The Journal of Geometric Analysis
Volume 8, Number 4, 1998

A Remark on Unique Continuation

By *Yifei Pan and Thomas Wolff*

ABSTRACT. In this paper a unique continuation result is proved for differential inequality of second order.

1. Introduction

We prove the following result:

Theorem 1.1.

There is a numerical constant $\epsilon > 0$ making the following true. Assume that u is a function defined on a (connected) neighborhood U of the origin in $\mathbb{R}^n$ for some $n \geq 2$, which belongs to the Sobolev space $W^{2,2}$ and satisfies the differential inequality

$$|\Delta u| \leq \frac{A}{|x|^2}|u| + \frac{\epsilon}{|x|}|\nabla u|$$

for some constant A. Assume also that

$$\lim_{r \to 0} r^{-k} \int_{|x| \leq r} |u|^2 = 0$$

for all $k < \infty$. Then $u = 0$.

The proof will be done using the standard Carleman method and essentially well-known L^2 estimates (cf. [2, 6], and the application in [4]) and the result is perhaps not really new. However, it does not appear in the literature and may be of some interest as representing the borderline situation where such a statement is true. We now briefly discuss this. It is important to make a distinction between real-valued functions u on the one hand, and complex- or vector-valued functions on the other.

For real-valued functions, the situation is as follows: in [8] an example is constructed in any dimension ≥ 4 of a smooth real-valued function satisfying $|\Delta u| \leq \frac{C}{|x|}|\nabla u|$ for a certain constant

Math Subject Classifications. 35J10, 35B60.
Key Words and Phrases. unique continuation, Laplace operator.
Acknowledgements and Notes. First author partially supported by a grant from the Purdue Research Foundation. Second author partially supported by NSF.

ISSN 1050-6926

C and vanishing to infinite order at the origin. On the other hand, in [4] it is shown that this is impossible in the two-dimensional case, i.e., the theorem above is valid for any constant ϵ in that case, at least if $A = 0$.

In the complex case, the two-dimensional result just mentioned from [4] is no longer valid. This is due to Alinhac and Baouendi; actually it was proved some years ago although not circulated until recently [1]. On the other hand, since the proof of Theorem 1.1 just uses the Carleman method, it works equally well in both cases.

We thank X. Huang who showed us the paper of [1], while we were writing up the present paper for publication. After seeing [1], we worked out an alternate approach to their construction which is included at the end of the paper. Paper [1] also indicates that an observation similar to our Theorem 1.1 had been made by Regbaoui in his thesis [5], independently and somewhat earlier.

2. Proof of the theorem

Theorem 1.1 will be a corollary of the following Carleman inequality.

Proposition. *Let η_0 be a positive constant. Assume that $\tau \geq n$ and $dist(\tau - \frac{n}{2}, \mathbb{Z}) \geq \eta_0$. Let $u \in C_0^\infty(\mathbb{R}^n \backslash 0)$. Then*

$$\left\| |x|^{-\tau} u \right\|_2 \leq C\tau^{-1} \left\| |x|^{2-\tau} \Delta u \right\|_2$$
$$\left\| |x|^{1-\tau} \nabla u \right\|_2 \leq C \left\| |x|^{2-\tau} \Delta u \right\|_2$$

where C depends on η_0 only. □

As has already been mentioned, this proposition should be considered implicit in the literature, e.g., in [2]. However, we give the simple proof. In what follows, we always assume that τ satisfies the conditions in the proposition. The proposition will be a corollary of the following lemma:

Lemma. *If Y is a degree k spherical harmonic, b is a radial function belonging to $C_0^\infty(\mathbb{R}^n \backslash 0)$ and $u = bY$, then*

$$\left\| |x|^{-\tau} u \right\|_2 \leq C \min\left(k^{-1}, \tau^{-1}\right) \left\| |x|^{2-\tau} \Delta u \right\|_2$$
$$\left\| |x|^{1-\tau} \nabla u \right\|_2 \leq C \left\| |x|^{2-\tau} \Delta u \right\|_2$$

with C depending on η_0 only. □

Proof. We will write $x \lesssim y$ to mean that $x \leq Cy$ where C is a constant depending only on η_0. Following an approach discussed, e.g., in [3, p. 14] and [7, p. 147] we identify $\mathbb{R}^n$ with $S^{n-1} \times \mathbb{R}$ via $x = e^t \omega$, $t \in \mathbb{R}$, $\omega \in S^{n-1}$. This is an isometry if $S^{n-1} \times \mathbb{R}$ is given the metric $e^{2t}(dt^2 + d\omega^2)$ where $d\omega^2$ is the usual metric on the sphere. The Laplacian on $S^{n-1} \times \mathbb{R}$ is $e^{-2t}(\frac{d^2}{dt^2} + (n-2)\frac{d}{dt} + \Delta_S)$ where Δ_S is the Laplacian on the sphere, and the gradient satisfies $|\nabla_{S^{n-1} \times \mathbb{R}} u| = e^{-t}(|\frac{du}{dt}|^2 + |\nabla_S u|^2)^{\frac{1}{2}}$ where ∇_S is the gradient on the sphere. Furthermore, the volume element is $e^{nt} d\omega dt$ where $d\omega$ is the volume element on the sphere. These assertions follow from the basic formulas of Riemannian geometry, cf. [7]. Thus, what we have to prove is that if Y is an eigenfunction of Δ_S with eigenvalue $-k(k + n - 2)$ and if $b \in C_0^\infty(\mathbb{R})$, then

$$\left\| e^{-\tau t} b(t) Y(\omega) \right\|_{L^2(e^{nt} dt d\omega)} \lesssim \min(k^{-1}, \tau^{-1}) \left\| e^{-\tau t} \left(\frac{d^2}{dt^2} + (n-2)\frac{d}{dt} + \Delta_S \right) (b(t) Y(\omega)) \right\|_{L^2(e^{nt} dt d\omega)}$$

$$\left\|e^{-\tau t}Y(\omega)\tfrac{db}{dt}\right\|_{L^2(e^{nt}dtd\omega)} \lesssim \left\|e^{-\tau t}\left(\tfrac{d^2}{dt^2}+(n-2)\tfrac{d}{dt}+\Delta_S\right)(b(t)Y(\omega))\right\|_{L^2(e^{nt}dtd\omega)}$$

$$\left\|e^{-\tau t}b(t)\nabla_S Y(\omega)\right\|_{L^2(e^{nt}dtd\omega)} \lesssim \left\|e^{-\tau t}\left(\tfrac{d^2}{dt^2}+(n-2)\tfrac{d}{dt}+\Delta_S\right)(b(t)Y(\omega))\right\|_{L^2(e^{nt}dtd\omega)}.$$

Equivalently $(a(t)=e^{-(\tau-\frac{n}{2})t}b(t))$

$$\begin{aligned}
&\|a(t)Y(\omega)\|_{L^2(dtd\omega)}\\
&\quad\lesssim \min(k^{-1},\tau^{-1})\left\|e^{-(\tau-\frac{n}{2})t}\left(\frac{d^2}{dt^2}+(n-2)\frac{d}{dt}+\Delta_S\right)\left(e^{(\tau-\frac{n}{2})t}a(t)Y(\omega)\right)\right\|_{L^2(dtd\omega)}\\
&\left\|\left(\frac{da}{dt}+\left(\tau-\frac{n}{2}\right)a(t)\right)Y(\omega)\right\|_{L^2(dtd\omega)}\\
&\quad\lesssim \left\|e^{-(\tau-\frac{n}{2})t}\left(\frac{d^2}{dt^2}+(n-2)\frac{d}{dt}+\Delta_S\right)\left(e^{(\tau-\frac{n}{2})t}a(t)Y(\omega)\right)\right\|_{L^2(dtd\omega)}\\
&\|a(t)\nabla_S Y(\omega)\|_{L^2(dtd\omega)}\\
&\quad\lesssim \left\|e^{-(\tau-\frac{n}{2})t}\left(\frac{d^2}{dt^2}+(n-2)\frac{d}{dt}+\Delta_S\right)\left(e^{(\tau-\frac{n}{2})t}a(t)Y(\omega)\right)\right\|_{L^2(dtd\omega)}
\end{aligned}$$

which are also equivalent to

$$\begin{aligned}
&\|a(t)Y(\omega)\|_{L^2(dtd\omega)}\\
&\qquad\lesssim \min(k^{-1},\tau^{-1})\left\|\left[a_{tt}+2(\tau-1)a_t+\left(\left(\tau-\frac{n}{2}\right)^2+(n-2)\right.\right.\right.\\
&\qquad\qquad\left.\left.\left.\left(\tau-\frac{n}{2}\right)-k(k+n-2)\right)a\right]Y\right\|_{L^2(dtd\omega)}\\
&\left\|\left(a_t+\left(\tau-\frac{n}{2}\right)a\right)Y(\omega)\right\|_{L^2(dtd\omega)}\\
&\quad\lesssim \left\|\left[a_{tt}+2(\tau-1)a_t+\left(\left(\tau-\frac{n}{2}\right)^2+(n-2)\left(\tau-\frac{n}{2}\right)-k(k+n-2)\right)a\right]Y\right\|_{L^2(dtd\omega)}\\
&\|a(t)\nabla_S Y(\omega)\|_{L^2(dtd\omega)}\\
&\quad\lesssim \left\|\left[a_{tt}+2(\tau-1)a_t+\left(\left(\tau-\frac{n}{2}\right)^2+(n-2)\left(\tau-\frac{n}{2}\right)-k(k+n-2)\right)a\right]Y\right\|_{L^2(dtd\omega)}.
\end{aligned}$$

In other words, we want to show that

$$\|a\|_{L^2(dt)}\lesssim \min\left(k^{-1},\tau^{-1}\right)\|a_{tt}+\lambda a_t+\mu a\|_{L^2(dt)} \tag{2.1}$$

$$\left\|a_t+\left(\tau-\frac{n}{2}\right)a\right\|_{L^2(dt)}\lesssim \|a_{tt}+\lambda a_t+\mu a\|_{L^2(dt)} \tag{2.2}$$

$$\|a\|_{L^2(dt)}\lesssim \|a_{tt}+\lambda a_t+\mu a\|_{L^2(dt)}\frac{\|Y\|_{L^2(d\omega)}}{\|\nabla_S Y\|_{L^2(d\omega)}} \tag{2.3}$$

where $\lambda=2(\tau-1)$, $\mu=(\tau-\frac{n}{2}-k)(\tau+\frac{n}{2}+k-2)$. To prove (2.1), we take Fourier Transforms in t obtaining

$$\|a_{tt}+\lambda a_t+\mu a\|_{L^2(dt)}\gtrsim \min_{\xi\in\mathbb{R}}\left(\lambda|\xi|+\left|\mu-\xi^2\right|\right)\|a\|_{L^2(dt)}$$

Next we show that

$$\lambda|\xi|+\left|\mu-\xi^2\right|\gtrsim \max(\tau,k,|\xi|) \tag{2.4}$$

Since τ is large, it is clear that the left side of (2.4) is $\gtrsim|\xi|$. For the rest of (2.4), we first observe (using that $\tau-\frac{n}{2}$ is bounded away from the integers and that $\tau\le 2(\tau-\frac{n}{2})$) that

$$|\mu|\gtrsim\begin{cases}\max(\tau,k) & \text{always}\\ \max(\tau^2,k^2) & \text{if } k\le\frac{\tau}{4} \text{ or } k\ge 4\tau\end{cases}.$$

If $|\xi|^2\le\frac{|\mu|}{2}$, then it follows immediately that the second term on the left side of (2.4) is $\gtrsim\max(\tau,k)$. If $|\xi|^2\ge\frac{|\mu|}{2}$, then (using that τ is large) the first term on the left side of (2.4) is $\ge\frac{1}{3}\tau\sqrt{|\mu|}$. If $\frac{1}{4}\tau\le k\le 4\tau$, then this is clearly $\gtrsim\max(\tau,k)$. If $k\le\frac{1}{4}\tau$ or $k\ge 4\tau$, then we have seen that $\sqrt{|\mu|}\gtrsim\max(\tau,k)$, hence $\tau\sqrt{|\mu|}\gtrsim\max(\tau,k)$ also. So (2.4) is proved, and therefore (2.1) is proved also. For (2.2) we argue similarly: by taking Fourier Transform in t we obtain

$$\begin{aligned}\|a_{tt}+\lambda a_t+\mu a\|_{L^2(dt)} &\gtrsim \min_{\xi\in\mathbb{R}}\frac{\lambda|\xi|+|\mu-|\xi|^2|}{|\xi|+\tau-\frac{n}{2}}\left\|a_t+\left(\tau-\frac{n}{2}\right)a\right\|_{L^2(dt)}\\ &\gtrsim \min_{\xi\in\mathbb{R}}\frac{\max(\tau,|\xi|)}{|\xi|+\tau-\frac{n}{2}}\left\|a_t+\left(\tau-\frac{n}{2}\right)a\right\|_{L^2(dt)}\\ &\gtrsim \left\|a_t+\left(\tau-\frac{n}{2}\right)a\right\|_{L^2(dt)}\end{aligned}$$

and (2.2) follows. Equation (2.3) follows from (2.1) since

$$\int_S|\nabla_S Y|^2=-\int_S Y\Delta_S Y=k(k+n-2)\int_S Y^2$$

so that $\frac{\|\nabla_S Y\|_2}{k\|Y\|_2}=\sqrt{k(k+n-2)}\le\sqrt{k(k+\tau)}\le\sqrt{2}\max(k,\tau)$ □

Proof of the proposition. If u is expanded in a series of radial functions times spherical harmonics, $u(x)=\Sigma_{k,j}a_{kj}(|x|)Y_{kj}(x)$ where $\{Y_{k,j}\}$ is an orthonormal basis for the degree k harmonics with respect to integration over the unit sphere, then the terms $\Delta(a_{k,j}(|x|)Y_{k,j}(x))$ are orthogonal with respect to integration over any sphere S centered at 0, and similarly with the terms $\nabla(a_{k,j}(|x|)Y_{k,j}(x))$ regarded as elements of $L^2(S,\mathbb{C}^n)$. This is a well-known fact verified by integration by parts. We therefore have

$$\frac{\left\||x|^{-\tau}u\right\|_2}{\left\||x|^{2-\tau}\Delta u\right\|_2}\le\max_{k,j}\frac{\left\||x|^{-\tau}a_{k,j}Y_{k,j}\right\|_2}{\left\||x|^{2-\tau}\Delta\left(a_{k,j}Y_{k,j}\right)\right\|_2},$$

which is $\lesssim\tau^{-1}$ by the first part of the lemma. The first order inequality follows similarly using the second part of the lemma. □

Completion of proof of Theorem 1.1. We note first that the assumptions and Corollary 17.1.4 of [3] also imply that $\lim_{r\to 0}r^{-k}\int_{|x|<r}|\nabla u|^2=0$ for all k. In view of this, a standard limiting argument shows that the inequalities in the proposition above are applicable to ϕu for any $\phi\in C_0^\infty(U)$. We may therefore carry out the Carleman argument in the following way: choose η small enough so that $\{x:|x|\le\eta\}\subset U$, and choose $\phi\in C_0^\infty(U)$ with $\phi(x)=1$ when $|x|\le\eta$. Let $v=\phi u$, then $|\Delta v|\le\frac{\epsilon}{|x|}|\nabla v|+\frac{A}{|x|^2}|v|+E$ with $E\in L^2$ and E supported in $|\{x:|x|\ge\eta\}$. If τ is as in the proposition with (say) $\eta_0=\frac{1}{2}$, then we get $|||x|^{2-\tau}\Delta v||_2\le C(\epsilon+\frac{A}{\tau})|||x|^{2-\tau}\Delta v||_2+|||x|^{2-\tau}E||_2$, with C a fixed constant. If $\epsilon\le\frac{1}{2C}$ and τ is large enough, then it follows that $|||x|^{2-\tau}\Delta v||_2\le 3|||x|^{2-\tau}E||_2$, $||\left(\frac{|x|}{\eta}\right)^{2-\tau}\Delta v||_{L^2(\{x:|x|\le\eta\})}\le 3||E||_2$. Letting $\tau\to\infty$ we get that v is harmonic near 0. In view of the infinite order vanishing, it follows that $v=0$ near 0, then since u satisfies a nonsigular differential inequality away from the origin it follows that u vanishes identically. □

We now present an alternate approach to the counterexample of [1].

Theorem. *There is a smooth function $u : \mathbb{R}^2 \to \mathbb{C}$ which vanishes to infinite order at the origin and satisfies $|\Delta u(z)| \leq \frac{C}{|x|}|\nabla u(z)|$ for a certain constant C.* □

Remark. It should be pointed out that [1] gives a specific value for the constant C and also states several additional properties of u which we will not consider. On the other hand, the construction below (which is related to the construction in [8]) would appear to be simpler. □

Proof of the Theorem. We first show that if $k \in \mathbb{Z}^+$ is sufficiently large, then there is a smooth function $v_k : \mathbb{C} \to \mathbb{C}$ satisfying

(i) $v_k(z) = z^k$ when $|z| \geq 2$, $v_k(z) = z^{k+1}$ when $|z| \leq \frac{1}{2}$.

(ii) $|\Delta v_k| \leq C|\nabla v_k|$ with C independent of k.

Namely, choose two functions $\phi_1 : \mathbb{C} \to \mathbb{C}$ and $\phi_2 : \mathbb{C} \to \mathbb{C}$ such that $\phi_1(z)$ and $\phi_2(z)$ are both equal to 1 when $|z| \geq 2$ and to z when $|z| \leq \frac{1}{2}$ and so that ϕ_1 and ϕ_2 have no common zeroes in the region $\frac{1}{2} \leq |z| \leq 2$. Existence of such functions is obvious. Now let

$$v_k(z) = \mathrm{re}\left(z^k\phi_1(z)\right) + i\ \mathrm{im}\left(z^k\phi_2(z)\right)$$

Statement (i) is then clear and we must prove (ii). A simple calculation shows that $|\nabla \mathrm{re} v_k(z)| \geq C^{-1}k|z|^{k-1}|\phi_1(z)| - C|z|^k$ when $\frac{1}{2} \leq |z| \leq 2$, and similarly $|\nabla \mathrm{im} v_k(z)| \geq C^{-1}k|z|^{k-1}|\phi_2(z)| - C|z|^k$. Since ϕ_1 and ϕ_2 do not vanish simultaneously, it follows that $|\nabla v_k(z)| \geq C^{-1}k|z|^k$ for large k. Also a calculation with the product rule using that z^k is harmonic shows that $|\Delta v_k| \leq Ck|z|^k$ and (ii) follows.

To finish the proof of the theorem, we choose a sequence $\{r_k\}_{k=k_0}^{\infty}$ of positive numbers, tending rapidly to zero. Here k_0 should be large enough so that the function v_k above is defined for all $k \geq k_0$. We let $a_k = \frac{r_k^k}{\prod_{k_0 \leq j < k} r_j}$ and $\tilde{v}_k(z) = a_k v_k(r_k^{-1}z)$, and note that $\tilde{v}_k(z) = \tilde{v}_{k+1}(z)$ if $2r_{k+1} \leq |z| \leq \frac{r_k}{2}$. We now define a function u as follows: if $|z| > 2r_{k_0}$, then $u(z) = \tilde{v}_{k_0}(z)$; if $k \geq k_0$ and $2r_{k+1} < |z| \leq 2r_k$, then $u(z) = \tilde{v}_k(z)$; and finally $u(0) = 0$. Then u is smooth on the boundaries $|z| = 2r_k$, $k > k_0$ since $\tilde{v}_k$ and $\tilde{v}_{k+1}$ are equal when $|z|$ is slightly greater than $2r_{k+1}$. Furthermore, if $\{r_k\}$ decrease sufficiently rapidly, then all derivatives of u approach 0 as $z \to 0$. Namely, we have $|D^\alpha v_k(z)| \leq C_{k\alpha}|z|^{k-|\alpha|}$ for suitable constants $C_{k\alpha}$, and therefore by scaling

$$\left|D^\alpha \tilde{v}_k(z)\right| \leq C_{k\alpha}\frac{r_k^{k-|\alpha|}}{\prod_{k_0 \leq j < k} r_j}\left|r_k^{-1}z\right|^{k-|\alpha|} \leq C_{k\alpha}\frac{|z|^{k-|\alpha|}}{\prod_{k_0 \leq j < k} r_j}$$

Thus in the region $2r_{k+1} < |z| \leq 2r_k$ where u is equal to $\tilde{v}_k$, we have $|D^\alpha u| \leq 2^{k-|\alpha|}C_{k\alpha}\frac{r_k^{k-|\alpha|}}{\prod_{k_0 \leq j < k} r_j}$. If we take r_k small enough compared to r_{k-1}, then it is clear that this will approach 0 as $k \to \infty$ (i.e., as $|z| \to 0$) for all α, as claimed. It follows that u is smooth on $\mathbb{R}^2$. Property (ii) above implies by scalling that $|\Delta \tilde{v}_n| \leq \frac{C}{|z|}|\nabla \tilde{v}_n|$, so $|\Delta u| \leq \frac{C}{|z|}|\nabla u|$ and the proof is complete. □

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Received January 10, 1994

Indiana University–Purdue University at Fort Wayne, Fort Wayne, IN 46805
e-mail: pan@cvax.ipfw.inidiana.edu

University of California at Berkeley
e-mail: wolf@cco.caltech.edu