let us consider the equation

$$-\Delta u + \frac{c}{|x|^{1+\epsilon}} \nabla u = 0 \tag{31}$$

with c a constant vector. If we look for radial solutions of (31), we are led to consider the ode

$$r^2 u''(r) + (r - r^{1-\epsilon}) u'(r) = 0.$$

Letting $v = u'$, we come out with

$$r^2 v' + (r - r^{1-\epsilon})v = 0.$$

Now it is easy to see that

$$v = Ce^{-r^{-\epsilon}} \tag{32}$$

for some constant C. So u determined by (32) vanishes of infinite order at 0, but is identically zero.

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