Weighted Projective Spaces and a Generalization of Eves' Theorem

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Abstract

For a certain class of configurations of points in space, Eves' Theorem gives a ratio of products of distances that is invariant under projective transformations, generalizing the cross-ratio for four points on a line. We give a generalization of Eves' theorem, which applies to a larger class of configurations and gives an invariant with values in a weighted projective space. We also show how the complex version of the invariant can be determined from classically known ratios of products of determinants, while the real version of the invariant can distinguish between configurations that the classical invariants cannot.

1 Introduction

Eves' Theorem ([E]) is a generalization of two basic geometric results: Ceva's Theorem for triangles in Euclidean geometry, and the projective invariance of the cross-ratio in projective geometry. Both results, and more generally Eves' Theorem, assign an invariant ratio of products of distances to certain types of configurations of points in space.

The example shown in Figure 1, where eleven points lie on five lines, forming twelve directed segments, gives the general idea of Eves' Theorem.

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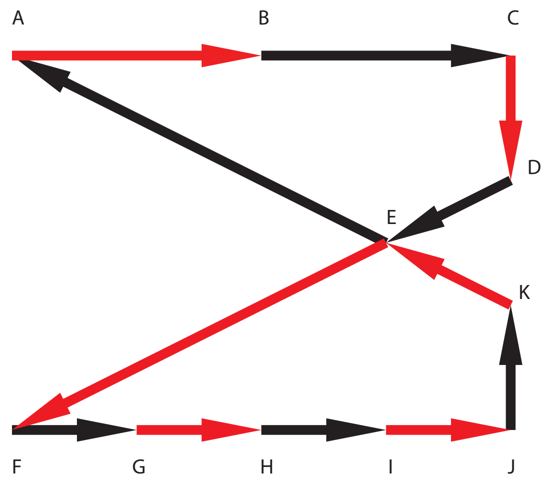

Figure 1: A configuration of 11 points, 5 lines, and 12 directed segments in the real Euclidean plane, to which Eves' Theorem applies.

The ratio of Euclidean signed distances

$$\frac{AB \cdot CD \cdot EF \cdot GH \cdot IJ \cdot KE}{BC \cdot DE \cdot FG \cdot HI \cdot JK \cdot EA}$$

is called (by Eves) an "h-expression," meaning that each point $A, \ldots, K$ occurs equally many times in the numerator and denominator (for example, E occurs twice), and each line determined by one of the twelve segments also occurs equally many times (for example, $\overleftrightarrow{FG} = \overleftrightarrow{HI}$ occurs twice in the numerator and twice in the denominator). The statement of Eves' Theorem is that the value of an h-expression is an invariant under projective transformations of the plane. Related identities for products of distances have been known in projective geometry since at least [P], but it is convenient to attribute the above formulation to Eves.

The notion of h-expression can also be more visually conveyed in terms of <u>coloring</u> the configuration — an idea demonstrated at a 2011 talk by Marc Frantz [F_1]. Each point in the configuration of Figure 1 is an endpoint of an equal number of red and black segments, and, dually, each line contains an equal number of red and black segments. Then the ratio of the product of red lengths to the product of black lengths is Eves' invariant.

Eves' Theorem, when stated in a purely projective way (using homogeneous coordinates, not Euclidean distances, as in Example 9) is itself a special case of a family of invariant ratios of products of determinants of homogeneous coordinates for points in projective space over a field $\mathbb{K}$. These ratios were well-known in 19^{th} century Invariant Theory ([B], [C], [Salmon]), but have been more recently used (and, sometimes, re-discovered) in projective geometry applied to computational topics such as vision and photogrammetry, or automated proofs ([BB], [CRG], [F_2], [RG]).

Eves' Theorem can be stated in terms of a function, where the input is a configuration of points $\mathcal{S}$ in projective space $\mathbb{K}P^D$, and the output is a ratio, i.e., an element of the projective line $\mathbb{K}P^1$; the content of the Theorem is that the ratio is well-defined (independent of certain choices made in specifying the configuration) and also invariant under projective transformations. Our new construction, Theorem 32, generalizes the target to a "weighted projective space"

$$\mathbb{K}P(\mathbf{p}) = \mathbb{K}P(p_0, \ldots, p_n),$$

so the projective line is the special case $\mathbb{K}P(1,1)$. In Section 4, we give a unified treatment of the configurations to which Theorem 32 applies, by a weighting, coloring, and indexing scheme. A configuration $\mathcal{S}$ of points in the projective space $\mathbb{K}P^D$ that satisfies a condition (Definition 28), depending on the weight vector $\mathbf{p} = (p_0, \ldots, p_n)$, is assigned an element $E_{\mathbf{p}}(\mathcal{S}) \in \mathbb{K}P(\mathbf{p})$, an invariant under "morphisms" of the configuration (Definition 26), which generalize projective transformations. The number of colors is $n+1$, so the classical case is the assignment of Eves' ratio $E_{(1,1)}(\mathcal{S}) \in \mathbb{K}P(1,1)$ to some two-color configurations $\mathcal{S}$, and the new weighted invariants apply to a larger category of multi-color configurations.

In Section 2 we review the definition and some elementary properties of weighted projective spaces — these properties are well-known in the complex case, but the real case is different in some ways we intend to exploit, so we are careful to present all the necessary details. Section 3 introduces a new notion of "reconstructibility" for a weighted projective space; the two main results are that complex weighted projective spaces are all reconstructible, and that some real weighted projective spaces are not. In Section 5 we review a connection between real projective and Euclidean geometry, and state a Euclidean version of Theorem 32. Section 6 applies the notion of reconstructibility to show that in the complex case, the weighted invariant $E_{\mathbf{p}}$ of a multi-color configuration can be determined by finding the (classical) $E_{(1,1)}$ ratios for a finite list of related two-color configurations. However, in the real case, Examples 13, 14, 15 give pairs of configurations with different weighted invariants in $\mathbb{R}P(\mathbf{p})$, but which cannot be distinguished by applying the reconstruction method to the $E_{(1,1)}$ ratios in $\mathbb{R}P^1$.

2 Weighted projective spaces

This Section reviews the definition of weighted projective spaces and some of their elementary properties. For the complex case, these properties (in particular, Example 2 and Lemmas 6 and 10), are well-known; we give elementary proofs with the intent of showing how the complex case is different from the relatively lesser-known real case. Only the objects' set-theoretic properties are of interest here, not their structure as topological or analytic spaces, algebraic varieties, or orbifolds. The applications in subsequent Sections use only $\mathbb{R}$ and $\mathbb{C}$, but to start in a general way, let $\mathbb{K}$ be any field.

2.1 The basic construction

The ingredients are $n \in \mathbb{N}$, the vector space $\mathbb{K}^{n+1}$, and a weight $\mathbf{p} = (p_0, p_1, \ldots, p_n) \in \mathbb{N}^{n+1}$. Denote $\mathbb{K}_*^{n+1} = \mathbb{K}^{n+1} \setminus \{\mathbf{0}\}$, and for elements $\mathbf{z} = (z_0, \ldots, z_n)$, $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{K}_*^{n+1}$, define a relation $\sim_{\mathbf{p}}$ so that $\mathbf{z} \sim_{\mathbf{p}} \mathbf{w}$ means there exists $\lambda \in \mathbb{K}_*^1$ such that:

$$w_0 = \lambda^{p_0} z_0, \quad w_1 = \lambda^{p_1} z_1, \ \ldots, \ w_n = \lambda^{p_n} z_n.$$

This is an equivalence relation on $\mathbb{K}_*^{n+1}$ because $\mathbb{K}$ is a field.

Definition 1. Let $\mathbb{K}P(\mathbf{p})$ denote the set of equivalence classes for $\sim_{\mathbf{p}}$.

$$\mathbb{K}P(\mathbf{p}) = \mathbb{K}P(p_0, \ldots, p_n)$$

is the weighted projective space corresponding to the weight $\mathbf{p}$.

Notation 2. Let $\pi_{\mathbf{p}} : \mathbb{K}_*^{n+1} \to \mathbb{K}P(\mathbf{p})$ denote the canonical quotient map, defined so that $\pi_{\mathbf{p}}(\mathbf{z})$ is the equivalence class of $\mathbf{z}$. It is convenient to use the same letter for elements of the weighted projective space, and square brackets for weighted homogeneous coordinates:

$$\pi_{\mathbf{p}}(\mathbf{z}) = z = [z_0 : z_1 : \ldots : z_n]_{\mathbf{p}}.$$

Example 1. For $\mathbf{p} = (1, 1, \ldots, 1)$, $\mathbb{K}P(1, 1, \ldots, 1)$ is the usual projective space, denoted $\mathbb{K}P^n$, with homogeneous coordinates $\pi : (z_0, \ldots, z_n) \mapsto [z_0 : \ldots : z_n]$ (omitting the subscripts).

Example 2. For $\mathbb{K} = \mathbb{C}$ and $\mathbf{p} = (p, p, \ldots, p)$, $\mathbf{z}$ and $\mathbf{w} \in \mathbb{C}_*^{n+1}$ are $\sim_\mathbf{p}$-equivalent if and only if they are related by non-zero complex scalar multiplication, so the following sets are exactly equal: $\mathbb{C}P(p, p, \ldots, p) = \mathbb{C}P(1, 1, \ldots, 1) = \mathbb{C}P^n$.

Example 3. For $\mathbb{K} = \mathbb{R}$ and $\mathbf{p} = (2k+1, 2k+1, \ldots, 2k+1)$, $\mathbf{z}$ and $\mathbf{w} \in \mathbb{R}_*^{n+1}$ are $\sim_\mathbf{p}$-equivalent if and only if they are related by non-zero real scalar multiplication, so the following sets are exactly equal: $\mathbb{R}P(2k+1, 2k+1, \ldots, 2k+1) = \mathbb{R}P(1, 1, \ldots, 1) = \mathbb{R}P^n$.

Example 4. For $\mathbb{K} = \mathbb{R}$ and $\mathbf{p} = (2k, 2k, \ldots, 2k)$, the restriction of $\pi_\mathbf{p}$ to the unit sphere $S^n \subseteq \mathbb{R}_*^{n+1}$ is a one-to-one function onto $\mathbb{R}P(\mathbf{p})$. It is not inconvenient to identify the sets: $\mathbb{R}P(2k, 2k, \ldots, 2k) = S^n$.

2.2 Mappings

Here we present some properties of rational maps between weighted projective spaces, in an elementary, set-theoretic way, directly using Definition 1 (in contrast with an algebraic-geometric approach, as in [Delorme], [Dolgachev]). The elementary proofs of Lemmas 6 and 10 are provided as preparation for similar methods to be used in the Proof of Theorem 16.

Let $\mathbb{K}$ and $\mathbb{F}$ be fields, and let $\mathbf{p} \in \mathbb{N}^{n+1}$, $\mathbf{q} \in \mathbb{N}^{N+1}$ be weights. Consider any function $\mathbf{f} : \mathbb{K}_*^{n+1} \to \mathbb{F}^{N+1}$. Given $\mathbf{z} \in \mathbb{K}_*^{n+1}$, suppose $\mathbf{f}$ has the following two properties: first,

$$\mathbf{f}(\mathbf{z}) = (w_0, w_1, \ldots, w_N) \neq \mathbf{0}, \tag{1}$$

and second, for any $\lambda \in \mathbb{K}_*^1$, there exists $\mu \in \mathbb{F}_*^1$ so that

$$\mathbf{f}(\lambda^{p_0} z_0, \lambda^{p_1} z_1, \ldots, \lambda^{p_n} z_n) = (\mu^{q_0} w_0, \mu^{q_1} w_1, \ldots, \mu^{q_N} w_N). \tag{2}$$

Then $\mathbf{f}$ also has these two properties at every point $\mathbf{z}' \in \mathbb{K}_*^{n+1}$ in the same equivalence class as $\mathbf{z}$, and if $\mathbf{z}' \sim_\mathbf{p} \mathbf{z}$, then $\mathbf{f}(\mathbf{z}') \sim_\mathbf{q} \mathbf{f}(\mathbf{z})$. Let $\mathbf{U} \subseteq \mathbb{K}_*^{n+1}$ be the set of points where $\mathbf{f}$ has the two properties, and let $U = \pi_\mathbf{p}(\mathbf{U})$. Then we say "$\mathbf{f}$ induces a map from $\mathbb{K}P(\mathbf{p})$ to $\mathbb{F}P(\mathbf{q})$ which is well-defined on the set U," and denote the induced map, which takes $\pi_\mathbf{p}(\mathbf{z}) \in U$ to $\pi_\mathbf{q}(\mathbf{f}(\mathbf{z}))$, by $f : z \mapsto f(z)$. For $z \notin U$, $f(z)$ is undefined.

Lemma 3. *For $\mathbf{f}$, f, and $\mathbf{U}$ as above, and an element $w \in \mathbb{F}P(\mathbf{q})$, let $\mathbf{w} \in \mathbb{F}_*^{N+1}$ be any representative $\mathbf{w} \in w = \pi_\mathbf{q}(\mathbf{w})$. Then, the inverse image of w is:*

$$f^{-1}(w) = \pi_\mathbf{p}(\{\mathbf{z} \in \mathbf{U} : \mathbf{f}(\mathbf{z}) \sim_\mathbf{q} \mathbf{w}\}). \tag{3}$$

Proof. The inverse image is

$$f^{-1}(w) = \{z \in \mathbb{K}P(\mathbf{p}) : z \in \pi_{\mathbf{p}}(\mathbf{U}) \text{ and } f(z) = w\}.$$

The first condition is that $\exists \mathbf{x} \in \mathbf{U} : z = \pi_{\mathbf{p}}(\mathbf{x})$, and the second condition is that the $\sim_{\mathbf{q}}$-equivalence class of $\mathbf{w}$ is the same as the $\sim_{\mathbf{q}}$-equivalence class of $\mathbf{f}(\mathbf{z})$ for some $\mathbf{z} \in z$. So,

$$\begin{aligned} & f^{-1}(w) \\ = \ & \{z : (\exists \mathbf{x} \in \mathbf{U} : \mathbf{x} \in z) \text{ and } (\exists \mathbf{z} \in z : \mathbf{f}(\mathbf{z}) \sim_{\mathbf{q}} \mathbf{w})\}. \end{aligned}$$

From (3), denote the RHS set (depending on w but not the choice of $\mathbf{w}$):

$$\begin{aligned} & A_w = \pi_{\mathbf{p}}(\{\mathbf{z} \in \mathbf{U} : \mathbf{f}(\mathbf{z}) \sim_{\mathbf{q}} \mathbf{w}\}) \\ = \ & \{z \in \mathbb{K}P(\mathbf{p}) : \exists \mathbf{z} \in \mathbf{U} : (\pi_{\mathbf{p}}(\mathbf{z}) = z \text{ and } \mathbf{f}(\mathbf{z}) \sim_{\mathbf{q}} \mathbf{w})\}. \end{aligned}$$

If $z \in A_w$, letting $\mathbf{x} = \mathbf{z}$ shows $z \in f^{-1}(w)$. Conversely, if $z \in f^{-1}(w)$, then $\exists \mathbf{x} \in \mathbf{U} : \mathbf{x} \in z$ and $\exists \mathbf{z} \in z : \mathbf{f}(\mathbf{z}) \sim_{\mathbf{q}} \mathbf{w}$. Since $\mathbf{x} \in \mathbf{U}$ has properties (1) and (2), and $\mathbf{z} \sim_{\mathbf{p}} \mathbf{x} \in z$, $\mathbf{z}$ also has the two properties, so $\mathbf{z} \in \mathbf{U}$, and $z \in A_w$. ∎ ∎

Similar reasoning with the above data leads to the following equivalences:

Proposition 4. *A map $f : \mathbb{K}P(\mathbf{p}) \to \mathbb{F}P(\mathbf{q})$ which is well-defined on the set U is an onto map if and only if: for every $\mathbf{w} \in \mathbb{F}_*^{N+1}$, there exists $\mathbf{z} \in \mathbf{U}$ such that $\mathbf{f}(\mathbf{z}) \sim_{\mathbf{q}} \mathbf{w}$.* ∎

Proposition 5. *A map $f : \mathbb{K}P(\mathbf{p}) \to \mathbb{F}P(\mathbf{q})$ which is well-defined on the set U is a one-to-one map if and only if: for all $\mathbf{z}$, $\mathbf{z}' \in \mathbf{U}$, if $\mathbf{f}(\mathbf{z}) \sim_{\mathbf{q}} \mathbf{f}(\mathbf{z}')$, then $\mathbf{z} \sim_{\mathbf{p}} \mathbf{z}'$.* ∎

The following Lemma generalizes the well-known equality of sets

$$\mathbb{C}P(q_1, \ldots, q_n) = \mathbb{C}P(mq_0, \ldots mq_n).$$

Lemma 6. *For $m \in \mathbb{N}$ and two weights:*

$$\begin{aligned}\mathbf{q} &= (q_0, q_1, q_2, \ldots, q_n),\\ \mathbf{p} &= (mq_0, mq_1, mq_2, \ldots, mq_n),\end{aligned}$$

let $\mathbf{f}$ *be the inclusion*

$$\begin{aligned}\mathbf{f} &: \mathbb{K}_*^{n+1} \to \mathbb{K}^{n+1}\\ &: (z_0, z_1, z_2, \ldots, z_n) \mapsto (z_0, z_1, z_2, \ldots, z_n).\end{aligned}$$

The induced map $f : \mathbb{K}P(\mathbf{p}) \to \mathbb{K}P(\mathbf{q})$ is well-defined and onto. In the following three cases,

- $\mathbb{K} = \mathbb{C}$*; or*
- $\mathbb{K} = \mathbb{R}$ *and m is odd; or*
- $\mathbb{K} = \mathbb{R}$ *and all the integers q_0, …, q_n are even,*

f is the identity map and the sets $\mathbb{K}P(\mathbf{p})$ and $\mathbb{K}P(\mathbf{q})$ are equal.

Proof. $\mathbf{f}$ clearly satisfies (1) at every point $\mathbf{z} \in \mathbb{K}_*^{n+1}$, and also (2) with $\mu = \lambda^m$, so $\mathbf{U} = \mathbb{K}_*^{n+1}$. The induced map $f : \mathbb{K}P(\mathbf{p}) \to \mathbb{K}P(\mathbf{q})$ is well-defined on $U = \mathbb{K}P(\mathbf{p})$, and is an onto map as in Proposition 4. f is also one-to-one if it satisfies the condition of Proposition 5: for all $\mathbf{z}$, $\mathbf{z}' \in \mathbb{K}_*^{n+1}$, if $(z_0', \ldots, z_n') = (\lambda^{q_0} z_0, \ldots, \lambda^{q_n} z_n)$ for some $\lambda \in \mathbb{K}_*^1$, then there exists $\mu \in \mathbb{K}_*^1$ such that $(z_0', \ldots, z_n') = (\mu^{mq_0} z_0, \ldots, \mu^{mq_n} z_n)$. So, if $\mathbb{K}$ and m have the property that $\forall \lambda \neq 0\ \exists \mu : \mu^m = \lambda$, then f is one-to-one. This happens for the first two cases: $\mathbb{K} = \mathbb{C}$ and any m, and also for $\mathbb{K} = \mathbb{R}$ and odd m. Another situation in which f is one-to-one is the case where $\mathbb{K} = \mathbb{R}$ and all the integers q_0, …, q_n are even: for any $\lambda \neq 0$, let $\mu = |\lambda|^{1/m}$, then for $k = 0, \ldots, n$, $\mu^{mq_k} = (|\lambda|^{1/m})^{mq_k} = |\lambda|^{q_k} = \lambda^{q_k}$.

Because in each of the three cases, we have $\mathbf{z} \sim_{\mathbf{p}} \mathbf{z}' \iff \mathbf{z} \sim_{\mathbf{q}} \mathbf{z}'$, the equivalence classes are the same, f is the identity map, and these sets are equal: $\mathbb{K}P(\mathbf{p}) = \mathbb{K}P(\mathbf{q})$. ∎

Theorem 7. *For any weight* $\mathbf{q} = (q_0, \ldots, q_n)$*, let*

$$\mathbf{p} = (2q_0, \ldots, 2q_n).$$

Let $f : \mathbb{R}P(\mathbf{p}) \to \mathbb{R}P(\mathbf{q})$ *be induced by the inclusion*

$$\mathbf{f} : \mathbb{R}_*^{n+1} \to \mathbb{R}^{n+1} : \mathbf{z} \mapsto \mathbf{z}$$

as in Lemma 6. If q_j *is odd, then the restriction of* f *to the set* $\pi_{\mathbf{p}}(\{\mathbf{z} : z_j \neq 0\})$ *is two-to-one.*

Proof. f is well-defined and onto by Lemma 6. Take any $\mathbf{w} \in \mathbb{R}_*^{n+1}$ with $w_j \neq 0$, and let $w = \pi_{\mathbf{q}}(\mathbf{w})$. By Lemma 3,

$$\begin{aligned} f^{-1}(w) &= \pi_{\mathbf{p}}(\{\mathbf{z} \in \mathbb{R}_*^{n+1} : \mathbf{z} \sim_{\mathbf{q}} \mathbf{w}\}) \\ &= \pi_{\mathbf{p}}(\{(\mu^{q_0} w_0, \ldots, \mu^{q_n} w_n) : \mu \in \mathbb{R}_*^1\}). \end{aligned}$$

Two points $(\mu_1^{q_0} w_0, \ldots, \mu_1^{q_n} w_n)$, $(\mu_2^{q_0} w_0, \ldots, \mu_2^{q_n} w_n)$ are $\sim_{\mathbf{p}}$-equivalent if and only if there exists $\lambda \in \mathbb{R}_*^1$ such that

$$\mu_1^{q_0} w_0 = \lambda^{2q_0} \mu_2^{q_0} w_0, \ldots, \mu_1^{q_n} w_n = \lambda^{2q_n} \mu_2^{q_n} w_n. \tag{4}$$

For $w_j \neq 0$ and q_j odd, $\mu_1^{q_j} w_j = \lambda^{2q_j} \mu_2^{q_j} w_j \iff \lambda^2 = \mu_1/\mu_2$, which is equivalent to the system of equations (4). So, the two points are $\sim_{\mathbf{p}}$-equivalent if and only if μ_1 and μ_2 have the same sign: there are two $\sim_{\mathbf{p}}$-equivalence classes. ∎

Example 5. If all the q_j are odd, then f as in Theorem 7 is globally two-to-one. An important special case is the map $f : \mathbb{R}P(2, 2, \ldots, 2) \to \mathbb{R}P(1, 1, \ldots, 1)$, which is exactly the well-known two-to-one covering $S^n \to \mathbb{R}P^n$, the "antipodal identification." An example with the q_j not all odd is the map $f : \mathbb{R}P(4, 2) \to \mathbb{R}P(2, 1)$. For this f, $f^{-1}([1 : z_1]_{(2,1)}) = \{[1 : z_1]_{(4,2)}, [1 : -z_1]_{(4,2)}\}$, a two-element set for $z_1 \neq 0$, but a singleton for $z_1 = 0$.

Lemma 8. *For* $m \in \mathbb{N}$ *and two weights:*

$$\begin{aligned} \mathbf{q} &= (q_0, q_1, q_2, \ldots, q_n), \\ \mathbf{p} &= (q_0, mq_1, mq_2, \ldots, mq_n), \end{aligned}$$

let $\mathbf{f}$ *be the monomial map*

$$\begin{aligned} \mathbf{f} &: \mathbb{K}_*^{n+1} \to \mathbb{K}^{n+1} \\ &: (z_0, z_1, z_2, \ldots, z_n) \mapsto (z_0^m, z_1, z_2, \ldots, z_n). \end{aligned}$$

If $\mathbb{K} = \mathbb{C}$, or if $\mathbb{K} = \mathbb{R}$ and m is odd, then the induced map

$$f : \mathbb{K}P(\mathbf{p}) \to \mathbb{K}P(\mathbf{q})$$

is well-defined and onto.

Proof. For any $\mathbb{K}$ and m, $\mathbf{f}$ clearly satisfies (1) at every point $\mathbf{z} \in \mathbb{K}_*^{n+1}$, and also (2) with $\mu = \lambda^m$, so $\mathbf{U} = \mathbb{K}_*^{n+1}$, and the induced map $f : \mathbb{K}P(\mathbf{p}) \to \mathbb{K}P(\mathbf{q})$ is well-defined on $U = \mathbb{K}P(\mathbf{p})$. f is an onto map if it satisfies the condition of Proposition 4: for every $\mathbf{w} = (w_0, \ldots, w_n) \in \mathbb{K}_*^{n+1}$, there exist $\mathbf{z}$ and λ such that

$$(w_0, \ldots, w_n) = (\lambda^{q_0} z_0^m, \ldots, \lambda^{q_n} z_n).$$

Under the hypothesis that $\mathbb{K} = \mathbb{C}$, or $\mathbb{K} = \mathbb{R}$ and m is odd, then given $\mathbf{w}$, one can choose $\lambda = 1$, any z_0 with $z_0^m = w_0$, and $z_k = w_k$ for $k = 1, \ldots, n$. ∎

Lemma 9. *For $w_0 \in \mathbb{C}_*^1$, and N, $P \in \mathbb{N}$, suppose*

$$\{\zeta_0, \ldots, \zeta_{N-1}\}$$

are the N distinct complex roots of the equation $\zeta^N = w_0$. Then the number of distinct elements in the set

$$\{\zeta_0^P, \ldots, \zeta_{N-1}^P\}$$

is $\mathrm{lcm}(P, N)/P$.

Proof. In polar form, $w_0 = \rho e^{i\theta}$ for a unique $\rho > 0$, $\theta \in [0, 2\pi)$. By re-labeling if necessary,

$$\zeta_j^P = \rho^{P/N} e^{i(\theta + 2\pi j)P/N}$$

for $j = 0, \ldots, N-1$. Let j be the smallest integer such that $jP/N \in \mathbb{N}$. It follows that $j = \mathrm{lcm}(P, N)/P$, the elements ζ_k^P are distinct for $k = 0, \ldots, j-1$, and $\zeta_k^P = \zeta_{k+j}^P$. ∎

Lemma 10. *For $m \in \mathbb{N}$ and two weights:*

$$\begin{aligned} \mathbf{q} &= (q_0, q_1, q_2, \ldots, q_n), \\ \mathbf{p} &= (q_0, mq_1, mq_2, \ldots, mq_n), \end{aligned}$$

let $\mathbf{f}$ *be the monomial map*

$$\begin{aligned} \mathbf{f} &: \mathbb{C}_*^{n+1} \to \mathbb{C}^{n+1} \\ &: (z_0, z_1, z_2, \ldots, z_n) \mapsto (z_0^m, z_1, z_2, \ldots, z_n). \end{aligned}$$

If the integers m and q_0 are relatively prime, then the induced map $f : \mathbb{C}P(\mathbf{p}) \to \mathbb{C}P(\mathbf{q})$ is invertible.

Proof. The map f is well-defined and onto by Lemma 8. To establish the one-to-one property as in Proposition 5, we have to show that for any $\mathbf{z}$, $\mathbf{z}' \in \mathbb{C}_*^{n+1}$, if there exists $\lambda \neq 0$ such that

$$(\lambda^{q_0} z_0^m, \lambda^{q_1} z_1, \ldots, \lambda^{q_n} z_n) = ((z_0')^m, z_1', \ldots, z_n'), \tag{5}$$

then there exists $\mu \neq 0$ so that

$$(\mu^{q_0} z_0, \mu^{mq_1} z_1, \ldots, \mu^{mq_n} z_n) = (z_0', z_1', \ldots, z_n'). \tag{6}$$

The algebra problem is: given λ, $\mathbf{z}$, $\mathbf{z}'$, find μ. If $z_0' = 0$, then $z_0 = 0$ and we can pick any μ satisfying $\mu^m = \lambda$. If $z_0' \neq 0$, then $z_0 \neq 0$, and there are m different roots $\{\mu_k : k = 0, \ldots, m-1\}$ satisfying $\mu_k^m = \lambda$. For $j = 1, \ldots, n$, each μ_k satisfies $\mu_k^{mq_j} z_j = \lambda^{q_j} z_j = z_j'$. Each μ_k also satisfies

$$(\mu_k^{q_0})^m = \mu_k^{mq_0} = \lambda^{q_0} = (z_0')^m / z_0^m = (z_0'/z_0)^m,$$

so each element of the set $R_1 = \{\mu_0^{q_0}, \ldots, \mu_{m-1}^{q_0}\}$ is also one of the m elements of the set $R_2 = \{\xi : \xi^m = (z_0'/z_0)^m\}$. One of the elements of R_2 is z_0'/z_0. Using the assumption that m and q_0 are relatively prime and Lemma 9, R_1 has m distinct elements, so there is some k such that $\mu_k^{q_0} = z_0'/z_0$. This μ_k is the μ required in (6), to show f is one-to-one. ∎

For any space of the form $\mathbb{C}P(q_0, mq_1, \ldots, mq_n)$, if q_0 and m are not already relatively prime, then a common factor can be divided out as in Case 1 of Lemma 6 without changing the set $\mathbb{C}P(\mathbf{p})$, and then Lemma 10 can be applied.

3 Reconstructibility

This Section introduces a new notion, Definition 14, describing a property that a weighted projective space may or may not have, in terms of certain $\mathbb{K}P^1$-valued rational functions on $\mathbb{K}P(\mathbf{p})$. This property of $\mathbb{K}P(\mathbf{p})$ is not used in Section 4 when the generalized Eves expression $E_{\mathbf{p}}$ is defined, but rather only in Section 6, to understand how the invariant $E_{\mathbf{p}}(\mathcal{S}) \in \mathbb{K}P(\mathbf{p})$ of a multi-color configuration $\mathcal{S}$ is related to the invariants $E_{(1,1)} \in \mathbb{K}P^1$ of some two-color configurations. Whether the $E_{\mathbf{p}}$ invariant is determined by the $E_{(1,1)}$ invariants turns out to depend on whether $\mathbb{K}P(\mathbf{p})$ is "reconstructible," an intrinsic property whose definition is motivated by, but does not depend on, and is not required by, the topic of invariants of configurations.

Given a weight $\mathbf{p}$, and two indices $i < j$ in $\{0, 1, \ldots, n\}$, consider numbers a_{ij}, $b_{ij} \in \mathbb{N}$ and a mapping $\mathbf{c}_{ij} : \mathbb{K}_*^{n+1} \to \mathbb{K}^2$ defined by the formula

$$\mathbf{c}_{ij}(z_0, z_1, \ldots, z_i, \ldots, z_j, \ldots, z_n) = (z_i^{a_{ij}}, z_j^{b_{ij}}).$$

The function $\mathbf{c}_{ij}$ satisfies (1) on the complement of the set $\{\mathbf{z} : z_i = z_j = 0\}$, and if the products are equal: $p_i a_{ij} = p_j b_{ij}$, then it also satisfies (2) for weights $\mathbf{p}$ and $\mathbf{q} = (1, 1)$.

Definition 11. For $\mathbf{p}$, a_{ij}, b_{ij} as above, the function $\mathbf{c}_{ij}$ induces a map

$$\begin{array}{rcl} c_{ij} : \mathbb{K}P(\mathbf{p}) & \to & \mathbb{K}P^1 : \\ [z_0 : z_1 : \ldots : z_i : \ldots : z_j : \ldots : z_n]_{\mathbf{p}} & \mapsto & \left[z_i^{a_{ij}} : z_j^{b_{ij}}\right], \end{array}$$

which is well-defined on the complement of the set

$$\{[z_0 : \ldots : z_n]_{\mathbf{p}} : z_i = z_j = 0\}.$$

We call such a map an <u>axis projection</u>.

Lemma 12. *Given a weight* $\mathbf{p}$ *and indices* i, j, *let*

$$\ell_{ij} = \mathrm{lcm}(p_i, p_j).$$

Then

$$h_{ij} : \mathbb{K}P(\mathbf{p}) \to \mathbb{K}P^1 : z \mapsto [z_i^{\ell_{ij}/p_i} : z_j^{\ell_{ij}/p_j}] \tag{7}$$

is an axis projection. For any axis projection c_{ij} *as in Definition 11, there exists* $k_{ij} \in \mathbb{N}$ *such that* c_{ij} *factors as* $c_{ij} = G_{ij} \circ h_{ij}$*, where the function* $G_{ij} : \mathbb{K}P^1 \to \mathbb{K}P^1$ *is given by the formula*

$$G_{ij} : [w_0 : w_1] \mapsto [w_0^{k_{ij}} : w_1^{k_{ij}}].$$

Proof. G_{ij} is well-defined on $\mathbb{K}P^1$, by checking (1) and (2). ℓ_{ij} is the least common multiple of p_i and p_j. By elementary number theory ([O] Ch. 3), any other common multiple is divisible by ℓ_{ij}, so there exists k_{ij} so that $p_i a_{ij} = p_j b_{ij} = k_{ij}\ell_{ij}$. ∎

Notation 13. Let I be the set of index pairs $\{(i,j) : 0 \le i < j \le n\}$. Let $D_{\mathbf{p}} \subseteq \mathbb{K}P(\mathbf{p})$ be the set of points where all the coordinates are non-zero: $\{z_0 \neq 0,\ z_1 \neq 0,\ \ldots,$ and $z_n \neq 0\}$. Given axis projections c_{ij} for $(i,j) \in I$, let $\prod c_{ij}$ denote the map

$$\begin{array}{rcl} \mathbb{K}P(\mathbf{p}) & \to & \mathbb{K}P^1 \times \mathbb{K}P^1 \times \cdots \times \mathbb{K}P^1 \\ z & \mapsto & (c_{01}(z), c_{02}(z), \ldots, c_{ij}(z), \ldots, c_{n-1,n}(z)) \, . \end{array}$$

The target space in the above Notation has one $\mathbb{K}P^1$ factor for each of the elements of I ($\#I = n(n+1)/2$), so the output formula assumes some ordering of I and lists an axis projection for every index pair (i,j). The map $\prod c_{ij}$ is well-defined at every point in $D_{\mathbf{p}}$, and possibly at some points not in $D_{\mathbf{p}}$.

Definition 14. A weighted projective space

$$\mathbb{K}P(\mathbf{p}) = \mathbb{K}P(p_0, \ldots, p_n)$$

is <u>reconstructible</u> means: there exist axis projections such that the restriction of the map $\prod c_{ij}$ to the domain $D_{\mathbf{p}}$ is one-to-one.

The idea is to try to use a list of unweighted ratios,

$$c_{ij}(z) \in \mathbb{K}P^1, \quad (i,j) \in I,$$

as a coordinatization of the space $\mathbb{K}P(\mathbf{p})$. A reconstructible space is one where any point z (with non-zero coordinates) can be uniquely "reconstructed" from a list of its values under some axis projections. The use of the domain $D_{\mathbf{p}}$ in the Definition omits consideration of points z with a zero coordinate; as already seen in Example 5, such points can exhibit exceptional behavior, and we are interested in properties of generic points.

Lemma 15. *Given* $\mathbf{p}$*, the following are equivalent.*

1. $\mathbb{K}P(\mathbf{p})$ *is reconstructible;*
2. *for the axis projections* h_{ij} *from* (7)*, the map* $\prod_{(i,j)\in I} h_{ij}$ *is one-to-one on* $D_{\mathbf{p}}$*;*
3. *there exist a subset* $J \subseteq I$ *and axis projections* c_{ij} *so that* $\prod_{(i,j)\in J} c_{ij}$ *is one-to-one on* $D_{\mathbf{p}}$*;*
4. *there exists a subset* $J \subseteq I$ *so that* $\prod_{(i,j)\in J} h_{ij}$ *is one-to-one on* $D_{\mathbf{p}}$*.*

Proof. The implications $2 \implies 1 \implies 3$ and $2 \implies 4 \implies 3$ are logically trivial. To show $3 \implies 2$, given c_{ij} for $J \subseteq I$, pick any axis projections c_{ij} for the remaining indices not in J; then

$$\prod_{(i,j)\in J} c_{ij} = F \circ \prod_{(i,j)\in I} c_{ij},$$

where $F : (\mathbb{K}P^1)^{\#I} \to (\mathbb{K}P^1)^{\#J}$ forgets entries with non-J indices. Then, by Lemma 12, there exist factorizations $c_{ij} = G_{ij} \circ h_{ij}$, so

$$\prod_{(i,j)\in J} c_{ij} = F \circ \left(\prod_{(i,j)\in I} G_{ij} \right) \circ \left(\prod_{(i,j)\in I} h_{ij} \right)$$

(where the product map $\prod G_{ij} : (\mathbb{K}P^1)^{\#I} \to (\mathbb{K}P^1)^{\#I}$ is defined in the obvious way for the composition to make sense). If $\prod h_{ij}$ is not one-to-one on $D_{\mathbf{p}}$, then $\prod_{(i,j)\in J} c_{ij}$ is also not one-to-one on $D_{\mathbf{p}}$. ∎

Example 6. For any field $\mathbb{K}$, the space $\mathbb{K}P(1, p_1, \ldots, p_n)$ is reconstructible. Only n axis projections are needed for a one-to-one product map: let $J = \{(0, j) : j = 1, \ldots, n\}$, and consider $\mathbf{h}_{0j}(z_0, z_1, \ldots, z_n) = (z_0^{p_j}, z_j)$. If

$$z = \pi_{\mathbf{p}}(\mathbf{z}),\ z' = \pi_{\mathbf{p}}(\mathbf{z}') \in D_{\mathbf{p}}$$

satisfy $\mathbf{h}_{0j}(\mathbf{z}) \sim_{(1,1)} \mathbf{h}_{0j}(\mathbf{z}')$ for $j = 1, \ldots, n$, then there exist $\lambda_{0j} \neq 0$ such that $(z_0')^{p_j} = \lambda_{0j} z_0^{p_j}$ and $z_j' = \lambda_{0j} z_j$. Let $\mu = z_0'/z_0$ (using the assumption that $z_0 \neq 0$); then $\mu z_0 = z_0'$ and $\mu^{p_j} z_j = (z_0'/z_0)^{p_j} z_j = \lambda_{0j} z_j = z_j'$, so $\mathbf{z} \sim_{\mathbf{p}} \mathbf{z}'$.

Theorem 16. *For $\mathbb{K} = \mathbb{C}$ and any weight $\mathbf{p}$, $\mathbb{C}P(\mathbf{p})$ is reconstructible.*

Proof. Case 1: $n = 1$.

Any complex weighted projective line $\mathbb{C}P(q_0, q_1)$ is reconstructible; in fact, a stronger result holds: there is an axis projection which is one-to-one on the entire space. Let $g_{01} = \gcd(q_0, q_1)$ and $\ell_{01} = \operatorname{lcm}(q_0, q_1)$, so $q_0 = g_{01}p_0$, $q_1 = g_{01}p_1$, and $\ell_{01} = g_{01}p_0p_1$, where p_0, p_1 are relatively prime. The map

$$\mathbf{h}_{01} : \mathbb{C}^2_* \to \mathbb{C}^2 : (z_0, z_1) \mapsto (z_0^{\ell_{01}/q_0}, z_1^{\ell_{01}/q_1}) = (z_0^{p_1}, z_1^{p_0})$$

induces an axis projection h_{01} as in (7), so that the following diagram is commutative.

$$\begin{tikzcd}
\mathbb{C}P(q_0, q_1) \arrow[r, "h_{01}"] \arrow[d, "Id"'] & \mathbb{C}P^1 \\
\mathbb{C}P(p_0, p_1) \arrow[r, "(z_0^{p_1}, z_1)"'] & \mathbb{C}P(p_0, 1) \arrow[u, "(z_0, z_1^{p_0})"']
\end{tikzcd}$$

The left arrow, labeled Id, represents the identity map as in Case 1 of Lemma 6 with $m = g_{01}$. The map indicated by the lower arrow is induced by the polynomial map $\mathbb{C}^2_* \to \mathbb{C}^2 : (z_0, z_1) \mapsto (z_0^{p_1}, z_1)$. Both maps, indicated by the lower and right arrows, are (globally) one-to-one by Lemma 10, so we can conclude h_{01} is one-to-one on the entire domain $\mathbb{C}P(q_0, q_1)$.

Case 2: $n > 1$.

We use a product of axis projections as in statement *2.* from Lemma 15. For $(i, j) \in I$, recall the notation $\ell_{ij} = \operatorname{lcm}(p_i, p_j)$, and fix

$$a_{ij} = \ell_{ij}/p_i, \quad b_{ij} = \ell_{ij}/p_j, \tag{8}$$

and $g_{ij} = \gcd(p_i, p_j)$. Consider the product map

$$\prod_{(i,j)\in I} h_{ij} : [z_0 : \ldots : z_n]_{\mathbf{p}} \mapsto \prod_{(i,j)\in I} [z_i^{a_{ij}} : z_j^{b_{ij}}]. \tag{9}$$

To show this product map is one-to-one on $D_{\mathbf{p}}$, suppose we are given $\mathbf{z}$, $\mathbf{z}'$ (with no zero components), and constants $\lambda_{ij} \in \mathbb{C}^1_*$ such that $\lambda_{ij}(z_i')^{a_{ij}} = z_i^{a_{ij}}$ and $\lambda_{ij}(z_j')^{b_{ij}} = z_j^{b_{ij}}$. The algebra problem is then to find $\mu \in \mathbb{C}^1_*$ such that

$$\mu^{p_j} z_j' = z_j \quad \text{for } j = 0, \ldots, n. \tag{10}$$

There are p_0 distinct elements $\{\mu_k : k = 0, \ldots, p_0 - 1\}$ satisfying $\mu_k^{p_0} = z_0/z_0'$. For each k and for any $j = 1, \ldots, n$,

$$\begin{aligned}(\mu_k^{p_j} z_j')^{b_{0j}} &= \mu_k^{p_j b_{0j}} (z_j')^{b_{0j}} = \mu_k^{p_0 a_{0j}} (z_j')^{b_{0j}} \\ &= \left(\frac{z_0}{z_0'}\right)^{a_{0j}} (z_j')^{b_{0j}} = \lambda_{0j} (z_j')^{b_{0j}} = z_j^{b_{0j}}. \end{aligned} \tag{11}$$

By Lemma 9,

$$\begin{aligned} &\#\{\mu_k^{p_j} z_j' : k = 0, \ldots, p_0 - 1\} \\ = \ &\#\{\mu_k^{p_j}\} = \frac{\operatorname{lcm}(p_0, p_j)}{p_j} = \frac{\ell_{0j}}{p_j} = b_{0j}, \end{aligned}$$

which is equal to the number of roots in $\{\xi : \xi^{b_{0j}} = z_j^{b_{0j}}\}$, and so for each $j = 1, \ldots, n$, there exists some index k_j such that $\mu_{k_j}^{p_j} z_j' = z_j$. At this point we note that if all the $k_1, \ldots, k_n$ index values were the same, $\mu = \mu_{k_j}$ would satisfy (10) and we would be done. One case where this happens in a trivial way is $p_0 = 1$; this was already observed in Example 6.

The rest of the Proof does not attempt to show the k_j values are equal to each other; instead we use their existence to establish the existence of some other index x such that μ_x is the required solution of (10).

For $i, j = 1, \ldots, n$ with $i < j$, μ_{k_i} and μ_{k_j} satisfy:

$$\begin{aligned} (\mu_{k_i}^{p_i} z_i')^{a_{ij}} &= \mu_{k_i}^{\ell_{ij}} (z_i')^{a_{ij}} = z_i^{a_{ij}} = \lambda_{ij} (z_i')^{a_{ij}}, \\ (\mu_{k_j}^{p_j} z_j')^{b_{ij}} &= \mu_{k_j}^{\ell_{ij}} (z_j')^{b_{ij}} = z_j^{b_{ij}} = \lambda_{ij} (z_j')^{b_{ij}} \\ \implies \lambda_{ij} &= \mu_{k_i}^{\ell_{ij}} = \mu_{k_j}^{\ell_{ij}}. \end{aligned} \tag{12}$$

By re-labeling the roots if necessary, as in the Proof of Lemma 9, we may assume $\mu_k = r^{1/p_0} e^{i(\theta + 2\pi k)/p_0}$ for $k = 0, \ldots, p_0 - 1$. Then (12) implies the congruence

$$k_j \ell_{ij} \equiv k_i \ell_{ij} \mod p_0. \tag{13}$$

We are looking for an index x such that for every $j = 1, \ldots, n$, $\mu_x^{p_j} = \mu_{k_j}^{p_j}$, so x is an integer solution to the following system of linear congruences, where p_j and k_j are known:

$$x p_j \equiv k_j p_j \mod p_0 \quad \text{for } j = 1, \ldots, n. \tag{14}$$

Dividing each congruence by $\gcd(p_0, p_j)$ does not change the solution set:

$$\begin{aligned} \frac{xp_j}{g_{0j}} &\equiv \frac{k_j p_j}{g_{0j}} \mod \frac{p_0}{g_{0j}} \\ \iff a_{0j}x &\equiv a_{0j}k_j \mod b_{0j}, \end{aligned}$$

which is equivalent, since a_{0j} and b_{0j} are relatively prime, to:

$$x \equiv k_j \mod b_{0j}. \tag{15}$$

By (an elementary generalization of) the Chinese Remainder Theorem ([O], Thm. 10–4), there exists an integer solution x of the system (15) if and only if for all pairs $1 \le i < j \le n$,

$$k_i \equiv k_j \mod \gcd(b_{0i}, b_{0j}). \tag{16}$$

Property (16) follows from (13): each congruence (16) is equivalent to

$$k_i \ell_{ij} \equiv k_j \ell_{ij} \mod \gcd(b_{0i}, b_{0j})\ell_{ij}.$$

The following equalities are elementary ([O] Chs. 3, 5); one step uses the property that a_{ij} and b_{ij} are relatively prime:

$$\begin{aligned} \gcd(b_{0i}, b_{0j})\ell_{ij} &= \gcd(b_{0i}\ell_{ij}, b_{0j}\ell_{ij}) \\ &= \gcd(b_{0i}p_i a_{ij}, b_{0j}p_j b_{ij}) \\ &= \gcd(\ell_{0i}a_{ij}, \ell_{0j}b_{ij}) = \gcd(\ell_{0i}, \ell_{0j}) \\ &= \gcd(\operatorname{lcm}(p_0, p_i), \operatorname{lcm}(p_0, p_j)) \\ &= \operatorname{lcm}(p_0, \gcd(p_i, p_j)) \\ &= \operatorname{lcm}(p_0, g_{ij}). \end{aligned}$$

By definition, ℓ_{ij} is a multiple of g_{ij}, and by (13), $k_i\ell_{ij} - k_j\ell_{ij}$ is a multiple of p_0. It follows that $k_i\ell_{ij} - k_j\ell_{ij}$ is a common multiple of p_0 and g_{ij}, and so a multiple of $\operatorname{lcm}(p_0, g_{ij})$, which implies (16). ∎

Theorem 17. *For $\mathbb{K} = \mathbb{R}$, $\mathbb{R}P(\mathbf{p})$ is reconstructible if and only if $p_0, \ldots, p_n$ are not all even.*

Proof. To establish reconstructibility, assume, WLOG, p_0 is odd. We can proceed with the same notation as Case 2 of the Proof of Theorem 16, and use a product of axis projections as in (9), although as in Example 6, only

Figure 2: The diagram for the Proof of Theorem 17

$$\begin{tikzcd}
\mathbb{R}P(\mathbf{p}) \arrow[rr] \arrow[d, "Id"'] & & \prod_{(i,j)\in I} \mathbb{R}P^1 \\
\mathbb{R}P(2q_0, 2^{e_1-e_0+1}q_1, \ldots, 2^{e_n-e_0+1}q_n) \arrow[r, "2:1"'] & \mathbb{R}P(q_0, 2^{e_1-e_0}q_1, \ldots, 2^{e_n-e_0}q_n) \arrow[ur] &
\end{tikzcd}$$

n axis projections, indexed by $(i,j) = (0,j)$, are needed for a one-to-one product map. Given real $\mathbf{z}$, $\mathbf{z}'$, and λ_{0j}, the algebra problem is to find a real solution μ of Equation (10). Since p_0 is odd and $z_0 \neq 0$, the equation $\mu^{p_0}z_0' = z_0$ has a unique real solution for μ. For each j, $b_{0j} = p_0/g_{0j}$ is odd, and using the unique solution for μ in Equation (11) gives $(\mu^{p_j}z_j')^{b_{0j}} = z_j^{b_{0j}}$, which implies $\mu^{p_j}z_j' = z_j$, so Equation (10) is satisfied.

For the converse, suppose $p_j = 2^{e_j}q_j$ with $e_j > 0$ and q_j odd for $j = 0, \ldots, n$. To show statement *2.* from Lemma 15 is false, we show that the product of axis projections as in (7), (9) is exactly two-to-one on $D_{\mathbf{p}}$; let this map be denoted by the top arrow in the diagram from Figure 2. WLOG, assume e_0 is the smallest of the e_j exponents. By Case 3 of Lemma 6, dividing the weight $\mathbf{p}$ by $m = 2^{e_0-1}$ does not change the weighted projective space; this identity map is shown as the left arrow in the diagram (Figure 2). The lower arrow is the map from Theorem 7; it is induced by the inclusion $\mathbb{R}_*^{n+1} \to \mathbb{R}^{n+1}$, and is two-to-one on the set $\{z : z_0 \neq 0\}$, which contains $D_{\mathbf{p}}$. The upward arrow on the right is defined as in statement *2.* from Lemma 15; this was shown to be one-to-one on $D_{(q_0,\ldots,2^{e_n-e_0}q_n)}$ in the first part of this Proof. The diagram is commutative (the top arrow is the composite of the other arrows) because the axis projections use the same exponents. For the top arrow,

$$\begin{aligned}
a_{ij} &= \frac{\operatorname{lcm}(p_i, p_j)}{p_i} = \frac{\operatorname{lcm}(2^{e_i}q_i, 2^{e_j}q_j)}{2^{e_i}q_i} \\
&= \frac{\operatorname{lcm}(2^{e_i-e_0}q_i, 2^{e_j-e_0}q_j)2^{e_0}}{2^{e_i}q_i}.
\end{aligned}$$

For the right arrow, the corresponding exponent is

$$\frac{\operatorname{lcm}(2^{e_i-e_0}q_i, 2^{e_j-e_0}q_j)}{2^{e_i-e_0}q_i},$$

which is the same, and similarly for the exponents b_{ij}. ∎

4 Generalizing Eves' Theorem

Eves' Theorem, as described in the Introduction, refers to a collection of segments lying on lines in a projective space. Our generalization requires some technical indexing, but we start by informally introducing the notation for the original case as in Figure 1. The dimension of the projective space is D. There are ℓ lines (possibly repeated; in Figure 1, $\ell = 6$). On each line, with index $K = 1, \ldots, \ell$, there is one segment of each color, indexed by $c = 0, 1$: one black segment $\overrightarrow{s}_0^K$ and one red segment, $\overrightarrow{s}_1^K$. Each segment $\overrightarrow{s}_c^K$ has a pair of endpoints, $(s_c^{K,1}, s_c^{K,2})$, with $s_c^{K,e}$ indexed by $e = 1, 2$. Let $\mathcal{S}_c$ be the set (possibly with repeats) of ℓ segments of color c. The pair $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1)$ will be denoted (in Definition 22) a $((1,1), 2, \ell, D)$-configuration.

The generalization allows multiple ($n + 1 \geq 2$) colors, indexed by $c = 0, 1, \ldots, n$. The two endpoints of a segment generalize to r points in an ordered r-tuple:

$$\overrightarrow{s}_c^K = (s_c^{K,1}, \ldots, s_c^{K,e}, \ldots, s_c^{K,r}).$$

The ℓ lines generalize to $(r-1)$-dimensional subspaces of $\mathbb{K}P^D$, and for each subspace there is not just one, but a fixed number, p_c, of generalized segments (r-tuples) of color c. So, the number of r-tuples with color c is ℓp_c, forming a collection $\mathcal{S}_c$. Definition 22 will precisely arrange all this data, up to immaterial re-indexings, into a set $\mathcal{S}$ called a $((p_0, \ldots, p_n), r, \ell, D)$-configuration.

4.1 Configurations in projective space

Some combinatorial notation is needed to keep track of various indices.

Notation 18. Two ordered N-tuples

$$(x_1, \ldots, x_N),\ (y_1, \ldots, y_N)$$

are equivalent up to re-ordering if there exists a permutation σ of the index set $\{1, \ldots, N\}$ such that $y_i = x_{\sigma(i)}$ for $i = 1, \ldots, N$. This is an equivalence relation; we denote the equivalence class of $(x_1, \ldots, x_N)$ with square brackets,

$$[x_1, \ldots, x_N],$$

and call it an unordered list.

When it is necessary to index the entries in an unordered list, it is convenient to first pick an ordered representative. Using the following notation, we describe some configurations of points in (non-weighted) projective space.

Definition 19. Given $D, r \in \mathbb{N}$, and points

$$\alpha^1, \ldots, \alpha^e, \ldots, \alpha^r \in \mathbb{K}P^D,$$

denote an ordered r-tuple of points

$$\overrightarrow{s} = (\alpha^1, \ldots, \alpha^e, \ldots, \alpha^r).$$

Such an ordered r-tuple is an independent r-tuple means: there exist representatives for the points, $\boldsymbol{\alpha}^1, \ldots, \boldsymbol{\alpha}^e, \ldots, \boldsymbol{\alpha}^r$, which form a linearly independent set of r vectors in $\mathbb{K}^{D+1}$ (so $r \leq D+1$).

Notation 20. In the case $r = 2$, we call the ordered, independent pair $\overrightarrow{s} = (\alpha^1, \alpha^2)$ a directed segment, and the two points its endpoints. In the case $r = 3$, ordered, independent triples are triangles $\overrightarrow{s} = \triangle(\alpha^1\alpha^2\alpha^3)$ with three vertices.

Definition 21. Given an independent r-tuple $\overrightarrow{s}$, there is a unique r-dimensional subspace $\mathbf{L}$ of $\mathbb{K}^{D+1}$ that is spanned by any independent set of representatives for the points in $\overrightarrow{s}$. The image $\pi(\mathbf{L} \setminus \{\mathbf{0}\}) = L$ is a $(r-1)$-dimensional projective subspace of $\mathbb{K}P^D$, which we call the span of $\overrightarrow{s}$.

It is convenient to also refer to the $\mathbb{K}$-linear subspace $\mathbf{L}$ as $\pi^{-1}(L)$, and to $L = \pi(\mathbf{L})$, even though π is not defined at $\mathbf{0}$.

Definition 22. Given a weight $\mathbf{p} = (p_0, \ldots, p_n)$ as in Section 2 and some other numbers $D, \ell, r \in \mathbb{N}$ with $r \leq D+1$, a $(\mathbf{p}, r, \ell, D)$-configuration (or, just "configuration" when the $\mathbf{p}$, r, ℓ, and D are understood) is an ordered $(n+1)$-tuple $\mathcal{S}$,

$$\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_c, \ldots, \mathcal{S}_n), \tag{17}$$

where each $\mathcal{S}_c$ is an unordered list (possibly with repeats) of $\ell \cdot p_c$ ordered, independent r-tuples of points in $\mathbb{K}P^D$:

$$\mathcal{S}_c = [\overrightarrow{s}_c^1, \ldots, \overrightarrow{s}_c^{\ell p_c}].$$

We remark that it is possible for some $\mathcal{S}$ to be both a $(\mathbf{p}, r, \ell, D)$-configuration and a $(\mathbf{p}', r, \ell', D)$-configuration with $\mathbf{p} \neq \mathbf{p}'$ and $\ell \neq \ell'$, although if $\mathbf{p}$ is given, then ℓ is determined by the length of the lists $\mathcal{S}_c$.

As an aid to visualization and drawing, the indices $c = 0, \ldots, n$ can correspond to colors: $c = 0 =$ black, $c = 1 =$ red, $c = 2 =$ green, etc. So, for $r = 2$, $\mathcal{S}_0$ is a list of ℓp_0 black segments, $\mathcal{S}_1$ is a list of ℓp_1 red segments, etc.

Notation 23. Given a $(\mathbf{p}, r, \ell, D)$-configuration $\mathcal{S}$, define the following sets:

- $\mathcal{P}(\mathcal{S})$ is the set of points $z \in \mathbb{K}P^D$ such that z is one of the r components of some $\overrightarrow{s}_c^K$ in $\mathcal{S}_c$, for some $c = 0, \ldots, n$;
- $\mathcal{L}(\mathcal{S})$ is the set of $(r-1)$-dimensional projective subspaces which are the spans of the r-tuples $\overrightarrow{s}_c^K$;
- $\mathcal{U}(\mathcal{S})$ is the following union of r-dimensional subspaces in $\mathbb{K}^{D+1}$:

$$\mathcal{U}(\mathcal{S}) = \bigcup_{L \in \mathcal{L}(\mathcal{S})} \pi^{-1}(L).$$

All three of the above sets depend only on the set of r-tuples in $\mathcal{S}$, not on the ordering in (17), nor on $\mathbf{p}$ and ℓ.

For example, when $r = 2$, any directed segment lies on a unique projective line, so $\mathcal{L}(\mathcal{S})$ is a (finite) set of projective lines in $\mathbb{K}P^D$. Since the same point may appear in several different r-tuples, it is possible for the size of $\mathcal{P}(\mathcal{S})$ to be small compared to the number of r-tuples.

Definition 24. Given a $(\mathbf{p}, r, \ell, D)$-configuration $\mathcal{S}$, for $c = 0, \ldots, n$, choose an ordered representative

$$S_c = (\overrightarrow{s}_c^1, \ldots, \overrightarrow{s}_c^K, \ldots, \overrightarrow{s}_c^{\ell p_c}) \tag{18}$$

of the equivalence class $\mathcal{S}_c$. Define the $\underline{c\text{-degree}}$ of a point $z \in \mathbb{K}P^D$ to be

$$deg_c(z) =$$
$$\#\{K : z \text{ is one of the } r \text{ points of the } r\text{-tuple } \overrightarrow{s}_c^K \text{ in } S_c\}.$$

According to the color scheme indexed by c, every point in the configuration has a black degree, a red degree, etc. Definition 24 is stated in a way so that possibly repeated r-tuples are counted with multiplicity. The assumption that each r-tuple is independent implies that z appears at most once in an r-tuple. The number $deg_c(z)$ does not depend on the choice of ordered representative S_c for $\mathcal{S}_c$, nor on $\mathbf{p}$ and ℓ if $\mathcal{S}$ admits another description as a $(\mathbf{p}', r, \ell', D)$-configuration. For all but finitely many points in $\mathbb{K}P^D$, the c-degree is zero.

The following Definition is dual to Definition 24.

Definition 25. For $\mathcal{S}$ and S_c as in Definition 24, define the <u>c-degree</u> of a projective $(r-1)$-subspace L of $\mathbb{K}P^D$ to be

$$deg_c(L) =$$
$$\#\{K : \text{all } r \text{ points of the } r\text{-tuple } \overrightarrow{s}_c^K \text{ in } S_c \text{ lie on } L\}.$$

The following Definition of a morphism of a configuration was motivated by, but is different from, a notion of isomorphic plane configurations considered by [Shephard].

Definition 26. Given a $(\mathbf{p}, r, \ell, D)$-configuration

$$\mathcal{S} = (\mathcal{S}_0, \dots, \mathcal{S}_n),$$

and a $(\mathbf{p}, r, \ell, D')$-configuration

$$\mathcal{T} = (\mathcal{T}_0, \dots, \mathcal{T}_n),$$

$\mathcal{A}$ is a <u>morphism</u> from $\mathcal{S}$ to $\mathcal{T}$ means $\mathcal{A}$ is a function $\mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{T})$ such that:

1. For indexing purposes, for any ordered representative for each $\mathcal{S}_c$, $c = 0, \dots, n$,
$$(\overrightarrow{s}_c^1, \dots, \overrightarrow{s}_c^K, \dots, \overrightarrow{s}_c^{\ell p_c}), \tag{19}$$
there is an ordered representative for $\mathcal{T}_c$,
$$(\overrightarrow{t}_c^1, \dots, \overrightarrow{t}_c^K, \dots, \overrightarrow{t}_c^{\ell p_c}); \tag{20}$$
and,

2. There exists a function $\mathbf{A} : \mathcal{U}(\mathcal{S}) \to \mathbb{K}^{D'+1}$ such that the restriction of $\mathbf{A}$ to each of the subspaces $\mathbf{L} = \pi^{-1}(L)$ for $L \in \mathcal{L}(\mathcal{S})$ is one-to-one and $\mathbb{K}$-linear, and induces a map $A_L : L \to \mathbb{K}P^{D'}$ which satisfies, for every $\overrightarrow{s}_c^K$ that spans L:

$$\begin{aligned} & A_L(\overrightarrow{s}_c^K) \\ = \;& A_L\left(\left(s_c^{K,1}, \ldots, s_c^{K,e}, \ldots, s_c^{K,r}\right)\right) \\ = \;& \left(A_L(s_c^{K,1}), \ldots, A_L(s_c^{K,e}), \ldots, A_L(s_c^{K,r})\right) \\ = \;& \left(\mathcal{A}(s_c^{K,1}), \ldots, \mathcal{A}(s_c^{K,e}), \ldots, \mathcal{A}(s_c^{K,r})\right) \qquad (21) \\ = \;& \left(t_c^{K,1}, \ldots, t_c^{K,e}, \ldots, t_c^{K,r}\right) = \overrightarrow{t}_c^K. \end{aligned}$$

As a consequence of the Definition, a morphism defines a one-to-one correspondence between the lists (19) and (20) of r-tuples of color c, $c = 0, \ldots, n$. A morphism $\mathcal{A}$ is necessarily an onto map on the sets of points, $\mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{T})$, but is not necessarily one-to-one, and the number $\#\mathcal{L}(\mathcal{T})$ may also be less than $\#\mathcal{L}(\mathcal{S})$. Our notion of morphism is a little stronger than just an incidence-preserving collection of projective linear mappings A_L of the projective subspaces in $\mathcal{L}(\mathcal{S})$; the maps must all be induced by the same $\mathbf{A}$.

Proposition 27. *Given $\mathbb{K}$, $\mathbf{p}$, r, ℓ, the $D = r-1, \ldots, \infty$ union of the sets of $(\mathbf{p}, r, \ell, D)$-configurations, together with the above notion of morphism, forms a category.*

Proof (sketch). There is an identity morphism from any $\mathcal{S}$ to itself. It is straightforward to check that the usual composition of maps of sets $\mathcal{A} : \mathcal{P}(\mathcal{S}) \to \mathcal{P}(\mathcal{T})$ and $\mathbf{A} : \mathcal{U}(\mathcal{S}) \to \mathcal{U}(\mathcal{T})$ defines an associative composition of morphisms. ∎

Example 7. The classical notion of projective equivalence is an important special case of morphism, as follows. Let $D' = D$, and let $\mathbf{A}$ be an invertible $\mathbb{K}$-linear map $\mathbb{K}^{D+1} \to \mathbb{K}^{D+1}$. The induced map

$$A : \mathbb{K}P^D \to \mathbb{K}P^D$$

is a projective transformation and a configuration $\mathcal{S}$ is projectively equivalent to its image $A(\mathcal{S})$. The restriction of A to $\mathcal{P}(\mathcal{S})$ is a morphism $\mathcal{A}$ from $\mathcal{S}$ to $A(\mathcal{S})$ as in Definition 26. First, for any ordered representative of $\mathcal{S}_c$, $c = 0, \ldots, n$, index the r-tuples in $A(\mathcal{S}_c) = \mathcal{T}_c$ by setting $\overrightarrow{t}_c^K = A(\overrightarrow{s}_c^K)$. The

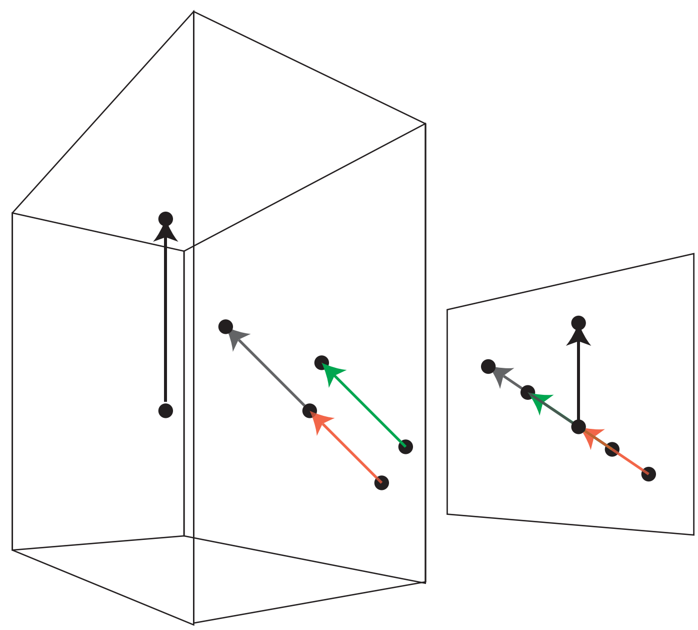

Figure 3: The projection from left to right defines a morphism from a configuration of 7 points, 3 lines, and 4 segments in three dimensions to a configuration of 6 points, 2 lines, and 4 segments in two dimensions.

map $\mathbf{A} : \mathcal{U}(\mathcal{S}) \to \mathbb{K}^{D+1}$ from the Definition is just the restriction of the given linear map to $\mathcal{U}(\mathcal{S})$, and restricts further to $\mathbf{L} = \pi^{-1}(L)$ for $L \in \mathcal{L}(\mathcal{S})$, so $\mathbf{A}|_{\mathbf{L}}$ is one-to-one and linear, satisfying (1) and (2), so it induces $A_L : L \to \mathbb{K}P^D$. For each independent r-tuple $\overrightarrow{s}_c^K$ with span L, the induced map A_L takes $\overrightarrow{s}_c^K$ to an independent r-tuple $A_L(\overrightarrow{s}_c^K) = \overrightarrow{t}_c^K$.

Example 8. In Example 7, checking Definition 26 did not require that $D' = D$, nor that $\mathbf{A}$ was invertible. The same argument applies to any $\mathbb{K}$-linear $\mathbf{A} : \mathbb{K}^{D+1} \to \mathbb{K}^{D'+1}$, which is not necessarily one-to-one or onto, but which is one-to-one when restricted to subspaces $\mathbf{L} = \pi^{-1}(L)$ for $L \in \mathcal{L}(\mathcal{S})$. As shown in Figure 3, the induced map A could be a projection from a subset of a higher-dimensional projective space to a lower-dimensional space, and would define a morphism $\mathcal{A}$ from a configuration $\mathcal{S}$ to $A(\mathcal{S})$ as long as the image of every $(r-1)$-dimensional projective subspace in $\mathcal{L}(\mathcal{S})$ is still $(r-1)$-dimensional.

Eves' notion of "h-expression," as described in the Introduction, applies to two-color configurations where the red degree of points (and dually, lines) matches the black degree; the following Definition generalizes the h-expression's degree condition by allowing multiple colors and weights.

Definition 28. A $(\mathbf{p}, r, \ell, D)$-configuration

$$\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_n)$$

is a weight $\mathbf{p}$ h-configuration means:

1. At every point $z \in \mathbb{K}P^D$, these numbers are integers and are equal to each other:

$$\frac{deg_0(z)}{p_0} = \cdots = \frac{deg_c(z)}{p_c} = \cdots = \frac{deg_n(z)}{p_n}; \tag{22}$$

2. For every projective $(r-1)$-subspace $L \subseteq \mathbb{K}P^D$, these numbers are integers and are equal to each other:

$$\frac{deg_0(L)}{p_0} = \cdots = \frac{deg_c(L)}{p_c} = \cdots = \frac{deg_n(L)}{p_n}. \tag{23}$$

See Figure 7 in Section 6 for an example of a weight $(2, 2, 4)$ h-configuration with $n + 1 = 3$ colors and $\ell = 3$ lines in the $D = 2$, $\mathbb{K} = \mathbb{R}$ plane.

For a weight $\mathbf{p}$ h-configuration $\mathcal{S}$, we have the following geometric interpretation of the parameter ℓ: if a $(r-1)$-dimensional projective subspace L in $\mathcal{L}(\mathcal{S})$ has $deg_c(L) = m_L p_c$, then by (23), m_L does not depend on c. There is an unordered ℓ-tuple of subspaces, $[L_1, \ldots, L_k, \ldots, L_\ell]$, where each L_k is incident with exactly p_c r-tuples with color c, and L_k occurs in the unordered list with multiplicity m_{L_k}.

Lemma 29. *If $\mathcal{S}$ is a weight $\mathbf{p}$ h-configuration and*

$$\mathcal{A} : \mathcal{S} \to \mathcal{T}$$

is a morphism, then $\mathcal{T}$ is a weight $\mathbf{p}$ h-configuration.

Proof. Let the ordered ℓp_c-tuple S_c be an ordered representative for $\mathcal{S}_c$; then let T_c be the corresponding ordered representative of $\mathcal{T}_c$ as in (20). The r points in $\overrightarrow{t}_c^K$ are indexed, using (21),

$$t_c^{K,e} = \mathcal{A}(s_c^{K,e}), \tag{24}$$

for $K = 1, \ldots, \ell p_c$ and $e = 1, \ldots, r$.

To check part 1. of Definition 28, suppose $z \in \mathbb{K}P^{D'}$. If $z \notin \mathcal{P}(\mathcal{T})$, then $deg_c(z) = 0$ for all c. If $z \in \mathcal{P}(\mathcal{T})$, then $\mathcal{A}^{-1}(z)$ is a finite set of points in $\mathcal{P}(\mathcal{S})$. There is no r-tuple $\overrightarrow{s}_c^K$ that contains more than one point of $\mathcal{A}^{-1}(z)$, since $\mathcal{A}(\overrightarrow{s}_c^K)$ is the independent r-tuple $\overrightarrow{t}_c^K$. An r-tuple $\overrightarrow{t}_c^K$ has z as one of its r points if and only if the corresponding r-tuple $\overrightarrow{s}_c^K$ has some element of $\mathcal{A}^{-1}(z)$ as one of its r points. For each c, the cardinality of the disjoint union of indices K is:

$$deg_c(z) = \sum_{w \in \mathcal{A}^{-1}(z)} deg_c(w).$$

The equalities in (22) for z follow from the assumed equalities for all the points w.

Dually, projective $(r-1)$-subspaces not in $\mathcal{L}(\mathcal{T})$ have $deg_c = 0$ for all c. By (21), every projective $(r-1)$-subspace in $\mathcal{L}(\mathcal{T})$ is of the form $A_{L'}(L')$, and if L' is the span of $\overrightarrow{s}_c^{K'}$, then all r points $t_c^{K',e}$ lie on $A_{L'}(L')$. The set $\mathcal{L}' = \{L \in \mathcal{L}(\mathcal{S}) : A_L(L) = A_{L'}(L')\}$ is finite, and there is no r-tuple $\overrightarrow{s}_c^K$ lying on more than one of these subspaces L. An r-tuple $\overrightarrow{t}_c^K$ lies on $A_{L'}(L')$

if and only if the corresponding r-tuple $\overrightarrow{s}_c^K$ lies on one of the subspaces $L \in \mathcal{L}'$. For each c, the cardinality of the disjoint union of indices K is:

$$deg_c(A_{L'}(L')) = \sum_{L \in \mathcal{L}'} deg_c(L).$$

The equalities in (23) for $A_{L'}(L')$ follow from the assumed equalities for all $L \in \mathcal{L}'$. ∎

4.2 The Invariant

Definition 30. Given an $(r-1)$-dimensional projective subspace L in $\mathbb{K}P^D$, let $\mathbf{L}$ be the r-dimensional subspace of $\mathbb{K}^{D+1}$ such that $L = \pi(\mathbf{L})$, and let $\mathcal{B}$ be any ordered basis $\mathcal{B} = (\mathbf{b}_0, \ldots, \mathbf{b}_{r-1})$ for $\mathbf{L}$. Given a linearly independent set of vectors $\mathbf{s}^e \in \mathbf{L}$, $e = 1, \ldots, r$, with $\pi(\mathbf{s}^e) = s^e$ on L, let $\overrightarrow{s}$ be the ordered r-tuple $(s^1, \ldots, s^e, \ldots, s^r)$, and define the Peano bracket of $\overrightarrow{s}$,

$$[\![\overrightarrow{s}]\!]_{\mathcal{B}} \in \mathbb{K}^1_*,$$

by the following procedure. The vectors have coordinates in the $\mathcal{B}$ basis:

$$\begin{aligned} \mathbf{s}^e &= s^{e,0}\mathbf{b}_0 + \ldots + s^{e,r-1}\mathbf{b}_{r-1} \\ \Longrightarrow \; [\mathbf{s}^e]_{\mathcal{B}} &= \begin{bmatrix} s^{e,0} \\ \vdots \\ s^{e,r-1} \end{bmatrix}_{\mathcal{B}} \in \mathbb{K}^r_*. \end{aligned} \tag{25}$$

By stacking columns into a square matrix, denote

$$[\![\overrightarrow{s}]\!]_{\mathcal{B}} = \det\left(\left[[\mathbf{s}^1]_{\mathcal{B}} \cdots [\mathbf{s}^e]_{\mathcal{B}} \cdots [\mathbf{s}^r]_{\mathcal{B}} \right]_{r \times r} \right). \tag{26}$$

For example, in the $r = 2$ case,

$$[\![\overrightarrow{s}]\!]_{\mathcal{B}} = s^{1,0}s^{2,1} - s^{2,0}s^{1,1}. \tag{27}$$

The r-dimensional vector space $\mathbf{L}$, together with the extra structure in the RHS of (26), is called a Peano space by [BBR]. The Peano bracket of $\overrightarrow{s}$ as we have defined it in (26) depends on the choices of basis and representative points, and also on the ordering of points in $\overrightarrow{s}$. Note that picking a different representative $\lambda \cdot \mathbf{s}^e$ for the point s^e and $\lambda \neq 0$ transforms $[\![\overrightarrow{s}]\!]_{\mathcal{B}}$ to $\lambda \cdot [\![\overrightarrow{s}]\!]_{\mathcal{B}}$.

Definition 31. Given a $(\mathbf{p}, r, \ell, D)$-configuration $\mathcal{S}$, for each $c = 0, \ldots, n$, choose an ordered representative S_c of $\mathcal{S}_c$ as in (18). For each point z in the set of points $\mathcal{P}(\mathcal{S}) = \{s_c^{K,e}\}$, choose one representative vector $\mathbf{z} = \mathbf{s}_c^{K,e}$. For each projective $(r-1)$-subspace L in the set $\mathcal{L}(\mathcal{S})$, choose one ordered basis $\mathcal{B}_L$ for the r-subspace $\mathbf{L} = \pi^{-1}(L)$, and if the span of $\overrightarrow{s}_c^K$ is L, denote $\mathcal{B}_{c,K} = \mathcal{B}_L$. Then we call the following element of $\mathbb{K}P(\mathbf{p})$ a generalized Eves expression.

$$E_{\mathbf{p}}(\mathcal{S}) =$$
$$\left[\prod_{K=1}^{\ell p_0} [\![\overrightarrow{s}_0^K]\!]_{\mathcal{B}_{0,K}} : \ldots : \prod_{K=1}^{\ell p_c} [\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}} : \ldots : \prod_{K=1}^{\ell p_n} [\![\overrightarrow{s}_n^K]\!]_{\mathcal{B}_{n,K}}\right]_{\mathbf{p}} .$$

Theorem 32. *Let $\mathcal{S}$ be a $(\mathbf{p}, r, \ell, D)$-configuration. If $\mathcal{S}$ is a weight $\mathbf{p}$ h-configuration, then the generalized Eves expression $E_{\mathbf{p}}(\mathcal{S}) \in \mathbb{K}P(\mathbf{p})$ is well-defined, depending only on $\mathcal{S}$ and $\mathbf{p}$, and not on any of the choices made in the construction of Definition 31. Further, if $\mathcal{T}$ is a $(\mathbf{p}, r, \ell, D')$-configuration and $\mathcal{A} : \mathcal{S} \to \mathcal{T}$ is a morphism, then*

$$E_{\mathbf{p}}(\mathcal{S}) = E_{\mathbf{p}}(\mathcal{T}).$$

Proof. The choice of ordering S_c as in (18) is used only for well-defined indexing; the first thing to prove is that the $E_{\mathbf{p}}$ expression does not depend on this choice. The second part of the Proof is to show the expression does not depend on the choices made in computing the bracket (26). The third part of the Proof is verifying the invariance under morphism.

First, for each ordered r-tuple $\overrightarrow{s}_c^K$ in S_c, formula (26) shows that the quantity $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}} \in \mathbb{K}^1_*$ depends on a choice of basis $\mathcal{B}_{c,K}$ and a choice of representative vectors for the r points. By the independence property, the r points of each $\overrightarrow{s}_c^K$ span a unique projective $(r-1)$-subspace L, for which a unique basis $\mathcal{B}_L$ was chosen, by the construction of Definition 31. So, the basis used to compute $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$ depends only on the r points of $\overrightarrow{s}_c^K$ in $\mathbb{K}P^D$. Each of the points $s_c^{K,e}$ has a representative in $\mathbb{K}^{D+1}_*$ that does not depend on the color index c or the assignment of K index to the r-tuple $\overrightarrow{s}_c^K$. The construction of Definition 31 requires picking the same representative vector $\mathbf{z}$ when a point appears more than once in the $\mathcal{S}$ configuration, in r-tuples with different indices or colors: if $z = s_c^{K,e} = s_{c'}^{K',e'}$ then $\mathbf{z} = \mathbf{s}_c^{K,e} = \mathbf{s}_{c'}^{K',e'}$. We can conclude that $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$ is computed using representative vectors of the points and a basis, both depending only on the r-tuple of points and not on the index K coming from S_c. By commutativity, the product $\prod_{K=1}^{\ell \cdot p_c} [\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$ does not depend on the choice of ordered representative S_c for $\mathcal{S}_c$, nor on $\mathbf{p}$, since $\ell \cdot p_c$ is uniquely determined by $\mathcal{S}$. The element

$$\left[\prod_{K=1}^{\ell p_0} [\![\overrightarrow{s}_0^K]\!]_{\mathcal{B}_{0,K}} : \ldots : \prod_{K=1}^{\ell p_n} [\![\overrightarrow{s}_n^K]\!]_{\mathcal{B}_{n,K}} \right]_{\mathbf{p}} \in \mathbb{K}P(\mathbf{p})$$

may depend on $\mathbf{p}$, as in Theorem 7. We can conclude so far that the above expression depends only on $\mathcal{S}$ and $\mathbf{p}$, not on any of the choices of S_c.

By the independence property, the quantities $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$ are all non-zero, so each of the $n+1$ components in the $E_{\mathbf{p}}$ expression is non-zero: $E_{\mathbf{p}}(\mathcal{S}) \in D_{\mathbf{p}}$.

For the second part of the Proof, as previously mentioned, for each point z occurring with any multiplicity in the $\mathcal{S}$ configuration, the construction of Definition 31 requires choosing a fixed representative $\mathbf{z}$. Changing the choice of representative for that point, $\lambda \cdot \mathbf{z}$ instead of $\mathbf{z}$, changes each $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$ quantity to $\lambda \cdot [\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$, as remarked after Definition 30, for every $\overrightarrow{s}_c^K$ that has z as one of its r points (and only one, by independence). In each expression

$$\prod_{K=1}^{\ell \cdot p_c} [\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$$

(with color index c), there are $deg_c(z)$ (possibly repeated) r-tuples $\overrightarrow{s}_c^K$ with z as one of its r points, so changing $\mathbf{z}$ to $\lambda \cdot \mathbf{z}$ changes the product expression by a factor of $\lambda^{deg_c(z)}$. By part 1. of Definition 28, there is some integer y_z depending on z but not c, so that $deg_c(z) = y_z \cdot p_c$. Since for each c, the product changes by a factor of $(\lambda^{y_z})^{p_c}$, the $\sim_{\mathbf{p}}$-equivalence class of the $E_{\mathbf{p}}$ expression does not depend on the choice of λ or $\mathbf{z}$.

For a projective $(r-1)$-subspace L, the value of the bracket $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$ depends on the choice of ordered basis $\mathcal{B}_{c,K} = \mathcal{B}_L = (\mathbf{b}_0, \ldots, \mathbf{b}_{r-1})$ in the following way: let $\mathcal{B}'_L$ be another ordered basis of the same r-dimensional space $\mathbf{L}$. Then there exists a $r \times r$ invertible matrix Q which changes $\mathcal{B}_L$-coordinates to $\mathcal{B}'_L$-coordinates, via matrix multiplication: if the $\mathcal{B}_L$-coordinate column vector of $\mathbf{s}_c^{K,e}$ is as in (25), then the $\mathcal{B}'_L$-coordinate column vector is $[\mathbf{s}_c^{K,e}]_{\mathcal{B}'_L} = Q[\mathbf{s}_c^{K,e}]_{\mathcal{B}_L}$. Applying the Q coordinate change matrix to each column in the determinant (26) transforms the bracket by the well-known formula

$$\begin{aligned}
[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}'_L} &= \det\left(\left[(Q[\mathbf{s}_c^{K,1}]_{\mathcal{B}_L})\cdots(Q[\mathbf{s}_c^{K,r}]_{\mathcal{B}_L})\right]_{r\times r}\right)\\
&= \det(Q)\det\left(\left[[\mathbf{s}_c^{K,1}]_{\mathcal{B}_L}\cdots[\mathbf{s}_c^{K,r}]_{\mathcal{B}_L}\right]_{r\times r}\right)\\
&= \det(Q)[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_L}.
\end{aligned}$$

We can conclude that for any L, changing the choice of ordered basis $\mathcal{B}_L$ to a new basis $\mathcal{B}'_L$, and using this new basis for every bracket expression for an r-tuple on L, results in changing each expression with color index c, $\prod_{K=1}^{\ell p_c}[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$, by a factor of $(\det(Q))^{deg_c(L)}$, where $deg_c(L) = m_L p_c$, and m_L does not depend on c, by part 2. of Definition 28. Since for each c, the product changes by a factor of $(\det(Q)^{m_L})^{p_c}$, the $\sim_{\mathbf{p}}$-equivalence class of $E_{\mathbf{p}}$ is unchanged. This shows that $E_{\mathbf{p}}$ does not depend on the choices made as in Definition 31, which are required to compute the brackets $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_L}$.

Thirdly, by Lemma 29, if there is a morphism $\mathcal{A}:\mathcal{S}\to\mathcal{T}$, then $\mathcal{T}$ is also a weight $\mathbf{p}$ h-configuration, and so the expression $E_{\mathbf{p}}(\mathcal{T})$ is well-defined by the previous part of this Proof. As in the Proof of Lemma 29, an ordering for $\mathcal{S}_c$ corresponds to one for $\mathcal{T}_c$, giving an indexing as in (24).

For each projective $(r-1)$-subspace $A_L(L)$ in the set $\mathcal{L}(\mathcal{T})$, pick an ordered basis $\mathcal{C}$ for the linear r-subspace $\mathbf{A}|_{\mathbf{L}}(\mathbf{L})$, as in Definition 31 applied to $\mathcal{L}(\mathcal{T})$. For any $\mathbf{L}'$ with $A_{L'}(L') = A_L(L)$, $\mathbf{A}|_{\mathbf{L}'}$ is linear and one-to-one, so $(\mathbf{A}|_{\mathbf{L}'})^{-1}(\mathcal{C})$ is an ordered basis for $\mathbf{L}'$, and setting $\mathcal{B}_{L'} = (\mathbf{A}|_{\mathbf{L}'})^{-1}(\mathcal{C})$ satisfies the uniqueness property of Definition 31 applied to $\mathcal{L}(\mathcal{S})$. Dually, for each point $w = t_c^{K,e}$ in the set $\mathcal{P}(\mathcal{T})$, pick a representative $\mathbf{w} = \mathbf{t}_c^{K,e}$ in $\mathbb{K}_*^{D'+1}$ as in Definition 31. For an index (c,K,e), the point $s_c^{K,e}$ lies on a $(r-1)$-subspace $L_{c,K}$ spanned by $\overrightarrow{s}_c^K$, and satisfies $A_{L_{c,K}}(s_c^{K,e}) = t_c^{K,e}$, and has a representative vector $(\mathbf{A}|_{\mathbf{L}_{c,K}})^{-1}(\mathbf{t}_c^{K,e})$ in $\mathbb{K}_*^{D+1}$. To show that this representative vector depends only on the point and not on the index, suppose (c',K',e') is any other index with $s_c^{K,e} = s_{c'}^{K',e'}$; then the point is on both projective $(r-1)$-subspaces $L_{c,K}$ and $L_{c',K'}$, and $\mathbf{A}|_{\mathbf{L}_{c,K}}$ and $\mathbf{A}|_{\mathbf{L}_{c',K'}}$ agree on the intersection $\mathbf{L}_{c,K}\cap\mathbf{L}_{c',K'}$ because they are restrictions of the same map $\mathbf{A}$. Since $\mathbf{t}_c^{K,e} = \mathbf{t}_{c'}^{K',e'}$, we can conclude $(\mathbf{A}|_{\mathbf{L}_{c,K}})^{-1}(\mathbf{t}_c^{K,e}) = (\mathbf{A}|_{\mathbf{L}_{c',K'}})^{-1}(\mathbf{t}_{c'}^{K',e'})$, and denote this representative vector $\mathbf{s}_c^{K,e}$.

Now, fix an index pair (c,K) and consider corresponding r-tuples $\overrightarrow{s}_c^K$ and $\overrightarrow{t}_c^K$, lying on subspaces $L_{c,K}$ and $A_{L_{c,K}}(L_{c,K})$ as above. The coordinate vector of $\mathbf{t}_c^{K,e}$ with respect to the ordered basis $\mathcal{C} = (\mathbf{c}_0,\ldots,\mathbf{c}_{r-1})$ of

$\mathbf{A}|_{\mathbf{L}_{c,K}}(\mathbf{L}_{c,K})$ is related to the coordinate vector of $\mathbf{s}_c^{K,e}$ with respect to the ordered basis $\mathcal{B}_{c,K} = \left(\mathbf{A}|_{\mathbf{L}_{c,K}}\right)^{-1}(\mathcal{C})$ of $\mathbf{L}_{c,K}$, by the linearity of $\mathbf{A}|_{\mathbf{L}_{c,K}}$:

$$
\begin{aligned}
& (\mathbf{s}_c^{K,e}) \\
= \; & \left(\mathbf{A}|_{\mathbf{L}_{c,K}}\right)^{-1}(\mathbf{t}_c^{K,e}) \\
= \; & \left(\mathbf{A}|_{\mathbf{L}_{c,K}}\right)^{-1}(t_c^{K,e,0}\mathbf{c}_0 + \ldots + t_c^{K,e,r-1}\mathbf{c}_{r-1}) \\
= \; & t_c^{K,e,0}\left(\mathbf{A}|_{\mathbf{L}_{c,K}}\right)^{-1}(\mathbf{c}_0) + \ldots + t_c^{K,e,r-1}\left(\mathbf{A}|_{\mathbf{L}_{c,K}}\right)^{-1}(\mathbf{c}_{r-1}) \\
= \; & t_c^{K,e,0}\mathbf{b}_{c,K,0} + \ldots + t_c^{K,e,r-1}\mathbf{b}_{c,K,r-1},
\end{aligned}
$$

i.e., the $\mathcal{B}_{c,K}$-coordinates of $\mathbf{s}_c^{K,e}$ are the same as the $\mathcal{C}$-coordinates of $\mathbf{t}_c^{K,e}$, and

$$
\begin{aligned}
[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}} &= \det\left(\left[[\mathbf{s}_c^{K,1}]_{\mathcal{B}_{c,K}} \cdots [\mathbf{s}_c^{K,r}]_{\mathcal{B}_{c,K}}\right]_{r\times r}\right) \\
&= \det\left(\left[[\mathbf{t}_c^{K,1}]_{\mathcal{C}} \cdots [\mathbf{t}_c^{K,r}]_{\mathcal{C}}\right]_{r\times r}\right) \\
&= [\![\overrightarrow{t}_c^{\,K}]\!]_{\mathcal{C}}.
\end{aligned}
$$

Using these brackets to compute the products in the $E_{\mathbf{p}}(\mathcal{S})$ expression, and the previously established fact that $E_{\mathbf{p}}(\mathcal{S})$ does not depend on the indexing S_c, or the choices of $\mathcal{B}_L$ or representative vectors, the claimed equality

$$E_{\mathbf{p}}(\mathcal{S}) = E_{\mathbf{p}}(\mathcal{T})$$

is proved. ∎

Example 9. For $\mathbf{p} = (1,1)$, $n = 1$ and there are two colors. $E_{(1,1)}(\mathcal{S})$ is a ratio of products of ℓ determinants of size $r \times r$, which, as stated in the Introduction, would have been recognizable before Eves' time. The case $\mathbf{p} = (1,1)$, $r = 2$, of Theorem 32 can be called a purely projective, or algebraic, version of Eves' Theorem, in comparison to the Euclidean, or metric, version, Theorem 6.2.2 of [E]. The connection between the determinantal expression and Eves' formula involving Euclidean signed lengths in $\mathbb{R}^D$ is discussed in Section 5.

For $r = 2$, in a $((1,1),2,\ell,D)$-configuration $\mathcal{S}$, $\mathcal{S}_0$ is a list of ℓ black directed segments in $\mathbb{K}P^D$, and $\mathcal{S}_1$ is a list of ℓ red segments. If $\mathcal{S}$ is a weight $\mathbf{p}$ h-configuration (which in this $\mathbf{p} = (1,1)$, $r = 2$ case we just call an <u>h-configuration</u>), then there are ℓ (counting with multiplicity) lines

$[L_1, \ldots, L_\ell]$ with one black segment $\overrightarrow{s}_0^K$ and one red segment $\overrightarrow{s}_1^K$ on each line, and at each point, the black degree equals the red degree. The following element of $\mathbb{K}P^1$, where each expression $[\![\overrightarrow{s}_c^K]\!]_{\mathcal{B}_{c,K}}$ is calculated as in Definition 31, is well-defined and invariant under projective transformations of $\mathbb{K}P^D$:

$$E_{(1,1)}(\mathcal{S}) = \left[\prod_{K=1}^{\ell} [\![\overrightarrow{s}_0^K]\!]_{\mathcal{B}_{0,K}} : \prod_{K=1}^{\ell} [\![\overrightarrow{s}_1^K]\!]_{\mathcal{B}_{1,K}}\right].$$

Eves calls the ratio

$$\frac{\displaystyle\prod_{K=1}^{\ell} [\![\overrightarrow{s}_1^K]\!]_{\mathcal{B}_{1,K}}}{\displaystyle\prod_{K=1}^{\ell} [\![\overrightarrow{s}_0^K]\!]_{\mathcal{B}_{0,K}}}$$

an "h-expression": each line L_K occurs equally often (multiplicity m_{L_K}) in the numerator and denominator, and each point in $\mathcal{P}(\mathcal{S})$ occurs equally often in the numerator (red degree) and denominator (black degree).

Example 10. Consider four distinct points α, β, γ, δ on the projective line $\mathbb{K}P^1$. These can be organized into an h-configuration $\mathcal{S}$, with $\mathbf{p} = (1,1)$ and $r = 2$ as in Example 9, dimension $D = 1$, and $\ell = 2$. Let $\mathcal{S}_0 = [(\delta,\alpha),(\gamma,\beta)]$ be a list of black segments, and let $\mathcal{S}_1 = [(\gamma,\alpha),(\delta,\beta)]$ be a list of red segments, as shown in Figure 4. Then $\mathcal{L}(\mathcal{S})$ is the singleton set $\{L = \mathbb{K}P^1\}$; we could, as mentioned after Definition 28, consider the line occurring with multiplicity two in the unordered list $[L_1, L_2]$ with $L = L_1 = L_2$. Choose the standard ordered basis $\mathcal{B}_L = ((1,0),(0,1))$ of $\mathbb{K}^2$, so α has homogeneous coordinates $[\alpha_0 : \alpha_1]$, vector representative $\alpha_0\mathbf{b}_0 + \alpha_1\mathbf{b}_1$, and $\mathcal{B}_L$-coordinate vector $\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix}_{\mathcal{B}_L}$, and similarly for the other points. Let $S_0 = (\overrightarrow{s}_0^1 = (\delta,\alpha), \overrightarrow{s}_0^2 = (\gamma,\beta))$ be an ordered representative of $\mathcal{S}_0$ and let $S_1 = (\overrightarrow{s}_1^1 = (\gamma,\alpha), \overrightarrow{s}_1^2 = (\delta,\beta))$ be an ordered representative of $\mathcal{S}_1$. Each endpoint has black degree and red degree both equal to 1, and the line L satisfies Part 2. of Definition 28, with $deg_0(L) = deg_1(L) = 2$. Alternatively, we could assign one of the black segments and one of the red segments to $L_1 = L$, and the remaining segments to $L_2 = L$; there are various choices of such assignments, which would not affect the expression (28). The generalized

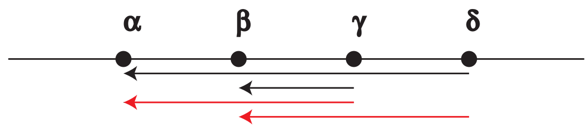

Figure 4: A configuration of 4 points, 1 line, and 4 ordered pairs, as indicated by the red and black arrows drawn offset from the line.

Eves expression is:

$$\begin{aligned}
& E_{(1,1)}(\mathcal{S}) \\
= \; & [\llbracket \overrightarrow{s}{}^{1}_{0} \rrbracket_{\mathcal{B}_L} \llbracket \overrightarrow{s}{}_{0}{}^{2} \rrbracket_{\mathcal{B}_L} : \llbracket \overrightarrow{s}{}^{1}_{1} \rrbracket_{\mathcal{B}_L} \llbracket \overrightarrow{s}{}_{1}{}^{2} \rrbracket_{\mathcal{B}_L}] \\
= \; & [(\alpha_1\delta_0 - \alpha_0\delta_1)(\beta_1\gamma_0 - \beta_0\gamma_1) : (\alpha_1\gamma_0 - \alpha_0\gamma_1)(\beta_1\delta_0 - \beta_0\delta_1)] \, ,
\end{aligned} \tag{28}$$

which is exactly the well-known cross-ratio of the ordered quadruple $(\alpha, \beta, \gamma, \delta)$.

In classical Invariant Theory, the fundamental property of projective invariance of the cross-ratio was often proved using determinants and algebraic methods similar to our Proof of Theorem 32 (e.g., [C]; [Salmon] Arts. XIII.136, 137, XVII.195). In projective geometry, the general idea that projective transformations introduce canceling factors in certain product expressions already appears in ([P] §20).

5 Metric versions

Eves' Theorem as stated in [E] is about ratios of signed lengths of directed segments, in the real Euclidean plane extended to include points at infinity. The earlier identities of [P], and interesting applications of Eves' Theorem, including Ceva's Theorem and others appearing in ([E] §6.2), [F_2], and [Shephard], also involve Euclidean distance between pairs of points. The constructions in Section 4 were developed in terms of linear algebra and projective geometry, avoiding any notion of distance. However, there are connections between projective geometry and Euclidean geometry — a contemporary treatment is given by [RG], relating Cartesian coordinates in affine neighborhoods, Peano bracket operations, and Euclidean notions of distance, area, volume, angles, etc.

We review some of these connections between projective and Euclidean geometry here, with two purposes: first, to show how the choices from Definitions 30 and 31 are related to Euclidean properties such as the choice of a unit of distance; and, second, to arrive at a generalized, but easy to use, statement of Eves' Theorem for directed segment lengths in $\mathbb{R}^D$ in Corollary 33. We start by incorporating a notion of distance as a bit of extra structure added to the projective coordinate system.

Consider, as in Example 1, $\mathbb{R}^{D+1}$ with coordinates $\mathbf{x} = (x_0, x_1, \ldots, x_D)$, the projection $\pi : \mathbb{R}^{D+1}_* \to \mathbb{R}P^D$, and homogeneous coordinates $x = [x_0 : x_1 : \ldots : x_D]$ for $\mathbb{R}P^D$. The restriction of π to the hyperplane $\{(1, x_1, x_2, \ldots, x_D)\}$ is one-to-one onto the image $\{x : x_0 \neq 0\}$ in $\mathbb{R}P^D$. We can refer to this affine neighborhood as $\mathbb{R}^D$, where a point in $\mathbb{R}^D$ has both homogeneous and affine coordinates: $x = [1 : x_1 : \ldots : x_D] = (x_1, \ldots, x_D)$, and also is the image of a representative vector: $x = \pi(\mathbf{x}) = \pi(1, x_1, \ldots, x_D)$.

The extra structure we initially assign to the affine neighborhood $\mathbb{R}^D$ is that of a normed vector space, where the vector space structure is the usual one from the affine coordinate system $(x_1, \ldots, x_D)$, and $\| \ . \ \|$ is any norm function. Then there is a distance function on $\mathbb{R}^D$: $\mathbf{d}(x, y) = \|y - x\|$.

In the $r = 2$ case, we are interested in directed segments on lines. Given a line L in $\mathbb{R}^D$ (meaning, a non-empty intersection of a projective line $L = \pi(\mathbf{L})$ with the $\{x : x_0 \neq 0\}$ neighborhood), it can be parametrized by choosing a start point b_0 and a non-zero direction vector v, so $L = \{b_0 + tv : t \in \mathbb{R}\}$. The choice of v also determines a direction for the line: an ordered pair of distinct points $(b_0 + t_1 v, b_0 + t_2 v)$ is a positively (or negatively) directed segment depending on the sign of $t_2 - t_1$. There exists a unique t value so that $t > 0$ and the point $b_1 = b_0 + tv$ satisfies $\mathbf{d}(b_0, b_1) = 1$. Choose these representative vectors in $\mathbb{R}^{D+1}_*$ for b_0 and b_1: $\mathbf{b}_0 = (1, b_0^1, \ldots, b_0^D)$ and $\mathbf{b}_1 = (1, b_1^1, \ldots, b_1^D)$. So, choosing a start point and a direction for the affine line L determines (and is determined by) an ordered basis $\mathcal{B} = (\mathbf{b}_0, \mathbf{b}_1)$ (with both points in $\{x_0 = 1\}$) for the plane $\mathbf{L}$.

Consider two distinct points α, β on the line L in $\mathbb{R}^D \subseteq \mathbb{R}P^D$. If we re-parametrize L using $b_1 - b_0$ as a direction vector,

$$\begin{aligned}
\alpha &= (b_0^1 + t_1(b_1^1 - b_0^1), \ldots, b_0^D + t_1(b_1^D - b_0^D)), \\
\beta &= (b_0^1 + t_2(b_1^1 - b_0^1), \ldots, b_0^D + t_2(b_1^D - b_0^D)).
\end{aligned}$$

The distance from α to β does not depend on the choice of start point b_0 nor the direction; it satisfies:

$$\begin{aligned}\mathbf{d}(\alpha,\beta) &= \|\beta-\alpha\| = \|(t_2-t_1)(b_1-b_0)\| \\ &= |t_2-t_1|\|b_1-b_0\| = |t_2-t_1|. \end{aligned}\tag{29}$$

The signed length of the directed segment $(\alpha,\beta)=\overrightarrow{\alpha\beta}$ is t_2-t_1, which depends on the direction but not the start point. Choosing the representative vectors

$$\begin{aligned}\boldsymbol{\alpha} &= (1, b_0^1+t_1(b_1^1-b_0^1),\ldots,b_0^D+t_1(b_1^D-b_0^D)) \\ &= (1-t_1)\mathbf{b}_0+t_1\mathbf{b}_1, \\ \boldsymbol{\beta} &= (1, b_0^1+t_2(b_1^1-b_0^1),\ldots,b_0^D+t_2(b_1^D-b_0^D)) \\ &= (1-t_2)\mathbf{b}_0+t_2\mathbf{b}_1,\end{aligned}\tag{30}$$

the signed length is exactly the same as the bracket formula (27):

$$[\![\overrightarrow{\alpha\beta}]\!]_{\mathcal{B}} = t_2(1-t_1)-(1-t_2)t_1 = t_2-t_1.$$

For directed segments $\overrightarrow{\alpha\beta}$ appearing in an h-configuration and the brackets $[\![\overrightarrow{\alpha\beta}]\!]_{\mathcal{B}}$ in a generalized Eves expression, Theorem 32 states that $E_{\mathbf{p}}(\mathcal{S})$ does not depend on the choice of representative vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ as long as representatives are chosen consistently (as in (30)), nor on the choice of $\mathcal{B}$ as long as that ordered basis is used for all directed segments on that line through α and β. Since the construction of Definition 31 requires that each line L is assigned a unique ordered basis $\mathcal{B}_L$, each line can have its own choice of direction determined by $\mathcal{B}_L$, and a unit of length depending on a norm $\|\ .\ \|_L$. So, it was not necessary at the start to pick one norm $\|\ \|$ for the whole space $\mathbb{R}^D$ to use for all the lines L; all that is needed is a distance function on each line such that (29) is valid. A metric version for the $r=2$ (directed segments) case of Theorem 32 can be stated as follows.

Corollary 33. *Given a $(\mathbf{p},2,\ell,D)$-configuration $\mathcal{S}$ of segments in $\mathbb{R}^D$, choose ordered representatives S_c as in* (18)*, and for each line L in the set $\mathcal{L}(\mathcal{S})$, choose a direction and unit of length, and denote by $[\![\overrightarrow{\alpha\beta}]\!]$ the signed length of a directed segment on that line. If $\mathcal{S}$ is a weight $\mathbf{p}$ h-configuration, then the following element of $\mathbb{R}P(\mathbf{p})$ does not depend on the choices of ordered representatives, directions, or unit lengths.*

$$E_{\mathbf{p}}(\mathcal{S}) = \left[\prod_{K=1}^{\ell p_0}[\![\overrightarrow{s}_0^K]\!] : \ldots : \prod_{K=1}^{\ell p_c}[\![\overrightarrow{s}_c^K]\!] : \ldots : \prod_{K=1}^{\ell p_n}[\![\overrightarrow{s}_n^K]\!]\right]_{\mathbf{p}}.$$

Further, $E_{\mathbf{p}}(\mathcal{S})$ is invariant under a morphism that maps the points in $\mathcal{S}$ into an affine neighborhood $\mathbb{R}^{D'}$. ∎

The Peano bracket also admits a Euclidean interpretation in the above coordinate system for configurations with $r = 3$ and $D = 2$ (see [BB], [CRG], in addition to the previously mentioned [RG]). However, in order for the bracket to define a Euclidean area in $\mathbb{R}^2$, we must use the Euclidean magnitude $\| \, . \, \|$, defined by the standard dot product in the affine coordinate system (x_1, x_2). Let L be the entire real projective plane $L = \mathbb{R}P^2$, and let $\mathbf{L} = \mathbb{R}^3$. Pick the standard ordered basis $\mathcal{B}$, so that three points $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$ in $\mathbb{R}^2 \subseteq \mathbb{R}P^2$ have representatives with $\mathcal{B}$-coordinates $\boldsymbol{\alpha} = \begin{bmatrix} 1 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}_{\mathcal{B}}$, etc. Then,

$$\begin{aligned} [\![(\alpha, \beta, \gamma)]\!]_{\mathcal{B}} &= \det \left(\left[\begin{bmatrix} 1 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \end{bmatrix}_{\mathcal{B}} \begin{bmatrix} 1 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}_{\mathcal{B}} \right]_{3\times 3} \right) \\ &= 2Area\triangle(\alpha\beta\gamma), \end{aligned}$$

twice the signed area of the triangle ([E] §2.1), which depends on the ordering of the three vertices and the (previously chosen) standard Euclidean structure on $\mathbb{R}^2$. The points with affine coordinates $(0, 0)$, $(1, 0)$, $(0, 1)$, in that order, form a counter-clockwise triangle with positive area $\frac{1}{2}$.

The following Example of ratios of areas was described by Clifford ([C]) as a "graphometric" quantity: a Euclidean measurement invariant under projective transformations.

Example 11. Consider six points, labeled 1, 2, 3, 4, 5, 6, in the plane $\mathbb{R}^2 \subseteq \mathbb{R}P^2$. They can be organized into a $((1, 1), 3, 2, 2)$-configuration $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1)$, where

$$\mathcal{S}_0 = [\triangle(124), \triangle(356)]$$

is a list of two black triangles, and

$$\mathcal{S}_1 = [\triangle(123), \triangle(456)]$$

is a list of two red triangles (assuming non-collinearity of the indicated triples), as in Figure 5. Then $\mathcal{L}(\mathcal{S}) = \{L = \mathbb{R}P^2\}$, and we choose the standard basis $\mathcal{B}$ as above. As in Example 10, $deg_0(L) = deg_1(L) = 2$, or we could consider a list $[L_1, L_2]$ with $L_1 = L_2 = L$, and one black triangle and one red triangle is assigned to each of L_1 and L_2. Each of the six points in $\mathcal{P}(\mathcal{S})$ is a vertex of one black triangle and one red triangle, so the black degree equals the red degree and $\mathcal{S}$ is a weight $(1,1)$ h-configuration. The generalized Eves expression is analogous to (28):

$$\begin{aligned} & E_{(1,1)}(\mathcal{S}) \\ = & \ [[\![\overrightarrow{s}_0^1]\!]_{\mathcal{B}}[\![\overrightarrow{s}_0^2]\!]_{\mathcal{B}} : [\![\overrightarrow{s}_1^1]\!]_{\mathcal{B}}[\![\overrightarrow{s}_1^2]\!]_{\mathcal{B}}] \\ = & \ [[\![\triangle(124)]\!]_{\mathcal{B}}[\![\triangle(356)]\!]_{\mathcal{B}} : [\![\triangle(123)]\!]_{\mathcal{B}}[\![\triangle(456)]\!]_{\mathcal{B}}] \,. \end{aligned}$$

We can conclude that the ratio of signed areas

$$\frac{(Area\triangle(123))(Area\triangle(456))}{(Area\triangle(124))(Area\triangle(356))}$$

is an invariant of the configuration $\mathcal{S}$ under projective transformations (that do not send any of the six points to infinity).

We remark that the property

$$E_{(1,1)}(\mathcal{S}^{(0,1)}) = [1:1],$$

or equivalently

$$[\![\triangle(124)]\!]_{\mathcal{B}}[\![\triangle(356)]\!]_{\mathcal{B}} - [\![\triangle(123)]\!]_{\mathcal{B}}[\![\triangle(456)]\!]_{\mathcal{B}} = 0,$$

admits a projective (not necessarily Euclidean) interpretation as the concurrence of the lines through $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ ([CRG], [RG] Ch. 6).

Example 12. In the configuration from Example 11, it is possible for $\mathcal{S}$ to be a weight $(1,1)$ h-configuration even if points 1 and 6 coincide, as in Figure 5. This gives another well-known $E_{(1,1)}$ projective invariant for five points in the (projective or Euclidean) plane ([RG] §10.2),

$$\frac{[\![\triangle(123)]\!]_{\mathcal{B}}[\![\triangle(451)]\!]_{\mathcal{B}}}{[\![\triangle(124)]\!]_{\mathcal{B}}[\![\triangle(351)]\!]_{\mathcal{B}}} = \frac{(Area\triangle(123))(Area\triangle(451))}{(Area\triangle(124))(Area\triangle(351))}.$$

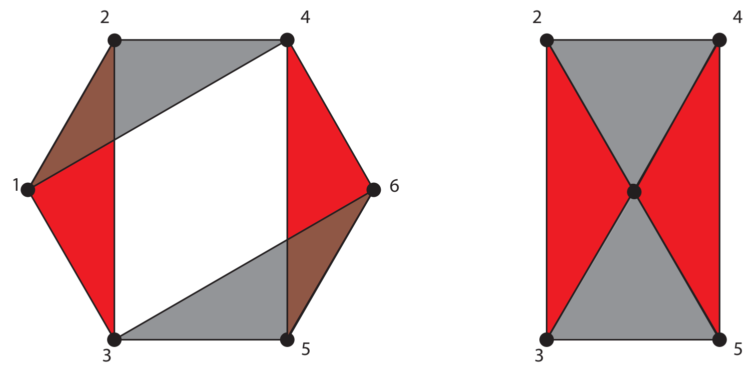

Figure 5: Left: A configuration of 6 points and 4 triangles in the real plane, from Example 11. Right: The points 1 and 6 coincide, as in Example 12.

6 Reconstruction

In this Section, we give some more examples of weight $\mathbf{p}$ h-configurations and their $E_{\mathbf{p}}$ invariants. If, for a configuration $\mathcal{S}$, we pick two out of the $n+1$ colors and look at only the segments or triangles with those colors, then the two-color configuration is still weighted. However, the classical, unweighted (meaning, weight $(1,1)$) invariant can be computed if some of the $\overrightarrow{s}_i^K$ are counted with enough multiplicity to balance the weights. The notion of reconstructibility, from Section 3, explains whether, and how, these unweighted invariants, considered for some or all color pairs, determine (reconstruct) the $E_{\mathbf{p}}$ invariant for the multi-color, weighted configuration $\mathcal{S}$.

Let

$$\mathcal{S} = (\mathcal{S}_0, \ldots, \mathcal{S}_n)$$

be a $(\mathbf{p}, r, \ell, D)$-configuration in $\mathbb{K}P^D$, and pick a pair of colors $(i,j) \in I$. The ordered pair $(\mathcal{S}_i, \mathcal{S}_j)$ is a $((p_i, p_j), r, \ell, D)$-configuration. If $\mathcal{S}$ is a weight $\mathbf{p}$ h-configuration, then $(\mathcal{S}_i, \mathcal{S}_j)$ is a weight (p_i, p_j) h-configuration. For a weight $\mathbf{p}$ h-configuration $\mathcal{S}$, the following are equivalent:

1. $(\mathcal{S}_i, \mathcal{S}_j)$ is a $((1,1), r, \ell{\cdot}p_i, D)$-configuration and a weight $(1,1)$ h-configuration;
2. $p_i = p_j$.

The goal of the following construction is to modify a $((p_i, p_j), r, \ell, D)$-configuration $(\mathcal{S}_i, \mathcal{S}_j)$ into a new $((1,1), r, \ell', D)$-configuration $\mathcal{S}^{(i,j)}$ in a way such that if

$p_i = p_j$, then the configuration does not change:

$$\mathcal{S}^{(i,j)} = (\mathcal{S}_i, \mathcal{S}_j),$$

and if $(\mathcal{S}_i, \mathcal{S}_j)$ is a weight (p_i, p_j) h-configuration, then $\mathcal{S}^{(i,j)}$ is a weight $(1,1)$ h-configuration.

Recall $\ell_{ij} = \mathrm{lcm}(p_i, p_j)$, and let $a_{ij} = \ell_{ij}/p_i$, $b_{ij} = \ell_{ij}/p_j$ as in (8).

Notation 34. Given a $((p_i, p_j), r, \ell, D)$-configuration $(\mathcal{S}_i, \mathcal{S}_j)$, define a new ordered pair

$$\mathcal{S}^{(i,j)} = (\mathcal{S}_i^{(i,j)}, \mathcal{S}_j^{(i,j)}),$$

where as in (17), each entry is an unordered list of r-tuples of points, one list with color i, the other with color j. Let $\mathcal{S}_i^{(i,j)}$ be the concatenation of a_{ij} copies of the list $\mathcal{S}_i$, so each of its $\ell \cdot p_i$ entries is repeated a_{ij} times. Similarly, let $\mathcal{S}_j^{(i,j)}$ be the concatenation of b_{ij} copies of $\mathcal{S}_j$.

The new configuration could be (but is not) descriptively denoted $(a_{ij}\mathcal{S}_i, b_{ij}\mathcal{S}_j)$. So far, $\mathcal{S}^{(i,j)}$ is a $((1,1), r, \ell \cdot \ell_{ij}, D)$-configuration, since both $\mathcal{S}_i^{(i,j)}$ and $\mathcal{S}_j^{(i,j)}$ have $\ell \cdot \ell_{ij}$ entries, and the independence property of each r-tuple is inherited.

Lemma 35. *If $(\mathcal{S}_i, \mathcal{S}_j)$ as above is a weight (p_i, p_j) h-configuration, then $\mathcal{S}^{(i,j)}$ is a weight $(1,1)$ h-configuration.*

Proof. Part 1. of Definition 28 is satisfied, with weight $(1,1)$: By construction, the i-degree of any point z in the $\mathcal{S}^{(i,j)}$ configuration is a_{ij} times $deg_i(z)$, the i-degree of the same point in the $\mathcal{S}$ configuration, and similarly for j, so:

$$\frac{deg_i(z) \cdot a_{ij}}{1} = \frac{deg_j(z) \cdot b_{ij}}{1} \iff \frac{deg_i(z)}{p_i} = \frac{deg_j(z)}{p_j}.$$

Dually, part 2. of Definition 28 is also satisfied, by the same calculation. ∎

The following identity applies Theorem 32 to $\mathcal{S}^{(i,j)}$. Recall

$$h_{ij} : \mathbb{K}P(\mathbf{p}) \to \mathbb{K}P^1 : z \mapsto [z_i^{\ell_{ij}/p_i} : z_j^{\ell_{ij}/p_j}]$$

is the axis projection (7) from Lemma 12.

Corollary 36. *If $\mathcal{S}$ is a weight $\mathbf{p}$ h-configuration, then*

$$E_{(1,1)}(\mathcal{S}^{(i,j)}) = h_{ij}\left(E_{\mathbf{p}}(\mathcal{S})\right).$$

Proof. Suppose the points and projective $(r-1)$-subspaces in the weight $\mathbf{p}$ h-configuration $\mathcal{S}$ have been assigned vector representatives $\mathbf{z}$ and bases $\mathcal{B}_L$ as in Definition 31. In the weight $(1,1)$ h-configuration $\mathcal{S}^{(i,j)}$, we can use the same representatives and bases, and then choose some ordered representative

$$S_i^{(i,j)} = \left(\overrightarrow{s}_{i,1}^{(i,j)}, \ldots, \overrightarrow{s}_{i,K}^{(i,j)}, \ldots, \overrightarrow{s}_{i,\ell \cdot \ell_{ij}}^{(i,j)} \right)$$

for $\mathcal{S}_i^{(i,j)}$ and similarly $S_j^{(i,j)}$ for $\mathcal{S}_j^{(i,j)}$. By the weight $(1,1)$ case of Theorem 32, the following $E_{(1,1)}$ ratio in $\mathbb{K}P^1$ is well-defined, and invariant under morphisms. The products can be expanded using the multiplicity of the r-tuples.

$$\begin{aligned}
& E_{(1,1)}(\mathcal{S}^{(i,j)}) \\
= \ & \left[\prod_{K=1}^{\ell \cdot \ell_{ij}} [\![\overrightarrow{s}_{i,K}^{(i,j)}]\!]_{\mathcal{B}_{i,K}} : \prod_{K=1}^{\ell \cdot \ell_{ij}} [\![\overrightarrow{s}_{j,K}^{(i,j)}]\!]_{\mathcal{B}_{j,K}} \right] \\
= \ & \left[\left(\prod_{K'=1}^{\ell p_i} [\![\overrightarrow{s}_i^{K'}]\!]_{\mathcal{B}_{i,K'}} \right)^{a_{ij}} : \left(\prod_{K'=1}^{\ell p_j} [\![\overrightarrow{s}_j^{K'}]\!]_{\mathcal{B}_{j,K'}} \right)^{b_{ij}} \right] \\
= \ & h_{ij}\left(E_{\mathbf{p}}(\mathcal{S}) \right).
\end{aligned}$$

∎

The analogue of the above construction in classical Invariant Theory is the formation of an absolute invariant as a ratio of powers of differently weighted relative invariants, as in ([Salmon] Art. XII.122).

Suppose $\mathbf{p}$ and $\mathbb{K}$ have the property that $\mathbb{K}P(\mathbf{p})$ is reconstructible. By Lemma 15, $E_{\mathbf{p}}(\mathcal{S})$ is uniquely determined by the set of ratios $h_{ij}\left(E_{\mathbf{p}}(\mathcal{S})\right)$, for $(i,j) \in I$. Corollary 36 shows that the weight $\mathbf{p}$ invariant $E_{\mathbf{p}}(\mathcal{S})$ can be uniquely reconstructed by finding the weight $(1,1)$ invariant for all (or possibly fewer) of the weight $(1,1)$ h-configurations $\mathcal{S}^{(i,j)}$. So, the $E_{\mathbf{p}}$ invariant has no more power to distinguish projectively inequivalent weight $\mathbf{p}$ h-configurations $\mathcal{S}$ than does the $E_{(1,1)}$ invariant, applied at most $n(n+1)/2$ times, two colors at a time, via the above construction.

However, if $\mathbb{K}P(\mathbf{p})$ is not reconstructible, then there may be weight $\mathbf{p}$ h-configurations with different $E_{\mathbf{p}}$ invariants, but which cannot be distinguished using only $E_{(1,1)}$ and the reconstruction process described in the previous paragraph. The following two Examples show this can happen when

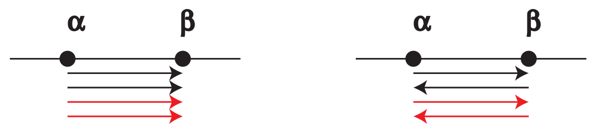

Figure 6: Configurations of 2 points and 4 segments on 1 real line, from Example 13. Left: configuration $\mathcal{S}$; Right: configuration $\mathcal{T}$.

$\mathbb{K} = \mathbb{R}$, $r = 2$, and Eves' Theorem is applied to signed distances in $\mathbb{R}^D$ as in Corollary 33.

Example 13. The simplest example of a non-reconstructible weighted projective space is $\mathbb{R}P(2,2)$, where there is only one axis projection in the product from Definition 14: let $h_{0,1} : \mathbb{R}P(2,2) \to \mathbb{R}P(1,1)$ be the two-to-one map induced by the inclusion $\mathbf{h}_{0,1}(z_0, z_1) = (z_0, z_1)$ as in Theorem 7 and Example 5. The simplest example of a weight $(2,2)$ h-configuration has $r = 2$, $D = 1$ and $\ell = 1$: one line $L = \mathbb{R}P^1$. Let α and β be distinct points on $\mathbb{R}^1 \subseteq \mathbb{R}P^1$, and consider the configuration with the directed segment (α, β) appearing with multiplicity 4: two black segments and two red segments. The indexing as in (17) is $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1)$, and $\mathcal{S}_0 = \mathcal{S}_1 = [(\alpha,\beta), (\alpha,\beta)]$. If we pick any unit of length in either direction, in order to define $[\![\overrightarrow{\alpha\beta}]\!]$ as the signed length of the directed segment (α, β), then the weighted invariant from Corollary 33 is

$$E_{(2,2)}(\mathcal{S}) = \left[[\![\overrightarrow{\alpha\beta}]\!]^2 : [\![\overrightarrow{\alpha\beta}]\!]^2 \right]_{(2,2)} = [1:1]_{(2,2)}.$$

The modification of $\mathcal{S}$ into a weight $(1,1)$ h-configuration $\mathcal{S}^{(0,1)}$ is only a change in point of view from a $((2,2),2,1,1)$-configuration to a $((1,1),2,2,1)$-configuration; there is no change in the lists of segments:

$$\mathcal{S}^{(0,1)} = (\mathcal{S}_0^{(0,1)}, \mathcal{S}_1^{(0,1)}) = (\mathcal{S}_0, \mathcal{S}_1) = \mathcal{S},$$

or the set of lines, $\{L\}$. By Corollary 36, the $(1,1)$ invariant of this h-configuration is:

$$\begin{aligned} E_{(1,1)}(\mathcal{S}^{(0,1)}) &= h_{0,1}(E_{(2,2)}(\mathcal{S})) \\ &= \left[[\![\overrightarrow{\alpha\beta}]\!]^2 : [\![\overrightarrow{\alpha\beta}]\!]^2 \right]_{(1,1)} = [1:1]. \end{aligned}$$

Now, let $\mathcal{T}$ be a new weight $(2,2)$ h-configuration: the same line L and points α, β as $\mathcal{S}$, but with two black segments in opposite directions, and two red segments also in opposite directions. The indexing as in (17) is

$$\begin{aligned} \mathcal{T} &= (\mathcal{T}_0, \mathcal{T}_1), \\ \mathcal{T}_0 &= \mathcal{T}_1 = [(\alpha, \beta), (\beta, \alpha)]. \end{aligned}$$

There is obviously no morphism $\mathcal{S} \to \mathcal{T}$, and the weighted invariant is a different element of $\mathbb{R}P(2,2)$:

$$E_{(2,2)}(\mathcal{T}) = \left[[\![\overrightarrow{\alpha\beta}]\!][\![\overrightarrow{\beta\alpha}]\!] : [\![\overrightarrow{\alpha\beta}]\!][\![\overrightarrow{\beta\alpha}]\!] \right]_{(2,2)} = [-1 : -1]_{(2,2)}.$$

$\mathcal{T}$ is also a weight $(1,1)$ h-configuration, with $(1,1)$ invariant:

$$\begin{aligned} E_{(1,1)}(\mathcal{T}^{(0,1)}) &= h_{0,1}(E_{(2,2)}(\mathcal{T})) \\ &= \left[[\![\overrightarrow{\alpha\beta}]\!][\![\overrightarrow{\beta\alpha}]\!] : [\![\overrightarrow{\alpha\beta}]\!][\![\overrightarrow{\beta\alpha}]\!] \right]_{(1,1)} \\ &= [-1 : -1] = [1 : 1]. \end{aligned} \tag{31}$$

The conclusion is that the $E_{(1,1)}$ invariant cannot distinguish between $\mathcal{S}^{(0,1)} = \mathcal{S}$ and $\mathcal{T}^{(0,1)} = \mathcal{T}$.

Example 14. Let α, β, γ be the vertices of a triangle in the Euclidean plane $\mathbb{R}^2$, and let α', β', $\gamma\,'$ be the midpoints on opposite sides. The following configuration

$$\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1, \mathcal{S}_2)$$

is a $((2,2,4),2,3,2)$-configuration and a weight $(2,2,4)$ h-configuration.

$$\begin{aligned} \mathcal{S}_0 &= [(\beta,\alpha'),(\beta,\alpha'),(\gamma,\beta'),(\gamma,\beta'),(\alpha,\gamma\,'),(\alpha,\gamma\,')] \\ \mathcal{S}_1 &= [(\alpha',\gamma),(\alpha',\gamma),(\beta',\alpha),(\beta',\alpha),(\gamma\,',\beta),(\gamma\,',\beta)] \\ \mathcal{S}_2 &= [(\beta,\alpha'),(\beta,\alpha'),(\alpha',\gamma),(\alpha',\gamma),(\gamma,\beta'),(\gamma,\beta'), \\ &\quad\ (\beta',\alpha),(\beta',\alpha),(\alpha,\gamma\,'),(\alpha,\gamma\,'),(\gamma\,',\beta),(\gamma\,',\beta)]. \end{aligned}$$

It is possible to pick a direction and unit of length for each of the three lines so all the directed segments have signed length $+1$. The invariant from Corollary 33 is:

$$E_{\mathbf{p}}(\mathcal{S}) = [1:1:1]_{(2,2,4)}.$$

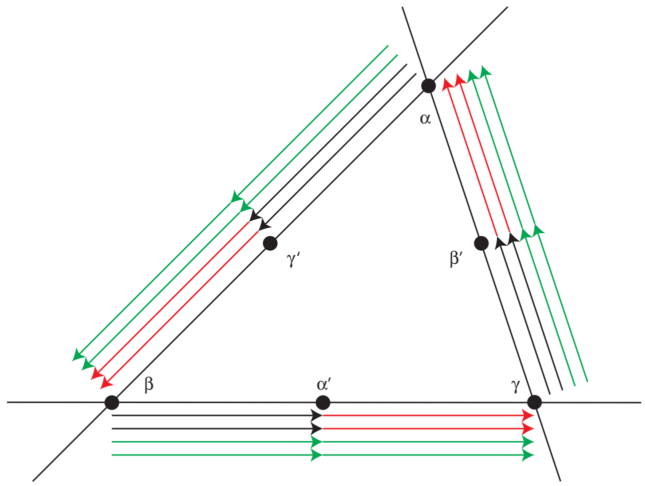

Figure 7: The configuration $\mathcal{S}$ from Example 14, of 6 points, 3 lines, and 24 directed segments in the real plane.

If we ignore the green segments and look only at the black and red segments,

$$\mathcal{S}^{(0,1)} = (\mathcal{S}_0, \mathcal{S}_1)$$

is an h-configuration with $E_{(1,1)}(\mathcal{S}^{(0,1)}) = [1 : 1]$. However, the other color pairs $(\mathcal{S}_0, \mathcal{S}_2)$ and $(\mathcal{S}_1, \mathcal{S}_2)$ are not h-configurations. The modification of $(\mathcal{S}_0, \mathcal{S}_2)$ into $\mathcal{S}^{(0,2)}$ is to duplicate all the black segments, so each line has four black segments and four green segments. Then

$$E_{(1,1)}(\mathcal{S}^{(0,2)}) = [1 : 1]$$

as in Corollary 36, and similarly $E_{(1,1)}(\mathcal{S}^{(1,2)}) = [1 : 1]$.

By Theorems 7 and 17, the product of axis projections,

$$\begin{aligned} \prod h_{ij} : \mathbb{R}P(2,2,4) &\to \mathbb{R}P^1 \times \mathbb{R}P^1 \times \mathbb{R}P^1 : \\ z &\mapsto (h_{01}(z), h_{02}(z), h_{12}(z)), \\ [z_0 : z_1 : z_2]_{(2,2,4)} &\mapsto ([z_0 : z_1], [z_0^2 : z_2], [z_1^2 : z_2]), \end{aligned}$$

is two-to-one on $D_{\mathbf{p}}$. In particular, $[1 : 1 : 1]_{(2,2,4)} \mapsto ([1 : 1], [1 : 1], [1 : 1])$, and the other point with that image is $[-1 : -1 : 1]_{(2,2,4)}$.

So, as in Example 13, it is possible to find projectively inequivalent weight $(2, 2, 4)$ h-configurations $\mathcal{S}$ and $\mathcal{T}$ with different $E_{(2,2,4)}$ invariants, but which

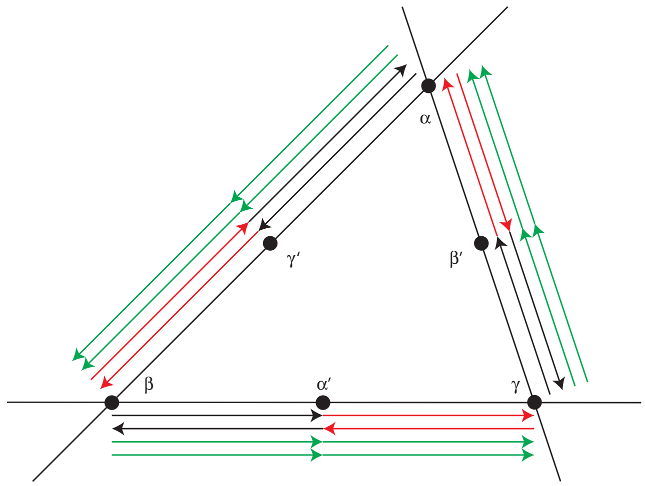

Figure 8: The configuration $\mathcal{T}$ from Example 14.

have the same $E_{(1,1)}$ invariants from applying Eves' Theorem to their three h-configurations $\mathcal{S}^{(i,j)}$ and $\mathcal{T}^{(i,j)}$. We can reverse some of the red and black directed segments from $\mathcal{S}$ to get a new configuration $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2)$,

$$\begin{aligned}
\mathcal{T}_0 &= [(\beta,\alpha'),(\alpha',\beta),(\gamma,\beta'),(\beta',\gamma),(\alpha,\gamma\,'),(\gamma\,',\alpha)] \\
\mathcal{T}_1 &= [(\alpha',\gamma),(\gamma,\alpha'),(\beta',\alpha),(\alpha,\beta'),(\gamma\,',\beta),(\beta,\gamma\,')] \\
\mathcal{T}_2 &= \mathcal{S}_2.
\end{aligned}$$

So, $E_{\mathbf{p}}(\mathcal{T}) = [-1 : -1 : 1]_{(2,2,4)}$, and all three (i,j) color pairs have $E_{(1,1)}(\mathcal{T}^{(i,j)}) = [1 : 1]$.

The next Example is a configuration considered by [B]; all six points are in the Euclidean plane, as in Examples 11, 12, but the configuration can be seen to have an octahedral pattern.

Example 15. Let $1, 2, 3, 4, 5, 6$, be six points in the Euclidean plane as in Example 11. Let $\mathcal{S} = (\mathcal{S}_0, \mathcal{S}_1)$ be a configuration of four black triangles and four red triangles:

$$\begin{aligned}
\mathcal{S}_0 &= [\triangle(465),\triangle(423),\triangle(512),\triangle(136)], \\
\mathcal{S}_1 &= [\triangle(123),\triangle(165),\triangle(245),\triangle(346)].
\end{aligned}$$

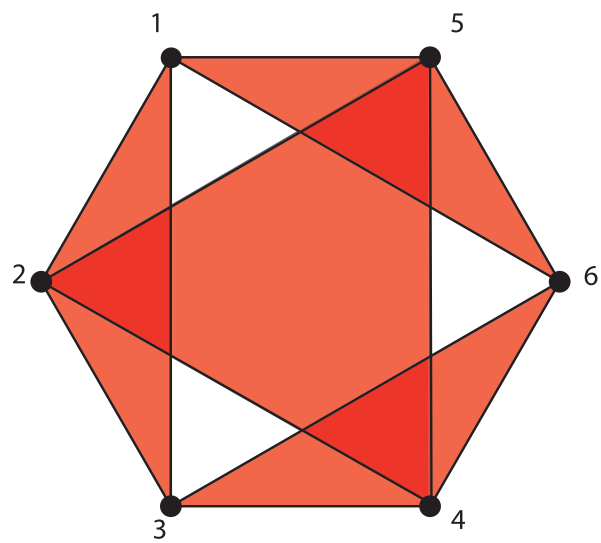

Figure 9: A configuration of 6 points in the real plane, showing the 4 red triangles from Example 15.

Each of the six points has black degree and red degree equal to 2, so, unlike Example 11, $\mathcal{S}$ can be viewed as either a $((2,2),3,2,2)$-configuration or a $((1,1),3,4,2)$-configuration. $\mathcal{S}$ is both a weight $(2,2)$ h-configuration and a weight $(1,1)$ h-configuration, equal to $\mathcal{S}^{(0,1)}$. In the plane coordinate system from Section 5, the $(2,2)$ invariant is:

$$\begin{aligned} & E_{(2,2)}(\mathcal{S}) \\ = \; & [[\![\triangle(465)]\!]_{\mathcal{B}}[\![\triangle(423)]\!]_{\mathcal{B}}[\![\triangle(512)]\!]_{\mathcal{B}}[\![\triangle(136)]\!]_{\mathcal{B}} : \\ & \quad [\![\triangle(123)]\!]_{\mathcal{B}}[\![\triangle(165)]\!]_{\mathcal{B}}[\![\triangle(245)]\!]_{\mathcal{B}}[\![\triangle(346)]\!]_{\mathcal{B}}]_{(2,2)} \\ = \; & [z_0 : z_1]_{(2,2)}. \end{aligned}$$

The $(1,1)$ invariant of the same configuration is:

$$E_{(1,1)}(\mathcal{S}^{(0,1)}) = [z_0 : z_1]_{(1,1)},$$

so $\frac{z_1}{z_0}$ can be interpreted as the ratio of signed areas:

$$\frac{(Area\triangle(123))(Area\triangle(165))(Area\triangle(245))(Area\triangle(346))}{(Area\triangle(465))(Area\triangle(423))(Area\triangle(512))(Area\triangle(136))}.$$

We remark that the property $E_{(1,1)}(\mathcal{S}^{(0,1)}) = [1:1]$ is equivalent to the projective property that six points lie on a conic ([CRG], [RG]).

Let $\mathcal{T} = (\mathcal{T}_0, \mathcal{T}_1)$ be a new configuration — the same six points, but changing the order of vertices in one of the black triangles and one of the red triangles to get the opposite signed areas:

$$\begin{aligned} \mathcal{T}_0 &= [\triangle(456), \triangle(423), \triangle(512), \triangle(136)], \\ \mathcal{T}_1 &= [\triangle(132), \triangle(165), \triangle(245), \triangle(346)]. \end{aligned}$$

Then $\mathcal{T}$ has the same $E_{(1,1)}$ invariant, and in the ratio of signed areas, the sign changes cancel, giving the same ratio as $\mathcal{S}$. The two configurations have different $(2,2)$ invariants, so they are projectively inequivalent:

$$E_{(2,2)}(\mathcal{T}) = [-z_0 : -z_1]_{(2,2)} \neq [z_0 : z_1]_{(2,2)} = E_{(2,2)}(\mathcal{S}).$$

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References

[AI] Adobe Illustrator CS5, Version 15.0.0.

[BBR] M. Barnabei, A. Brini, and G.-C. Rota, *On the exterior calculus of invariant theory*, J. Algebra (1) **96** (1985), 120–160. MR0808845 (87j:05002).

[B] J. Brill, *On certain analogues of anharmonic ratio*, Quarterly J. Pure and Appl. Math. **29** (1898), 286–302.

[BB] M. Brill and E. Barrett, *Closed-form extension of the anharmonic ratio to N-space*, Computer Vision, Graphics, and Image Processing (1) **23** (1983), 92–98.

[C] W. Clifford, *Analytical metrics*, Quarterly J. Pure and Appl. Math. (25) **7** (1865), (29) **8** (1866), (30) **8** (1866); reprinted in *Mathematical Papers by William Kingdon Clifford*, R. Tucker, ed., Chelsea, New York, 1968. MR 0238662 (39 #26).

[CRG] H. Crapo and J. Richter-Gebert, *Automatic proving of geometric theorems*, Invariant Methods in Discrete and Computational Geometry, 167–196, N. White, ed., Kluwer, 1995. MR 1368011 (97b:68196).

[Delorme] C. Delorme, *Espaces projectifs anisotropes*, Bull. Soc. Math. France (2) **103** (1975), 203–223. MR 0404277 (53 #8080a).

[Dolgachev] I. Dolgachev, *Weighted projective varieties*, Group Actions and Vector Fields (Vancouver, B.C., 1981), 34–71, LNM **956**, Springer, Berlin, 1982. MR 0704986 (85g:14060).

[E] H. Eves, *A Survey of Geometry*, Revised Ed., Allyn & Bacon, Boston, 1972. MR 0322653 (48 #1015).

[F_1] M. Frantz, "*The most underrated theorem in projective geometry*," presentation at the 2011 MAA MathFest, Lexington, KY, August 6, 2011.

[F_2] M. Frantz, *A car crash solved — with a Swiss Army knife*, Mathematics Magazine (5) **84** (2011), 327–338.

[O] Ø. Ore, *Number Theory and its History*, McGraw-Hill, New York, 1948. MR 0026059 (10,100b).

[P] J.-V. Poncelet, *Traité des Propriétés Projectives des Figures*, 1^{st} Ed., Bachelier, Paris, 1822.

[RG] J. Richter-Gebert, *Perspectives on Projective Geometry*, Springer, Berlin, 2011. MR 2791970 (2012e:51001).

[Salmon] G. Salmon, *Lessons Introductory to the Modern Higher Algebra*, 5^{th} Ed., Chelsea, New York, 1885.

[Shephard] G. Shephard, *Isomorphism invariants for projective configurations*, Canad. J. Math. (6) **51** (1999), 1277–1299. MR 1756883 (2001d:51014).