Unfolding CR Singularities

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Abstract

A notion of unfolding, or multi-parameter deformation, of CR singularities of real submanifolds in complex manifolds is proposed, along with a definition of equivalence of unfoldings under the action of a group of analytic transformations. In the case of real surfaces in complex 2-space, deformations of elliptic, hyperbolic, and parabolic points are analyzed by putting the parameter-dependent real analytic defining equations into normal forms up to some order. For some real analytic unfoldings in higher codimension, the method of rapid convergence is used to establish real algebraic normal forms.

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Unfolding CR singularities

1. Introduction

The local equivalence problem for real submanifolds of complex manifolds under local biholomorphic coordinate transformations can be approached by finding normal forms for the local defining equations. We follow that approach here in the case of real surfaces in $\mathbb{C}^2$ with complex tangents, and, more generally, real m-submanifolds of complex n-manifolds with $m \le n$. Then, we introduce a way to further understand the local extrinsic geometry by seeing how it changes under small perturbations of the embedding.

The general idea is to parallel the development of the singularity theory of differentiable maps, where the geometry of singularities is understood first by a classification of the local defining equations by finding normal forms, and secondly by an analysis of how the different types of singularities fit into parametrized families of maps (called "unfoldings"). The classification of unfoldings is, again, to find normal forms for the defining expressions, at least for the lowest degree terms, under an appropriate group of transformations.

Some analogies between the geometry of singular maps and the geometry of CR singularities of real submanifolds had already been noted at least since [**MW**] and [**Webster**$_1$], where a real analytic embedding of M in $\mathbb{C}^n$ is related to a certain singular holomorphic map $\mathbb{C}^m \to \mathbb{C}^n$ via a complexification construction, which we recall in Section 4.

Our starting point in the analysis of the normal form problem for the defining equations of a real surface M in $\mathbb{C}^2$ near a CR singularity will be the quadratic normal forms of [**Bishop**]. The well-known elliptic/parabolic/hyperbolic classification and some subsequent refinements are recalled in Section 5.1, and we contribute new quartic normal forms for some degenerate parabolic cases (Proposition 5.1).

Then, having some explicit equations for manifolds representing some of the simplest local normal forms, one of the basic issues in the deformation theory is the question of the topological stability of a CR singularity, that is, whether a submanifold with a CR singularity will still have a CR singularity after a small perturbation. If the singularity persists, then there is the geometric problem of what sort of properties of a given singular point will change or be preserved under perturbation.

One approach, also going back to [**Bishop**], uses a grassmannian variety construction to define a notion of general position, and to give an expected codimension formula for the locus of CR singularities. Section 2 will review some of the differential topology of submanifolds with CR singularities.

The bulk of this paper is devoted to consideration of the local problem, by the analysis of the local defining equations for real analytic submanifolds embedded in $\mathbb{C}^n$. To study the geometry of the deformation of a real m-submanifold M through k real parameters, we consider a real $(m+k)$-submanifold $\widehat{M}$ of $\mathbb{C}^{n+k}$ containing M as a submanifold. Briefly, if M is defined by a system of multivariable power series equations $\{\vec{e}(\vec{z}) = 0\}$, then $\widehat{M}$ is defined by introducing new terms depending on new parameters, $\{\vec{e}(\vec{z}, t) = 0\}$. The setup of the defining equations is straightforward — the interesting part is a group action; both are described in detail in Section 3. Our classification problem is then to find normal forms for $\widehat{M}$ under a certain group of holomorphic transformations, which respects the difference between the original (space direction) coordinates $\vec{z}$ and the new (time parameter) variables t.

Since the normal forms for n-manifolds in $\mathbb{C}^n$ are qualitatively different from the normal forms for m-manifolds in $\mathbb{C}^n$ with $m < n$, the two cases are considered separately in Sections 5 and 6, with the simplest representatives being surfaces in $\mathbb{C}^2$, and 4-manifolds in $\mathbb{C}^5$ (first considered by [$\mathbf{D}_1$], [**Beloshapka**], [$\mathbf{C}_1$]).

In Section 6, we consider a non-trivial unfolding of a real analytic manifold with a certain type of isolated, degenerate CR singularity in the case $\frac{2}{3}(n+1) = m < n$, and state Main Theorem 6.5, which claims the existence of a local coordinate transformation so that the unfolding $\widehat{M}$ (and therefore also the original manifold M) is real algebraic. The Proof, in Section 7, uses the technique of rapid convergence.

As a preview of Main Theorem 6.5, we state the following special case. Let M be a real 4-manifold in $\mathbb{C}^5$ given by the real analytic equations

$$\{y_2 = E_2, y_3 = E_3, z_4 = \bar{z}_1^2 + e_4, z_5 = (z_1 + x_2 + i x_3^2)\bar{z}_1 + e_5\},$$

where E_2, E_3, e_4, e_5 are higher order parts: series with terms of degree at least 4 in $z_1, \bar{z}_1, x_2, x_3$. This M has an isolated complex tangent at the origin, and it will be shown in Section 6 why this is a natural set of equations to consider, and that this type of singular point is unstable under perturbation in the following interesting way. We consider a real parameter t_1, and a submanifold $\widehat{M}$ of $\mathbb{C}^5 \times \mathbb{R}$ given by

$$\begin{aligned} y_2 &= E_2(z_1, \bar{z}_1, x_2, x_3, t_1) \\ y_3 &= E_3(z_1, \bar{z}_1, x_2, x_3, t_1) \\ z_4 &= \bar{z}_1^2 + e_4(z_1, \bar{z}_1, x_2, x_3, t_1) \\ z_5 &= (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 + e_5(z_1, \bar{z}_1, x_2, x_3, t_1), \end{aligned}$$

where a new quadratic term, $it_1\bar{z}_1$, appears, and the new higher order parts are the same as the old for $t_1 = 0$, but new terms depending on t_1 have been added, in an arbitrary (but still real analytic) way. $\widehat{M}$ will be called an unfolding of M; it has the property that $\widehat{M} \cap \{t_1 = 0\} = M \times \{0\}$, and for t_0 fixed but close to 0, $\widehat{M} \cap \{t_1 = t_0\}$ is a real 4-manifold inside $\mathbb{C}^5 \times \{t_0\}$, which we think of as being a small deformation of M. The result of the Main Theorem is that there is a coordinate change with identity linear part, defined near the origin $(\vec{0}, 0)$:

$$(\tilde{z}_1, \ldots, \tilde{z}_5) = (z_1, \ldots, z_5) + \vec{p}(z_1, \ldots, z_5, t_1),$$
$$\tilde{t}_1 = t_1 + P_1(t_1),$$

which is a local biholomorphic map from one $\{t_1 = \text{constant}\}$ slice to another (near the origin): $\mathbb{C}^5 \times \{t_1\} \to \mathbb{C}^5 \times \{\tilde{t}_1\}$, and simultaneously a real analytic re-parametrization of t_1, so that the defining equations of the real manifold $\widehat{M}$ in the new coordinate system are the real polynomials:

$$\{\tilde{y}_2 = 0,\ \tilde{y}_3 = 0\ \tilde{z}_4 = \bar{\tilde{z}}_1^2,\ \tilde{z}_5 = (\tilde{z}_1 + \tilde{x}_2 + i\tilde{t}_1 + i\tilde{x}_3^2)\bar{\tilde{z}}_1\}. \tag{1}$$

In this coordinate system, it will be easy to see (in Section 6.3) that slices of $\widehat{M}$ where $\tilde{t}_1$ is a small but positive constant are totally real, while slices of $\widehat{M}$ where $\tilde{t}_1 < 0$ have two nondegenerate CR singularities. Thinking of t_1 as a time parameter, this deformation represents a pair of CR singular points moving toward the origin, meeting in a degenerate CR singular point at time 0, and then cancelling so that there are no CR singular points after time 0.

The technique for constructing such an analytic transformation, that is, finding $\vec{p}$ and P_1 in terms of the given (E_2, E_3, e_4, e_5), is to solve a system of nonlinear functional equations, using a method of linear approximation and rapid convergence, as employed in similar problems by [**Moser**], [$\mathbf{C}_4$], [$\mathbf{C}_6$]. The exact solution of the equation is the composite of a sequence of approximate solutions, and a norm of each approximate solution in the sequence needs to be bounded in a certain way for the convergence argument to work. The approximate solutions are constructed by solving a linearized system of equations by a comparison of coefficients calculation, Theorem 7.6. This is the longest step in the proof of the Main Theorem 6.5, and the intricacy of the calculation is due in part to the large numbers of equations and variables, with the t (parameter) variables treated differently from the z, x variables. The x_3 variable is also distinguished because of the inhomogeneity of the cubic normal form. The more subtle and more serious difficulty regards the size of the domain of the approximation. The norm of one solution in the sequence of approximations might be estimated only on a strictly smaller domain than that of the previous approximation, so this shrinking of the domain must be controlled precisely. However, the linearized system has a large solution space of formal series, which includes divergent series solutions as well as solutions that converge on arbitrarily small sets; such approximate solutions are not suitable for an iterative process with the goal of converging toward an exact solution which is analytic on an open set. The calculation of Theorem 7.6 makes some choices to avoid the bad solutions and find a good solution, defined by a series expansion which converges on a suitably large set, and with a useful bound on its norm following from estimates for subseries and applications of Cauchy's Estimate and the Schwarz Lemma at several points of the construction.

2. Topological considerations

Mostly we will be looking at the local geometry of real submanifolds, however we start with the big picture by recalling some topological notions, without any claim of novelty with this paper, but with the intention to motivate and give a global context for some of the local constructions in the remaining Sections. These global notions will not be required in any later Proofs, but they do give a nice explanation for some of the choices of dimensions m, n, which might otherwise appear as merely technical restrictions (such as the hypothesis of Proposition 6.2) or computational conveniences (for example, Equation (105) in the middle of the lengthy calculation proving Theorem 7.6). The explicit normal forms for unfoldings give an opportunity to see, in a concrete way, how a global count of singularities can be conserved even under local deformations that create or destroy singular points, as in the example (1) from the Introduction.

2.1. A grassmannian construction.

Let M be an oriented real m-submanifold smoothly embedded in $\mathbb{C}^n$, with $m \leq n$. An embedding in general position will be *totally real* at most points $\mathbf{x}$: the tangent space $T_{\mathbf{x}}M$ will contain no complex lines. At other points $\mathbf{x}$, the tangent space may contain at least one complex line: such a point is a *CR singular* point of M (a point where M has a "complex tangent" is also known as a "complex point," "complex jump point," or "exceptional point"). The CR singular locus is stratified by the (complex) dimension j of the largest complex space tangent to M: denote

$$N_j = \{\mathbf{x} \in M : \dim_{\mathbb{C}}(T_{\mathbf{x}}M \cap iT_{\mathbf{x}}M) \geq j\}. \tag{2}$$

Recall (see [**Lai**], [**F**], [**G**]) that if G is the grassmannian variety of oriented real m-subspaces in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, then the real m-subspaces T such that $\dim_{\mathbb{C}} T \cap iT \geq j$ form a subvariety $\mathcal{D}_j$ of real codimension $2j(n-m+j)$ in G. Define the Gauss map $M \to G : \mathbf{x} \mapsto T_{\mathbf{x}}M$, so the set N_j is the inverse image of $\mathcal{D}_j$, and for M in general position, meaning that the Gauss map is transverse to each $\mathcal{D}_j \setminus \mathcal{D}_{j+1}$, the set N_j will have codimension $2j(n-m+j)$ in M. When M is in general position and its dimension is equal to the real codimension of $\mathcal{D}_j$ in G, the Gauss map will meet $\mathcal{D}_j \setminus \mathcal{D}_{j+1}$ in isolated points where we can assign an oriented intersection number, giving an index $+1$ or -1 for each isolated point in N_j.

Currently, the best understood CR singularities are those in $N_1 \setminus N_2$, the simplest and most generic type, where the tangent space contains a complex line but no complex plane. This paper will also consider only such points, leaving the study of points in N_2 as one of the open areas listed in Section 8. However, we will not restrict our attention to M in general position as defined above; some points $\mathbf{x} \in N_1$ where the Gauss map meets $\mathcal{D}_1 \setminus \mathcal{D}_2$ non-transversely will be of interest. The pairs (m, n) where the dimension m of M is equal to the real codimension $2(n-m+1)$ of $\mathcal{D}_1$ in G satisfy $m = \frac{2}{3}(n+1)$. The case $(2, 2)$, where a real surface in general position in $\mathbb{C}^2$ will either be totally real everywhere or will have isolated complex tangents, is considered in Section 5. When $m < n$, the smallest pair of dimensions where CR singularities are expected to occur is $m = 4$ and $n = 5$, where the codimension of $\mathcal{D}_1$ in G is 4, so a real 4-manifold M in $\mathbb{C}^5$ in general position is totally real except at isolated points in $N_1 \setminus N_2$. The next cases of $m = \frac{2}{3}(n+1)$ are $(6, 8)$, $(8, 11)$, ..., although their local geometry is expected to behave in about the same way as the $(4, 5)$ case since the local normal forms for nondegenerate singularities for all the pairs (m, n) with $\frac{2}{3}(n+1) \leq m < n$ fall into a common algebraic pattern, as shown in [$\mathbf{C}_6$].

When $m < \frac{2}{3}(n+1)$ (the case of very high codimension, such as a real surface in $\mathbb{C}^3$), CR singularities could still occur in a submanifold M, but are unstable, in the sense that one would expect most small perturbations of M to be totally real. So, we refer to the range $\frac{2}{3}(n+

$1) \le m \le n$ as the stable dimension range for N_1 CR singularities, and the later Sections will consider only these cases.

2.2. Global index sums.

An especially interesting one-parameter deformation phenomenon to be observed in Sections 5 and 6 is a pair creation/annihilation process, where the number of isolated, nondegenerate CR singular points increases or decreases by 2. At some critical intermediate point, one expects some sort of topological degeneracy where the Gauss map meets $\mathcal{D}_1 \setminus \mathcal{D}_2$ non-transversely. This cancellation property of certain types of pairs of CR singular points has been considered by [**F**] in the case of surfaces in $\mathbb{C}^2$, and by [$\mathbf{D}_1$] for 4-manifolds in $\mathbb{C}^5$, in both cases as examples of the h-principle method of Gromov: if M has a pair of CR singularities with opposite indices (and the same orientation, in the $m = n = 2$ case), then there is a homotopy of embeddings of M that deforms it into another submanifold with two fewer CR singular points but the same index sum. An analogue in the singularity theory of differentiable mappings would be the pairwise cancellation of cross-cap singularities of smooth maps from real surfaces to $\mathbb{R}^3$ as in [$\mathbf{Whitney}_2$]. In this paper, in Example 5.18 and Subsections 5.3, 6.2, 6.3, these cancellation processes are considered from a local point of view. At the moment of contact between colliding nondegenerate CR singularities of opposite index, there is still a CR singularity, but with a degenerate normal form.

For the ± 1 intersection indices of the Gauss map as previously defined, the index sum of all the isolated, nondegenerate CR singularities of a compact oriented real submanifold embedded in $\mathbb{C}^n$ is a topological invariant. More generally, the index sum can also be defined when M is smoothly immersed in an almost complex manifold, and is an invariant of the homotopy class of the immersion. Even more generally, let M be a d_M-dimensional connected, oriented, smooth manifold, let F be a smooth real vector bundle over M of real rank $2n$, with a complex structure operator J, so (F, J) is a complex vector bundle of complex rank n, and let T be an oriented real subbundle of F of real rank m. The CR singularities are points $\mathbf{x}$ on the base M where $T_{\mathbf{x}} \cap J_{\mathbf{x}} T_{\mathbf{x}}$ is a nonzero subspace (such subspaces are J-invariant, so they are complex subpaces of $F_{\mathbf{x}}$ with complex dimension $0 \le j \le n$). It makes sense (see [**DJ**]) to redefine G to be the bundle over M where the fiber over $\mathbf{x}$ is the grassmannian of real oriented subspaces of $F_{\mathbf{x}}$, to define a Gauss map $M \to G : \mathbf{x} \to T_{\mathbf{x}}$, and to say that T is in general position if the Gauss map transversely meets the smooth, oriented submanifolds $\mathcal{D}_j \setminus \mathcal{D}_{j+1}$ of G ($\mathcal{D}_j$ defined in each fiber as above). In the case

$d_M = m = \frac{2}{3}(n+1)$, M meets $\mathcal{D}_1 \setminus \mathcal{D}_2$ transversely in isolated points $\mathbf{x} \in N_1$ with intersection indices $\mathrm{ind}(\mathbf{x}) = \pm 1$. When M is compact, N_1 is a finite set and

$$\sum_{\mathbf{x} \in N_1} \mathrm{ind}(\mathbf{x}) = \int_M \mathbf{\Delta}_m(p(T), c(F,J)), \tag{3}$$

where $\mathbf{\Delta}_m$ is a polynomial in the pontrjagin classes of T, $p(T) = 1 + p_1 + \ldots + p_\ell$, $\ell \leq m/2$, and the chern classes of (F,J), $c(F,J) = 1+c_1+\ldots+c_n$. The polynomial expression is a Giambelli-type formula, specifically, $\mathbf{\Delta}_m$ is the degree m part of the following formal quotient of total chern classes:

$$\begin{aligned}
& c(F,J) \cdot (c(T \otimes \mathbb{C}))^{-1} \\
(4) \qquad = \; & (\sum_{a=0}^{n} c_a) \cdot (\sum_{b=0}^{\ell} (-1)^b p_b)^{-1} \\
(5) \qquad = \; & (\sum_{a=0}^{n} c_a) \cdot (\sum_{i=0}^{\infty} (\sum_{b=1}^{\ell} (-1)^{b+1} p_b)^i) \\
(6) \qquad = \; & 1 + c_1 + (c_2 + p_1) + (c_3 + c_1 p_1) + \\
& (c_4 + c_2 p_1 - p_2 + p_1^2) + \\
& (c_5 + c_3 p_1 - c_1 p_2 + c_1 p_1^2) + \\
& (c_6 + c_4 p_1 - c_2 p_2 + c_2 p_1^2 + p_3 - 2 p_1 p_2 + p_1^3) + \ldots
\end{aligned}$$

Step (4) is the definition of pontrjagin classes, and (5) is a formal series expansion. The terms of (6) are grouped by total (even) degree.

In the case where the subbundle T is the tangent bundle of M, the pontrjagin classes are topological invariants of M and do not depend on its orientation, however, the integration over M to get an integer does depend on the orientation. Reversing the orientation of M will switch the indices (± 1) on the LHS of (3), and reverse the sign of the characteristic number on the RHS. In the case where M is immersed in an almost complex manifold A, (F,J) is the restriction (or pullback by the immersion) of the ambient complex bundle (TA, J_A) to M, and the chern class $c(F,J)$ is an invariant of the homotopy class of the immersion. When $A = \mathbb{C}^n$, the chern class is trivial, $c(T\mathbb{C}^n) = 1 + 0 + \ldots + 0$.

Example 2.1. The case where $d_M = m = n = 2$ gives a formula of [**Webster**$_3$], where T is a real, oriented 2-subbundle of a complex 2-bundle (F,J) over a smooth, compact, oriented surface M. If T is in general position, the set N_1 of points $\mathbf{x}$ where the fiber $T_\mathbf{x}$ is complex (or "anticomplex," where the given orientation of T disagrees with its orientation as a complex subspace of (F, J)) is finite. The enumerative formula (3) then reads:

$$\sum_{\mathbf{x}\in N_1} \operatorname{ind}(\mathbf{x}) = \int_M c_1(F, J). \tag{7}$$

When T is the tangent bundle of M immersed in $\mathbb{C}^2$, F is trivial and the index sum is zero.

EXAMPLE 2.2. In a further special case of Example 2.1, suppose M is an immersed compact surface in general position in $\mathbb{R}^3$. Considering $\mathbb{R}^3$, with real coordinates x_1, y_1, x_2, as a real hyperplane in $\mathbb{C}^2$ with coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, the only complex lines contained in $\mathbb{R}^3$ are the horizontal planes parallel or equal to the z_1-axis. So, the complex tangents are the familiar critical points of the height function: elliptic points (local maxima and minima), and hyperbolic (saddle) points are the only types of critical points for M in general position. This coordinate system will be used in Section 5.3 to visualize CR singular surfaces in $\mathbb{C}^2$. The topological oriented intersection number counts elliptic complex points and hyperbolic anticomplex points as index $+1$, and hyperbolic complex points and elliptic anticomplex points as index -1.

We will recall the more general definition of elliptic and hyperbolic points for surfaces in $\mathbb{C}^2$ in Section 5. The above sign convention differs from that of [**F**], which assigns all elliptic points index $+1$ and all hyperbolic points index -1 (so the only difference is for the anticomplex points). In the notation of [**F**], the RHS of (7) is equal to $I_+ - I_-$. Formula (7) is not the only topological invariant of surfaces in $\mathbb{C}^2$ — a formula which involves the Euler characteristic of M and also applies to non-orientable surfaces is discussed by [**BF**], [**F**], [**IS**] App. IV, [**Slapar**]. For M embedded in $\mathbb{R}^3$ as in Example 2.2, the index sum using the [**F**] sign convention is $I_+ + I_- = \chi(M)$, a familiar result from Morse Theory.

EXAMPLE 2.3. The case $d_M = m = 4$, $n = 5$ gives a formula of [$\mathbf{D}_1$]:

$$\sum_{\mathbf{x}\in N_1} \operatorname{ind}(\mathbf{x}) = \int_M c_2(F, J) + p_1 T.$$

So, for a compact, oriented 4-manifold immersed in $\mathbb{C}^5$, the index sum of its CR singularities is equal to its pontrjagin number.

For example, it is easy to embed the 4-sphere S^4, with $p_1 S^4 = 0$, as a totally real submanifold of $\mathbb{C}^5$, by first embedding it in a totally real $\mathbb{R}^5$. In contrast, $\mathbb{C}P^2$ (considered only as a smooth, oriented 4-manifold) has no totally real immersion in $\mathbb{C}^5$, since $p_1\mathbb{C}P^2 = 3$. An embedding of $\mathbb{C}P^2$ with exactly three CR singularities and some immersions with exactly five CR singularities were constructed in [$\mathbf{C}_2$].

The topological cancellation of complex points of surfaces has recently found various geometric applications, for example, [**DJ**], [**IS**], [**Slapar**]. The relationship between pontrjagin classes and CR singularities has been considered since [**Wells**] and [**Lai**], and further formulas appear in [$\mathbf{Webster}_2$] and [$\mathbf{D}_2$]. For more on the interpretation of some of these characteristic numbers in terms of Giambelli-Thom-Porteous formulas for degeneracy loci of bundle maps, see [**HL**] and [$\mathbf{C}_1$].

3. Local defining equations and transformations

Here we set up the general framework and notation for the local defining equations of real submanifolds, and the action of the transformation groups.

3.1. Defining equations for m-submanifolds in $\mathbb{C}^n$.

For the rest of the paper, $m \le n$ and M is assumed to be a real analytic embedded m-submanifold of $\mathbb{C}^n$, which has coordinates $z_1 = x_1 + iy_1, \dots, z_n = x_n + iy_n$. It follows that for any point $\mathbf{x} \in N_1 \setminus N_2$ (as in (2), where there is exactly one complex line tangent to M at $\mathbf{x}$), there is some translation taking $\mathbf{x}$ to $\vec{0}$, and some complex linear transformation taking the tangent space $T_{\vec{0}}M$ to the real subspace with coordinates $x_1, y_1, x_2, \dots, x_{m-1}$, which contains the z_1-axis. Further, in some neighborhood of $\vec{0}$, M can be written as a graph over its tangent space:

$$
\begin{aligned}
y_2 &= H_2(z_1, \bar{z}_1, x), \dots, \\
y_{m-1} &= H_{m-1}(z_1, \bar{z}_1, x), \\
z_m &= h_m(z_1, \bar{z}_1, x), \dots, \\
z_n &= h_n(z_1, \bar{z}_1, x).
\end{aligned}
\tag{8}
$$

where the functions $H_2, \dots, H_{m-1}, h_m, \dots, h_n$ are real analytic (defined by convergent power series with complex coefficients, centered at the origin) functions of x_1, y_1, x_2, $x_3, \dots, x_{m-1}$, or equivalently, z_1, $\bar{z}_1 = x_1 - iy_1$, $x = (x_2, \dots, x_{m-1})$. Until Section 7, the size of the domain of convergence will not be of concern, only that the series are assumed to converge on some open neighborhood of the origin. The functions $H_2, \dots, H_{m-1}$ are real valued (as functions of $(z_1, \bar{z}_1, x)$), and the functions $h_m, \dots, h_n$ are complex valued. Since M is tangent to the z_1, x space at $\vec{0}$, the series for $H_2, \dots, h_n$ have no constant or linear terms, so each function could be labeled $O(2)$ according to the following Definition.

Definition 3.1. A (formal, with complex coefficient C) monomial of the form $Cz_1^a\zeta_1^b x^{\mathbf{I}}$ has *degree* $a+b+\mathbf{I}$, where $\mathbf{I}=(i_2,\dots,i_{m-1})$ is a multi-index, and $a+b+\mathbf{I}=a+b+i_2+\cdots+i_{m-1}$. A power series in m variables $e(z_1,\zeta_1,x)=\sum e^{ab\mathbf{I}}z_1^a\zeta_1^b x^{\mathbf{I}}$, is said to have *degree* d if $e^{ab\mathbf{I}}=0$ for all $(a,b,\mathbf{I})$ such that $a+b+\mathbf{I}<d$. Sometimes a series of degree d will be abbreviated $O(d)$.

Definition 3.2. Similarly for n variables, a monomial $Cz_1^{a_1}\cdots z_n^{a_n}$ has degree $a_1+\cdots+a_n$, but we will also work with the "weight," $a_1+\cdots+a_{m-1}+2a_m+\cdots+2a_n$. A series $p(\vec{z})=\sum p^{a_1\dots a_n}z_1^{a_1}\dots z_n^{a_n}$ has "weight" W if $p^{a_1\cdots a_n}=0$ when $a_1+\cdots+a_{m-1}+2a_m+\cdots+2a_n<W$.

Once $T_{\mathbf{x}}M$ is in standard position with local defining equations (8), the normal form problem is to find representatives of equivalence classes under biholomorphic changes of coordinates. Holomorphic transformations fixing the origin and the tangency of M to the $(z_1,x_2,\dots,x_{m-1})$ subspace take vectors $\vec{z}=(z_1,\dots,z_n)^T$ to

$$\tilde{z}=(\tilde{z}_1,\dots,\tilde{z}_n)^T=\mathbf{A}_{n\times n}\vec{z}+\vec{p}(\vec{z}), \tag{9}$$

where $\vec{p}(\vec{z})$ is a column vector of n functions of n variables $p_1(\vec{z})$, ..., $p_n(\vec{z})$, each of which is holomorphic in a neighborhood of $\vec{0}\in\mathbb{C}^n$ and has no constant or linear terms (so the weight is 2), and where $\mathbf{A}$, the invertible linear part of the transformation, has matrix representation of the form

$$\mathbf{A}_{n\times n}=\begin{pmatrix} a_1 & a_2\dots a_{m-1} & a_m\dots a_n \\ 0 & \mathbf{R}_{(m-2)\times(m-2)} & \mathbf{C}_{(m-2)\times(n-m+1)} \\ 0 & 0 & \mathbf{C}_{(n-m+1)\times(n-m+1)} \end{pmatrix}.$$

The block $\mathbf{R}$ has all real entries (and is invertible because $\mathbf{A}$ is), and the entries $a_1,\dots,a_n$, and in the blocks $\mathbf{C}$ are complex. The defining equations in the new $\tilde{z}$ coordinate system will still be of the form (8) but the goal is to find normal forms that expose the geometry of the equivalence classes.

3.2. Parametrized families of submanifolds.

The approach to constructing a deformation of a CR singular real submanifold of $\mathbb{C}^n$ will be to consider it as a slice of a higher-dimensional real submanifold of a higher-dimensional complex space. The deformation is parametrized by k real parameters, labeled $t=(t_1,\dots,t_k)$. For the sake of convenience in describing the ambient space, the parameters are the real parts of k new complex coordinates $w_1=t_1+is_1,\dots,w_k=$

$t_k + is_k$, although all the action will take place in the real subspace where $s_1 = \ldots = s_k = 0$. Consider $\mathbb{C}^{n+k}$ with (column vector) coordinates $(\vec{z}, \vec{w}) = (z_1, \ldots, z_n, w_1, \ldots, w_k)^T$, and a real submanifold $\widehat{M}$ of $\mathbb{C}^{n+k}$ with defining equations in a polydisc centered at $(\vec{0}, \mathbf{0})$ given in the form of a graph over the real $(m+k)$-space with coordinates $z_1, x_2, \ldots, x_{m-1}, t_1, \ldots, t_k$:

$$\begin{aligned} y_2 &= H_2(z_1, \bar{z}_1, x, t), \ldots, \\ y_{m-1} &= H_{m-1}(z_1, \bar{z}_1, x, t), \\ z_m &= h_m(z_1, \bar{z}_1, x, t), \ldots, \\ z_n &= h_n(z_1, \bar{z}_1, x, t), \\ s_1 = \ldots = s_k &= 0. \end{aligned} \tag{10}$$

For a fixed vector t_0 (sufficiently near $\mathbf{0} \in \mathbb{C}^k$), let $M_{t_0} = \widehat{M} \cap \{\vec{w} = t_0\}$ be a *slice* of $\widehat{M}$. Then M_{t_0} is a real m-submanifold of the n-dimensional complex space $\{\vec{w} = t_0\}$, and clearly any $M \subseteq \mathbb{C}^n$ as in (8) is of the form $M_{\mathbf{0}}$ for some $\widehat{M}$ as in (10). If $k = 1$, we can think of $t = (t_1)$ as a real time parameter, so that we are considering $\widehat{M}$ as a time evolution of $M = M_{\mathbf{0}}$, and in general, we think of each $M_{t_0} \subseteq \mathbb{C}^n \times \{t_0\}$ as a submanifold "close to" M when t_0 is close to $\mathbf{0}$. The slice M_{t_0} need not be CR singular, as examples will show.

We will consider only $\widehat{M}$ which is real analytic and which contains the origin, so that the real analytic functions $H_2, \ldots, h_n$ have no constant terms, and then by a complex linear transformation of $\mathbb{C}^{n+k}$ that fixes the $\vec{w}$ coordinates, we may assume that $M_{\mathbf{0}}$ is actually of the form (8), so that $H_2(z_1, \bar{z}_1, x, \mathbf{0})$, $\ldots$, $h_n(z_1, \bar{z}_1, x, \mathbf{0})$ are $O(2)$ in the $z_1, \bar{z}_1, x$ variables. This means the defining equations are of the form $H_2 = r_2^\alpha t_\alpha + O(2), \ldots, h_n = c_n^\alpha t_\alpha + O(2)$, where $r_2^\alpha, \ldots, r_{m-1}^\alpha$ are real coefficients summed over $\alpha = 1, \ldots, k$, $c_m^\alpha, \ldots, c_n^\alpha$ are complex coefficients, and the $O(2)$ notation now extends Definition 3.1 to include t in the count: the degree of $z_1^a \zeta_1^b x^{\mathbf{I}} t^{\mathbf{K}}$ is $a+b+i_2+\cdots+i_{m-1}+\mathbf{k}_1+\cdots+\mathbf{k}_k$.

A complex linear transformation of the form $\tilde{z}_1 = z_1$, $\tilde{z}_\sigma = z_\sigma - i r_\sigma^\alpha w_\alpha$ for $\sigma = 2, \ldots, m-1$, $\tilde{z}_u = z_u - c_u^\alpha w_\alpha$ for $u = m, \ldots, n$, $\tilde{w}_\alpha = w_\alpha$ for $\alpha = 1, \ldots, k$, will transform the defining equations of $\widehat{M}$ to eliminate the linear terms in t, so that H_2, $\ldots$, h_n in (10) are all $O(2)$. Such a transformation fixes $\mathbb{C}^n \times \{\mathbf{0}\}$ pointwise and acts as a translation on each parallel n-space $\mathbb{C}^n \times \{t_0\}$. Since $M = M_{\mathbf{0}}$ is not changed and each M_{t_0} is merely translated without changing its geometry, we will from this point work only with $\widehat{M}$ in <u>standard position</u>, where the linear coefficients r_σ^α, c_u^α have been normalized to 0, and the tangent space $T_{(\vec{0},\mathbf{0})}\widehat{M}$ is the subspace with coordinates z_1, $x_2, \ldots, x_{m-1}$, $t_1, \ldots, t_k$.

Evidently, $\widehat{M}$ itself is a CR singular $(m+k)$-submanifold of $\mathbb{C}^{n+k}$, and its tangent space at $(\vec{0},\mathbf{0})$ contains a complex line, but no complex plane.

3.3. The transformation group for unfoldings.

Given an unfolding $\widehat{M}$, one could try to put its defining equations (10) into a normal form by a biholomorphic transformation of a neighborhood of the origin in $\mathbb{C}^{n+k}$ in analogy with (9), but instead it is proposed to work with a subgroup of the full transformation group, that, roughly, preserves the distinction between the variables parametrizing $M = M_{\mathbf{0}}$ and those parametrizing the deformation.

NOTATION 3.3. Let $\mathcal{B}_{m,n+k}$ denote the group of germs near $(\vec{0},\mathbf{0})$ of transformations of $\mathbb{C}^{n+k}$ that fix the origin, are biholomorphic in $(\vec{z},\vec{w})$ near $(\vec{0},\mathbf{0})$, and whose linear part preserves the tangent space $T_{\vec{0}}\widehat{M}$ as in (9). Let $\mathcal{U}_{m,n,k}$ denote the subgroup obtained by further imposing the requirement that the transformation (11) preserve (near $(\vec{0},\mathbf{0})$) the real subspace $\{s_1 = \cdots = s_k = 0\}$ as a set, so that vectors of the form $(z_1,\ldots,z_n,t_1,\ldots,t_k)^T$, with the last k entries real, are taken to vectors of the same form.

Sometimes we will abuse notation by referring to transformations instead of germs, and spaces such as $\mathbb{C}^n$ instead of neighborhoods of the origin, although in Section 7 more care will be taken with the precise size of the domain of convergence of holomorphic maps.

The requirement that the subgroup $\mathcal{U}_{m,n,k}$ preserves the real subspace as a set means that its elements are transformations of the form:

$$(\tilde{z},\tilde{w}) = \mathbf{A}(\vec{z},\vec{w}) + \vec{p}(\vec{z},\vec{w}), \tag{11}$$

where the invertible linear part acts on column vectors

$$(\vec{z},\vec{w}) = (z_1,\ldots,z_n,w_1,\ldots,w_k)^T$$

by the $(n+k)\times(n+k)$ matrix representation

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2\ldots a_{m-1} & a_m\ldots a_n & a^1\ldots a^k \\ 0 & \mathbf{R}_{(m-2)\times(m-2)} & \mathbf{C}_{(m-2)\times(n-m+1)} & \mathbf{R}_{(m-2)\times k} \\ 0 & 0 & \mathbf{C}_{(n-m+1)\times(n-m+1)} & 0 \\ \mathbf{0} & 0 & 0 & \mathbf{R}_{k\times k} \end{pmatrix}. \tag{12}$$

and where the higher degree part is of the form

$$(p_1(\vec{z},\vec{w}),\ldots,p_n(\vec{z},\vec{w}),P_1(\vec{w}),\ldots,P_k(\vec{w}))^T,$$

with the quantities $P_1, \dots, P_k$ real valued functions on the subspace $\{w_1 = t_1, \dots, w_k = t_k\}$. Without the requirement that the t entries remain real, the last row of the above block matrix $\mathbf{A}$ would be $\mathbf{0}, \mathbf{R}_{k\times(m-2)}, \mathbf{C}_{k\times(n-m+1)}, \mathbf{R}_{k\times k}$.

As a consequence of the defining property of $\mathcal{U}_{m,n,k}$, the subgroup has the following property: for a fixed $t_0 \in \mathbb{R}^k$ (near $\mathbf{0}$) and transformation $(\vec{z}, \vec{w}) \mapsto (\tilde{z}, \tilde{w})$ in $\mathcal{U}_{m,n,k}$, all the points in the complex affine subspace $\{\vec{w} = t_0\} = \mathbb{C}^n \times \{t_0\}$ (near $(\vec{0}, t_0)$) are taken to another complex space $\mathbb{C}^n \times \{\tilde{t}_0\}$, with constant $\tilde{w} = \tilde{t}_0$. In particular, the subspace $\mathbb{C}^n \times \{\mathbf{0}\}$ is fixed as a set (near $(\vec{0}, \mathbf{0})$). So, if we think of $M_t \subseteq \mathbb{C}^n \times \{t\}$ as a t-dependent perturbation of $M = M_{\mathbf{0}}$, then the transformations under consideration are those that, for each t are holomorphic transformations of the ambient space of M_t, $\mathbb{C}^n \to \mathbb{C}^n$, and simultaneously the quantity t may be real analytically re-parametrized. The restriction of the transformation to $\mathbb{C}^n \times \{\mathbf{0}\}$ is exactly of the form (9), so it will be convenient to arrange for normal forms of $\widehat{M}$ that put M into already known normal forms.

In analogy with the deformation theory of singularities of maps, where the mapping variables and deformation parameters are transformed by groups that respect the distinction between the two types of coordinates by taking certain fibers to fibers ([**AVGL**] §§1.1, 3.1, [**Lu**] §3.4, [**Martinet**], [**PS**] §6.1, [**Wall**] §10), $\widehat{M} \subseteq \mathbb{C}^{n+k}$ and any of its normal forms under transformations of the form (11) will be called a k-parameter "unfolding" of the CR singular submanifold M. Manifolds $\widehat{M}_1$, $\widehat{M}_2 \subseteq \mathbb{C}^{n+k}$ related by a transformation in the above subgroup will be called *u-equivalent*.

Having set up the framework for the defining equations and transformation groups, from this point the reader could skip ahead to either the treatment of real surfaces and their unfoldings in Section 5, or to the case of higher codimension in Section 6 (which is different, and treated independently, from the surface case). In the next Section, we continue a general discussion of real analytic defining equations and their complexification.

4. A complexification construction

The previous Sections mentioned some analogies between the geometry of CR singularities and the theory of singularities of maps. It remains to be seen whether the analogies are merely rough and superficial, or can be made precise, and this paper will not give a complete answer to that question, the goal being instead to consider some of the simplest examples. As one possible framework for an investigation of the correspondence between CR singularities of real submanifolds and critical points of holomorphic maps, we recall a complexification construction ([**MW**], [**Webster$_1$**]), which relates a given real m-submanifold $M \subseteq \mathbb{C}^n$ to a certain parametric map $\mathbb{C}^m \to \mathbb{C}^n$ (or, more precisely, a neighborhood of the CR singularity in M to the germ of a map near the origin of $\mathbb{C}^m$). Our analysis of normal forms in later Sections will not explicitly refer to this construction, unlike [**MW**], where the geometry of the complexification is crucial for the theory. Instead, we will be using this complexification technique as a computational tool, for identifying the CR singular locus of some concrete examples in Subsections 5.3 and 6.3.

First, for an arbitrary power series $h = \sum h^{ab\mathbf{I}} z_1^a \zeta_1^b x^{\mathbf{I}}$ as in Definition 3.1, let $\hbar = \sum \hbar^{ab\mathbf{I}} z_1^a \zeta_1^b x^{\mathbf{I}}$ be the series with coefficients defined by the formula $\hbar^{ab\mathbf{I}} = \overline{h^{ba\mathbf{I}}}$. Then, the function $\hbar(z_1, \bar{z}_1, x)$ has the property that it is equal to $\overline{h(z_1, \bar{z}_1, x)}$ when $\zeta_1 = \bar{z}_1$ and $x = (x_2, \ldots, x_{m-1})$ is real.

Next, we want to consider the equations (8) of M, which are in the form of a graph over the tangent space, in two different ways, both as a parametric map and an implicit description.

M can be described implicitly in $\mathbb{C}^n$ by the following $2n - m$ equations:

$$
\begin{aligned}
&(13) \qquad & 0 &= y_2 - H_2(z_1, \bar{z}_1, x), \ldots, \\
& & 0 &= y_{m-1} - H_{m-1}(z_1, \bar{z}_1, x), \\
& & 0 &= z_m - h_m(z_1, \bar{z}_1, x), \ldots, \\
&(14) \qquad & 0 &= z_n - h_n(z_1, \bar{z}_1, x), \\
&(15) \qquad & 0 &= \bar{z}_m - \hbar_m(z_1, \bar{z}_1, x), \ldots, \\
&(16) \qquad & 0 &= \bar{z}_n - \hbar_n(z_1, \bar{z}_1, x).
\end{aligned}
$$

Equations (13–14) are exactly the equations (8), of which the $y_\sigma = H_\sigma$ equations are self-conjugate. Equations (15–16) are the complex conjugates of the $z_u = h_u$ equations. Now, consider $\mathbb{C}^{2n}$, with coordinates

$(\vec{z}, \vec{\zeta}) = (z_1, \dots, z_n, \zeta_1, \dots, \zeta_n)^T$. Taking Equations (13–16) and replacing every occurrence of $\bar{z}$ with ζ gives the following $2n - m$ equations:

$$\begin{aligned}
(17)\quad 0 &= \frac{z_2 - \zeta_2}{2i} - H_2(z_1, \zeta_1, \frac{z_2 + \zeta_2}{2}, \dots, \frac{z_{m-1} + \zeta_{m-1}}{2}), \dots, \\
0 &= \frac{z_{m-1} - \zeta_{m-1}}{2i} - H_{m-1}(z_1, \zeta_1, \frac{z_2 + \zeta_2}{2}, \dots, \frac{z_{m-1} + \zeta_{m-1}}{2}), \\
0 &= z_m - h_m(z_1, \zeta_1, \frac{z_2 + \zeta_2}{2}, \dots, \frac{z_{m-1} + \zeta_{m-1}}{2}), \dots, \\
0 &= z_n - h_n(z_1, \zeta_1, \frac{z_2 + \zeta_2}{2}, \dots, \frac{z_{m-1} + \zeta_{m-1}}{2}), \\
0 &= \zeta_m - \hbar_m(z_1, \zeta_1, \frac{z_2 + \zeta_2}{2}, \dots, \frac{z_{m-1} + \zeta_{m-1}}{2}), \dots, \\
0 &= \zeta_n - \hbar_n(z_1, \zeta_1, \frac{z_2 + \zeta_2}{2}, \dots, \frac{z_{m-1} + \zeta_{m-1}}{2}).
\end{aligned}$$

Near $(\vec{0}, \vec{0})$, these equations define a complex analytic m-submanifold M_c embedded in $\mathbb{C}^{2n}$. Denoting

$$\mathbf{\Delta} : \mathbb{C}^n \to \mathbb{C}^{2n} : \vec{z} \mapsto (z_1, \dots, z_n, \bar{z}_1, \dots, \bar{z}_n)^T,$$

the set $\{\zeta_1 = \bar{z}_1, \dots, \zeta_n = \bar{z}_n\} = \mathbf{\Delta}(\mathbb{C}^n)$ is a totally real $2n$-subspace of $\mathbb{C}^{2n}$, and $M_c \cap \mathbf{\Delta}(\mathbb{C}^n) = \mathbf{\Delta}(M)$. Denoting $\pi : \mathbb{C}^{2n} \to \mathbb{C}^n : (\vec{z}, \vec{\zeta}) \mapsto \vec{z}$, $(\pi \circ \mathbf{\Delta})(M) = \pi(M_c \cap \mathbf{\Delta}(\mathbb{C}^n)) = M$. In a neighborhood U of $\vec{0} \in \mathbb{C}^n$, $\pi(M_c)$ is a complex analytic variety (possibly all of U) containing M.

M can also be parametrically described as the image of the parametric map $\boldsymbol{\sigma} : \mathbb{R}^m \to \mathbb{C}^n$:

$$(z_1, \bar{z}_1, x) \mapsto (z_1, x_2 + iH_2, \dots, x_{m-1} + iH_{m-1}, h_m, \dots, h_n)^T.$$

The composite $\mathbf{\Delta} \circ \boldsymbol{\sigma} : \mathbb{R}^m \to \mathbb{C}^{2n}$ is a parametric map:

$$(z_1, \bar{z}_1, x) \mapsto (z_1, x_2 + iH_2, \dots, h_n, \bar{z}_1, x_2 - iH_2, \dots, \overline{h_n})^T.$$

The *complexification* of the parametrization is the holomorphic map $\Sigma : \mathbb{C}^m \to \mathbb{C}^{2n}$ defined by replacing $\bar{z}_1$ with ζ_1 and $x = (x_2, \dots, x_{m-1})$ with complex variables $\xi = (\xi_2, \dots, \xi_{m-1})$ in the expression for $\mathbf{\Delta} \circ \boldsymbol{\sigma}$:

$$\begin{aligned}
\Sigma : (z_1, \zeta_1, \xi) \;\; \mapsto \;\; & (z_1, \xi_2 + iH_2(z_1, \zeta_1, \xi), \dots, h_n(z_1, \zeta_1, \xi), \\
& \zeta, \xi_2 - iH_2(z_1, \zeta_1, \xi), \dots, \hbar_n(z_1, \zeta_1, \xi))^T.
\end{aligned}$$

Let $\boldsymbol{\delta}$ denote the inclusion of the totally real subspace

$$(18) \qquad \{\zeta_1 = \bar{z}_1, \xi_2 = \overline{\xi_2} = x_2, \dots, \xi_{m-1} = \overline{\xi_{m-1}} = x_{m-1}\}$$

in $\mathbb{C}^m$; then $\Sigma \circ \boldsymbol{\delta} = \mathbf{\Delta} \circ \boldsymbol{\sigma}$ and $\pi \circ \Sigma \circ \boldsymbol{\delta} = \pi \circ \mathbf{\Delta} \circ \boldsymbol{\sigma} = \boldsymbol{\sigma}$. For a neighborhood V of the origin in $\mathbb{C}^m$, Σ is a holomorphic embedding $V \to \mathbb{C}^{2n}$, and $\Sigma \circ \boldsymbol{\delta}$ is a totally real embedding. The connection between the parametric and implicit descriptions is that Σ is a local parametrization of the complex submanifold M_c: the image $\Sigma(V)$ is equal to M_c near $\vec{0} \in \mathbb{C}^n$. The composite $\pi \circ \Sigma : \mathbb{C}^m \to \mathbb{C}^n$:

$$(19) \qquad (z_1, \zeta_1, \xi) \quad \mapsto \quad (z_1, \xi_2 + iH_2(z_1, \zeta_1, \xi), \dots, h_n(z_1, \zeta_1, \xi))^T$$

is holomorphic, but not an embedding: the complex Jacobian drops rank along a critical point locus which includes the origin. For a point $\vec{u} = (z_1, \bar{z}_1, x)$ in the image of $\boldsymbol{\delta}$ in $\mathbb{C}^m$, if $\vec{u}$ is not a critical point of $\pi \circ \Sigma$, then $\pi \circ \Sigma$ is a local holomorphic embedding near $\vec{u}$, which takes a neighborhood of $\vec{u}$ in the totally real m-subspace (18) to a totally real neighborhood of $(\pi \circ \Sigma \circ \boldsymbol{\delta})(\vec{u}) = \boldsymbol{\sigma}(\vec{u})$ in M. So, if $\boldsymbol{\sigma}(\vec{u})$ is a CR singular point in M, then $\boldsymbol{\delta}(\vec{u})$ is a critical point of $\pi \circ \Sigma$.

This complexification construction has the practical advantage that the critical points of $\pi \circ \Sigma$ are easy to compute, and can be considered 'candidates' for CR singular points of a given real analytic parametrization. Generally speaking, the critical points of a complexified parametric map are also candidates for differential-topological singularities of the real map, and such points were observed in [**C**$_2$] and [**C**$_3$] for some parametrized images of real varieties in complex projective space. However, the local defining equations set up in Section 3 are always, being graphs of smooth functions, smooth embeddings.

In Sections 5 and 6, we will also complexify the defining equations (10) of $\widehat{M}$, treating the real t coordinates as more of the real x coordinates, so x and $\xi = (\xi_2, \dots, \xi_{m-1})$ in the above expressions will be extended to (x, t) and $(\xi, \omega) = (\xi_2, \dots, \xi_{m-1}, \omega_1, \dots, \omega_k)$.

The map $\pi \circ \Sigma$ is the one mentioned at the beginning of this Section; it seems from the form of (19) that there should be some connection between the geometry of its critical points, in terms of the better-understood singularity theory of holomorphic (or smooth) maps, and the geometry of the CR singularities of M. The classification of singularities of maps up to certain group actions may relate to the classification of CR singularities up to biholomorphic transformations (9).

However, the actions of the groups $\mathcal{B}_{m,n}$, $\mathcal{B}_{m,n+k}$, $\mathcal{U}_{m,n,k}$ (9, 11) on the defining equations of M and $\widehat{M}$ (8, 10) are not exactly the same as the group actions usually considered in the theory of singularities or their unfoldings ([**AVGL**] §3.1, [**Martinet**]). So while there may be a useful analogy between the singularity theory of maps and the geometry of CR singularities, it is already clear from previously known examples, and those to be considered in the next Sections, that the details will not be the same, and an understanding of the group action would be the key to making the analogy precise.

5. Real surfaces in $\mathbb{C}^2$

As previously mentioned, a real surface in $\mathbb{C}^2$ in general position is either totally real everywhere, or has isolated CR singularities.

5.1. Normal forms.

We begin by recalling some well-known normal forms for real surfaces in $\mathbb{C}^2$. The defining equation in standard position as in (8), with the manifold tangent to the z_1-axis, is:

$$z_2 = h_2(z_1, \bar{z}_1) = Q(z_1, \bar{z}_1) + C(z_1, \bar{z}_1) + e(z_1, \bar{z}_1),$$

where Q and C have terms of only degree 2 and 3 respectively, and the series $e = O(4)$.

PROPOSITION 5.1. *For any real analytic CR singular surface M there is exactly one normal form $\{z_2 = Q+C+e\}$ from the following list equivalent to M under a holomorphic transformation of a neighborhood of the CR singularity.*

Label	Q	C	e	comment
elliptic generic	$z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$	0	$O(4)$ real	$0 < \gamma < \frac{1}{2}$
elliptic cusp	$z_1\bar{z}_1$	$z_1^3 + \bar{z}_1^3$	$O(4)$ real	$\gamma = 0$
elliptic higher cusp	$z_1\bar{z}_1$	0	$z_1^s + \bar{z}_1^s$ $+O(s+1)$ real	$\gamma = 0$ $s \geq 4$
elliptic circular paraboloid	$z_1\bar{z}_1$	0	$\equiv 0$	$\gamma = 0$ $s = \infty$
parabolic nondegen.	$z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$	$i(z_1 - \bar{z}_1)z_1\bar{z}_1$	$O(4)$	$\gamma = \frac{1}{2}$
parabolic degenerate	$z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$	0	$\eta(z_1^2+\bar{z}_1^2)z_1\bar{z}_1$ $+O(5)$	$\gamma = \frac{1}{2}$ $\eta = \pm 1, 0$
hyperbolic generic	$z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$	0	$O(4)$	$\frac{1}{2} < \gamma < 1$ or $\gamma > 1$
hyperbolic nondegen. diophantine	$z_1\bar{z}_1 + z_1^2 + \bar{z}_1^2$	$(z_1 - \bar{z}_1)z_1\bar{z}_1$	$O(4)$	$\gamma = 1$
hyperbolic degenerate diophantine	$z_1\bar{z}_1 + z_1^2 + \bar{z}_1^2$	0	$O(4)$	$\gamma = 1$
hyperbolic fold	$z_1^2 + \bar{z}_1^2$	0	$O(4)$	$\gamma = \infty$
cubic	0	C	$O(4)$	

PROOF. The only new bit of information in the above table is the quartic part for the degenerate parabolic case, so that is the only calculation skteched here. The computation is typical of the procedure for putting M into normal form by a transformation (9). The other entries will be discussed in a later series of Examples.

For the degenerate parabolic case, begin with the defining equation of M in the form (8),

$$z_2 = h_2(z_1, \bar{z}_1) = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + e,$$

where $Q = \frac{1}{2}(z_1 + \bar{z}_1)^2$, $C \equiv 0$ and

$$e(z_1, \bar{z}_1) = e_2^{04}\bar{z}_1^4 + e_2^{13}\bar{z}_1^3 z_1 + e_2^{22}\bar{z}_1^2 z_1^2 + e_2^{31}\bar{z}_1 z_1^3 + e_2^{40} z_1^4 + O(5). \tag{20}$$

The transformations (9) which preserve the quadratic and cubic part of h_2, while contributing to the normalization of the quartic terms in e, are of the form

$$\begin{aligned} \tilde{z}_1 &= a_1 z_1 + a_2 z_2 + p_1^1 z_1^3 + p_1^2 z_1 z_2 \\ \tilde{z}_2 &= (a_1)^2 z_2 + p_2^1 z_1^4 + p_2^2 z_1^2 z_2 + p_2^3 z_2^2, \end{aligned} \tag{21}$$

where a_1 is a nonzero real number and a_2 is purely imaginary. Higher weight terms in the transformation would only contribute degree 5 or higher quantities to the transformed defining equation.

$$\begin{aligned} &\tilde{z}_2 - \frac{1}{2}(\tilde{z}_1 + \bar{\tilde{z}}_1)^2 \\ = \ &(a_1)^2 z_2 + p_2^1 z_1^4 + p_2^2 z_1^2 z_2 + p_2^3 z_2^2 \\ &-\frac{1}{2}(a_1 z_1 + a_2 z_2 + p_1^1 z_1^3 + p_1^2 z_1 z_2 + \overline{(a_1 z_1 + a_2 z_2 + p_1^1 z_1^3 + p_1^2 z_1 z_2)})^2, \end{aligned}$$

and for points $\mathbf{x} = (z_1, z_2)$ on M near $\vec{0}$, substituting $z_2 = h_2$ gives:

$$\begin{aligned} &\tilde{z}_1 \\ = \ &a_1 z_1 + a_2(\frac{1}{2}(z_1 + \bar{z}_1)^2 + e) + p_1^1 z_1^3 + p_1^2 z_1 \cdot (\frac{1}{2}(z_1 + \bar{z}_1)^2 + e), \end{aligned} \tag{22}$$

$$\begin{aligned} &\tilde{z}_2 - \frac{1}{2}(\tilde{z}_1 + \bar{\tilde{z}}_1)^2 \\ = \ &(a_1)^2(e_2^{04}\bar{z}_1^4 + e_2^{13}\bar{z}_1^3 z_1 + e_2^{22}\bar{z}_1^2 z_1^2 + e_2^{31}\bar{z}_1 z_1^3 + e_2^{40} z_1^4) \\ &+p_2^1 z_1^4 + p_2^2 z_1^2 \frac{1}{2}(z_1 + \bar{z}_1)^2 + p_2^3(\frac{1}{2}(z_1 + \bar{z}_1)^2)^2 \\ &-a_1(z_1 + \bar{z}_1) \cdot (p_1^1 z_1^3 + p_1^2 z_1 \frac{(z_1 + \bar{z}_1)^2}{2} + \overline{p_1^1}\bar{z}_1^3 + \overline{p_1^2}\bar{z}_1 \frac{(z_1 + \bar{z}_1)^2}{2}) \\ &+O(5). \end{aligned} \tag{23}$$

Substituting the inverse function of (22), $z_1 = \frac{1}{a_1}\tilde{z}_1 + O(2)$, into (23) gives the new defining equation in the $\tilde{z}$ coordinates:

$$
\begin{aligned}
(24)\ \tilde{z}_2 \ =\ & \frac{1}{2}(\tilde{z}_1+\bar{\tilde{z}}_1)^2 \\
& +\frac{1}{(a_1)^2}(e_2^{04}\bar{\tilde{z}}_1^4 + e_2^{13}\bar{\tilde{z}}_1^3\tilde{z}_1 + e_2^{22}\bar{\tilde{z}}_1^2\tilde{z}_1^2 + e_2^{31}\bar{\tilde{z}}_1\tilde{z}_1^3 + e_2^{40}\tilde{z}_1^4) \\
& +\frac{1}{(a_1)^4}(p_2^1\tilde{z}_1^4 + p_2^2\tilde{z}_1^2\frac{1}{2}(\tilde{z}_1+\bar{\tilde{z}}_1)^2 + p_2^3(\frac{1}{2}(\tilde{z}_1+\bar{\tilde{z}}_1)^2)^2) \\
& -\frac{\tilde{z}_1+\bar{\tilde{z}}_1}{(a_1)^3}\cdot(p_1^1\tilde{z}_1^3 + p_1^2\tilde{z}_1\frac{(\tilde{z}_1+\bar{\tilde{z}}_1)^2}{2} + \overline{p_1^1}\bar{\tilde{z}}_1^3 + \overline{p_1^2}\bar{\tilde{z}}_1\frac{(\tilde{z}_1+\bar{\tilde{z}}_1)^2}{2}) \\
& +O(5).
\end{aligned}
$$

Observe that the imaginary coefficient a_2 does not contribute to the degree 4 terms. The normal form problem is, given e, find $\vec{p}$ which simplifies (24). This is a (real) linear problem and a computer algebra system ([Maple]) is useful; the remainder of the calculation is broken into steps and just sketched here. The first step is that from the form of (24), it appears that one could find p_2^1, p_2^2, p_2^3 that eliminate the complex coefficients e_2^{04}, e_2^{22}, e_2^{40}, while p_1^1, p_1^2 are chosen to be 0 and $a_1 = 1$. In fact, the following choice works:

$$p_2^3 = -4e_2^{04},\ p_2^2 = -2e_2^{22} + 12e_2^{04},\ p_2^1 = e_2^{22} - e_2^{40} - 5e_2^{04}.$$

Applying such a transformation, and then dropping the tilde notation, gets M in the form (20), but with $e_2^{04} = e_2^{22} = e_2^{40} = 0$, and the new complex coefficients e_2^{13}, e_2^{31} possibly changed from the old ones.

The next transformation will be another with $p_1^2 = 0$, $a_1 = 1$, but will use p_1^1 to normalize e_2^{13} and e_2^{31}. To preserve the partial normal form $e_2^{04} = e_2^{22} = e_2^{40} = 0$, the coefficients of p_2 must be

$$p_2^2 = -12\overline{p_1^1}, p_2^1 = 5\overline{p_1^1} + p_1^1, p_2^3 = 4\overline{p_1^1}.$$

Then (24) becomes:

$$z_2 = Q + (3\overline{p_1^1} + e_2^{13})z_1\bar{z}_1^3 + (-p_1^1 + e_2^{31} - 8\overline{p_1^1})z_1^3\bar{z}_1 + O(5),$$

and for any e_2^{13}, e_2^{31}, there is some p_1^1 so that the above coefficients become equal.

The third transformation will use a_1 and all the unknown $\vec{p}$ coefficients, and again, to preserve the partial normal form $e_2^{04} = e_2^{22} =$

$e_2^{40}=0$, $e_2^{13}=e_2^{31}$, the coefficients of p_1 and p_2 must satisfy:

$$\begin{aligned}
p_1^1 &= -\frac{1}{5}(p_1^2-\overline{p_1^2}),\\
p_2^1 &= -\frac{1}{5}a_1(p_1^2-\overline{p_1^2}),\\
p_2^2 &= \frac{3}{5}a_1(p_1^2-\overline{p_1^2}),\\
p_2^3 &= \frac{2}{5}a_1(2p_1^2+3\overline{p_1^2}).
\end{aligned}$$

This brings (24) to:

$$z_2 = Q + \left(\frac{e_2^{13}}{(a_1)^2}+\frac{p_1^2}{10(a_1)^3}-\frac{\overline{p_1^2}}{10(a_1)^3}\right)(z_1^3\bar{z}_1+z_1\bar{z}_1^3)+O(5).$$

Only the imaginary part of p_1^2 contributes to these terms, so e_2^{13} can be normalized to be a real number, and then the positive scale factor $(a_1)^2$ will make e_2^{13} either $+1$, -1, or 0, as claimed. ∎

The classification of CR singularities of surfaces in $\mathbb{C}^2$ into elliptic, parabolic, and hyperbolic cases is well known, but a few brief remarks follow; for recent survey articles, see [**BER**], [**S-G**].

EXAMPLE 5.2. The higher degree terms for the generic elliptic case were considered by [**MW**], who showed that M has a real algebraic implicit normal form (to be recalled in Example 5.13).

EXAMPLE 5.3. The $\gamma=0$, $s=3,4,\ldots$ cases were considered by [**MW**] and [**Moser**], and also by [**HK**], who proved that there exists a holomorphic transformation so that h_2 is real valued, and M fits inside $\mathbb{R}^3\subseteq\mathbb{C}^2$. A real surface contained in $\mathbb{R}^3$ is said to be "holomorphically flat," and the transformation of a surface into a flat normal form is called "flattening." The term "cusp" describing the CR singularity is borrowed here from the terminology of singularity theory, since the $s=3$ form of h_2 bears a resemblance to the normal form for Whitney's cusp singularity of a map $\mathbb{C}^2\to\mathbb{C}^2:(z_1,\zeta_1)\mapsto(z_1,z_1\zeta_1+\zeta_1^3)$ ([**AVGL**] §3.1, [**GG**] §VI.2, [**Lu**] §2.6, [**Martinet**], [**Wall**]). The resemblance becomes more concrete in view of the complexification construction of Section 4.

EXAMPLE 5.4. The $\gamma = 0$, $s = \infty$ case was considered by [**Moser**], who showed that if h_2 can be transformed to $z_1\bar{z}_1 + O(s)$ for all $s = 3, 4, \ldots$, then there exists a holomorphic coordinate change transforming M into the real quadric variety.

EXAMPLE 5.5. Cubic normal forms for both the nondegenerate and degenerate parabolic cases were found by [**Webster**$_2$]. In the nondegenerate case, the normal form for C given by [**Webster**$_2$] is $C = -iz_1\bar{z}_1(z_1 + \bar{z}_1)$, however it is possible to choose the signs differently so that the Proposition's normal form gives a real valued $Q + C$.

EXAMPLE 5.6. In most of the hyperbolic cases, with $\gamma > \frac{1}{2}$, the cubic terms can be eliminated just as in the generic elliptic case. However, it was shown by [**MW**] that for γ satisfying a certain diophantine condition, M cannot always be flattened into $\mathbb{R}^3$ by making the higher degree terms real valued. The only value of γ for which this happens at the cubic terms is $\gamma = 1$, where the coefficient of the imaginary quantity $(z_1 - \bar{z}_1)z_1\bar{z}_1$ can be normalized to either 1 or 0.

EXAMPLE 5.7. The normal form $Q = z_1^2 + \bar{z}_1^2$ is considered the $\gamma \to +\infty$ limit of the hyperbolic normal forms. An alternative normal form would be the complex valued monomial $Q = \bar{z}_1^2$. The label "fold" again comes from its similarity to Whitney's normal form for the fold singularity, $\mathbb{C}^2 \to \mathbb{C}^2 : (z_1, \zeta_1) \mapsto (z_1, \zeta_1^2)$.

The presence of the continuous invariant γ interpolating between the fold and cusp normal forms is the most obvious way in which the classification problem for CR singularities of surfaces in $\mathbb{C}^2$ differs from the analogous problem for singularities of maps $\mathbb{C}^2 \to \mathbb{C}^2$.

For purposes of comparison with the $n = 2$ case, and also with the normal forms under u-equivalence in the next Subsection, the following table of normal forms for n-submanifolds of $\mathbb{C}^n$, $n \geq 3$, is recalled from [**MW**] and [**Webster**$_2$].

Proposition 5.8. *For $n \geq 3$ and any real analytic n-submanifold $M \subseteq \mathbb{C}^n$ with a CR singularity $\mathbf{x} \in N_1 \setminus N_2$, there is a normal form $\{z_n = h_n(z_1, \bar{z}_1, x), y_\sigma = H_\sigma(z_1, \bar{z}_1, x), \sigma = 2, \ldots, n-1\}$ from exactly one row in the following list equivalent to M under a holomorphic transformation of a neighborhood of $\mathbf{x}$.*

Label	normal form	comment
elliptic generic	$h_n = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + O(3)$ real $H_\sigma \equiv 0$	$0 < \gamma < \frac{1}{2}$
elliptic cusp	$h_n = z_1\bar{z}_1 + O(3)$ $H_\sigma = O(3)$	$\gamma = 0$
parabolic nondegenerate	$h_n = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + ic_n(z_1 - \bar{z}_1)x_2$ $-i(z_1 + \bar{z}_1)z_1\bar{z}_1 + \eta_n^\sigma x_\sigma z_1\bar{z}_1 + O(4)$ $H_\sigma = O(4)$	$\gamma = \frac{1}{2}$ $c_n = 0, 1$ η_n^σ real
parabolic degenerate	$h_n = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + ic_n(z_1 - \bar{z}_1)x_2$ $+\eta_n^\sigma x_\sigma z_1\bar{z}_1 + O(4)$ $H_\sigma = O(4)$	$\gamma = \frac{1}{2}$ $c_n = 0, 1$ η_n^σ real
hyperbolic generic	$h_n = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + O(3)$ $H_\sigma = O(3)$	$\gamma > \frac{1}{2}$
hyperbolic fold	$h_n = z_1^2 + \bar{z}_1^2 + O(3)$ $H_\sigma = b_\sigma z_1\bar{z}_1 + O(3)$	$\gamma = \infty$ $b_\sigma = 0, 1$
	$h_n = (z_1 + \bar{z}_1)x_2 + i(z_1 - \bar{z}_1)x_3 + O(3)$ $H_\sigma = b_\sigma z_1\bar{z}_1 + O(3)$	$n \geq 4$ $b_\sigma = 0, 1$
	$h_n = (z_1 + \bar{z}_1)x_2 + O(3)$ $H_\sigma = b_\sigma z_1\bar{z}_1 + O(3)$	$b_\sigma = 0, 1$
	$h_n = O(3)$ $H_\sigma = b_\sigma z_1\bar{z}_1 + O(3)$	$b_\sigma = 0, 1$

The quantities γ and c_n are biholomorphic invariants. ∎

The constants b_σ, η_n^σ in the above table are not necessarily invariants and could be normalized by the $\mathbf{R}_{(n-2)\times(n-2)}$ block of entries in (9).

Remark 5.9. The parabolic points of a submanifold M^n in general position in $\mathbb{C}^n$, $n \geq 3$, are characterized by [**Webster**$_2$] as the points $\mathbf{x}$ in $N_1 \setminus N_2$ where the real tangent space $T_{\mathbf{x}}N_1$ intersects the complex tangent line $T_{\mathbf{x}}M \cap iT_{\mathbf{x}}M$ in a real line, the "parabolic line." At the elliptic and hyperbolic points, these subspaces of TM intersect only at the origin. This property of parabolic CR singularities bears some resemblance to the $S_{r,s}$ system of classification of singularities of maps, as in [**GG**] §VI.4, but no connection will be pursued here.

5.2. Unfolding CR singularities of surfaces.

The $m = n = 2$ case of (10) is that $\widehat{M}$ is a $(2+k)$-submanifold of $\mathbb{C}^{2+k}$:

$$\begin{aligned} z_2 &= h_2(z_1, \bar{z}_1, t) = O(2) \\ s_1 = \ldots = s_k &= 0, \end{aligned} \tag{25}$$

and by a holomorphic transformation in z_1, z_2 only (leaving $\vec{w}$ fixed), we can assume that the quantity $h_2(z_1, \bar{z}_1, \mathbf{0})$ is in one of the normal forms from Proposition 5.1. So, the quadratic part of h_2 is

$$Q(z_1, \bar{z}_1, \mathbf{0}) + e_2^{10\alpha} t_\alpha z_1 + e_2^{01\alpha} t_\alpha \bar{z}_1 + e_2^{\alpha\beta} t_\alpha t_\beta, \tag{26}$$

with complex coefficients summed over α, $\beta = 1, \ldots, k$. The group of transformations $\mathcal{U}_{2,2,k}$ (11) is:

$$\begin{aligned} \tilde{z}_1 &= a_1 z_1 + a_2 z_2 + a^\alpha w_\alpha + p_1(z_1, z_2, \vec{w}) \\ \tilde{z}_2 &= a_{22} z_2 + p_2(z_1, z_2, \vec{w}) \end{aligned} \tag{27}$$

$$\tilde{w} = \mathbf{R}_{k\times k}\vec{w} + \vec{P}(\vec{w}). \tag{28}$$

Our approach to the u-equivalence problem is to find some normal forms for (25) under this group. The only differences between this problem and the normal form problem of Proposition 5.8 are that the functions H_σ are already identically zero in (25), and the full group from Proposition 5.8 would replace (28) by:

$$\tilde{w}_\alpha = r_\alpha^\beta w_\beta + a''_\alpha z_2 + P_\alpha(z_1, z_2, \vec{w}),$$

allowing complex a''_α and complex valued P_α.

The calculation proving the following result appears in [**Bishop**].

LEMMA 5.10. *Given $\widehat{M} \subseteq \mathbb{C}^{2+k}$ of the form* (25)*, with*

$$h_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + e_2^{10\alpha} t_\alpha z_1 + e_2^{01\alpha} t_\alpha \bar{z}_1 + e_2^{\alpha\beta} t_\alpha t_\beta + C(z_1, \bar{z}_1, t) + O(4),$$

with $\gamma \geq 0$ and $\gamma \neq \frac{1}{2}$, there exists a transformation of the form

$$\tilde{z}_1 = z_1 + a^\alpha w_\alpha, \quad \tilde{z}_2 = z_2 + p_2^\alpha z_1 w_\alpha + p_2^{\alpha\beta} w_\alpha w_\beta, \quad \tilde{w} = \vec{w},$$

such that the new defining equation is

$$h_2(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{t}) = \tilde{z}_1\bar{\tilde{z}}_1 + \gamma(\tilde{z}_1^2 + \bar{\tilde{z}}_1^2) + C(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{t}) + O(4),$$

where the corresponding coefficients of $C(\tilde{z}_1, \bar{\tilde{z}}_1, \mathbf{0})$ are the same as those of $C(z_1, \bar{z}_1, \mathbf{0})$. ∎

This means that in the elliptic or hyperbolic cases, the quadratic terms involving t can always be eliminated by an element of the group (27–28), without altering the cubic terms in $z_1, \bar{z}_1$ only. An analogous result holds for the $\gamma = \infty$ case.

The $\gamma = \frac{1}{2}$ case is considered in Example 5.18, but for any quadratic part $Q(z_1, \bar{z}_1, \mathbf{0})$ in (26), a transformation in the group (27–28) of the form $\tilde{z}_2 = z_2 + p_2^{\alpha} z_1 w_\alpha + p_2^{\alpha\beta} w_\alpha w_\beta$ will eliminate, or re-assign any complex value to, the coefficients $e_2^{10\alpha}$, $e_2^{\alpha\beta}$ in (26), without changing any other quadratic terms in h_2. More generally, terms in h_2 of the form $e_2^{a0\mathbf{K}} z_1^a t^{\mathbf{K}}$, without any $\bar{z}_1$ factor, can be assigned any complex coefficient by a transformation of the form $\tilde{z}_2 = z_2 + p_2^{a\mathbf{K}} z_1^a w^{\mathbf{K}}$ without changing any other terms of the same or lower degree. So, the defining equation (25) of $\widehat{M}$ is of the form $h_2 = Q(z_1, \bar{z}_1, t) + C(z_1, \bar{z}_1, t) + e(z_1, \bar{z}_1, t)$, where

$$\begin{aligned} C &= C(z_1, \bar{z}_1, \mathbf{0}) + e_2^{20\alpha} z_1^2 t_\alpha + e_2^{11\alpha} z_1 \bar{z}_1 t_\alpha + e_2^{02\alpha} \bar{z}_1^2 t_\alpha \\ &\quad + e_2^{10\alpha\beta} z_1 t_\alpha t_\beta + e_2^{01\alpha\beta} \bar{z}_1 t_\alpha t_\beta + e_2^{00\mathbf{K}} t^{\mathbf{K}}, \end{aligned}$$

and the coefficients $e_2^{20\alpha}$, $e_2^{10\alpha\beta}$, $e_2^{00\mathbf{K}}$ (where $\mathbf{K}$ is a degree three multi-index) can take any value after a weight 3 holomorphic transformation of z_2. However, the normalization of the other cubic coefficients depends on the quadratic part $Q(z_1, \bar{z}_1, t)$ and on $C(z_1, \bar{z}_1, \mathbf{0})$. The rest of this Subsection will consider a series of Examples of the most generic u-equivalence classes of unfoldings of the CR singularities with the various normal forms $Q+C$ from Proposition 5.1. For the sake of economy, we will assume the $e_2^{\alpha\beta} t_\alpha t_\beta$ and $e_2^{00\mathbf{K}} t^{\mathbf{K}}$ terms are eliminated at the beginning of each calculation and also at each step where they might re-appear, although we keep the terms $e_2^{10\alpha} z_1 t_\alpha$, $e_2^{20\alpha} z_1^2 t_\alpha$, $e_2^{10\alpha\beta} z_1 t_\alpha t_\beta$, since they may be used later to get a real valued normal form.

Example 5.11. For $M = M_{\mathbf{0}}$ with a generic elliptic or generic hyperbolic singularity, with $C(z_1, \bar{z}_1, \mathbf{0}) = 0$ as in Proposition 5.1, Lemma 5.10 applies to $\widehat{M}$, and (25) becomes $h_2 = Q + C(z_1, \bar{z}_1, t) + O(4)$, where

$$C = e_2^{20\alpha} z_1^2 t_\alpha + e_2^{11\alpha} z_1 \bar{z}_1 t_\alpha + e_2^{02\alpha} \bar{z}_1^2 t_\alpha + e_2^{10\alpha\beta} z_1 t_\alpha t_\beta + e_2^{01\alpha\beta} \bar{z}_1 t_\alpha t_\beta. \tag{29}$$

The transformations that could alter the cubic terms in C while preserving the normal form of $h_2(z_1, \bar{z}_1, \mathbf{0})$ are:

$$\begin{aligned} \tilde{z}_1 &= a_1 z_1 + p_1^{1\alpha} z_1 w_\alpha + p_1^{\alpha\beta} w_\alpha w_\beta \\ \tilde{z}_2 &= (a_1)^2 z_1 + p_2^{1\alpha} z_1^2 w_\alpha + p_2^{2\alpha} z_2 w_\alpha + p_2^{1\alpha\beta} z_1 w_\alpha w_\beta \\ \tilde{w} &= \mathbf{R}\vec{w}, \end{aligned} \tag{30}$$

where a_1 is a nonzero real scalar, and higher weight terms in $\vec{p}$, that would not contribute cubic terms in h_2, are omitted. Without stating all the details here, the coefficients $p_1^{\alpha\beta}$, $p_2^{1\alpha\beta}$ can eliminate the terms $e_2^{01\alpha\beta} \bar{z}_1 t_\alpha t_\beta$, $e_2^{10\alpha\beta} z_1 t_\alpha t_\beta$ (the cancellation is similar to that of Lemma 5.10). We choose to use coefficient $p_2^{2\alpha}$ to eliminate $e_2^{11\alpha} z_1 \bar{z}_1 t_\alpha$, and to use $p_2^{1\alpha} z_1^2 w_\alpha$ to equate the coefficients of $e_2^{20\alpha} z_1^2 t_\alpha$ and $e_2^{02\alpha} \bar{z}_1^2 t_\alpha$. The stabilizer of the partial normal form

$$C = e_2^{02\alpha}(z_1^2 + \bar{z}_1^2)t_\alpha$$

is of the form $(\tilde{z}, \tilde{w})$ as in (30), with $p_1^{\alpha\beta} = p_2^{1\alpha\beta} = 0$, $p_2^{2\alpha} = a_1(p_1^{1\alpha} + \overline{p_1^{1\alpha}})$, $p_2^{1\alpha} = 2a_1\beta(p_1^{1\alpha} - \overline{p_1^{1\alpha}})$. With $\mathbf{R}$ equal to the identity matrix $\mathbb{1}$, the transformed quantity C is

$$\left(e_2^{02\alpha} + \frac{\gamma}{a_1}(p_1^{1\alpha} - \overline{p_1^{1\alpha}})\right)(z_1^2 + \bar{z}_1^2)t_\alpha.$$

So, each $p_1^{1\alpha}$ only contributes its imaginary part, and $e_2^{02\alpha}$ can be normalized to be real. Unless all the $e_2^{02\alpha}$ are zero, the real matrix $\mathbf{R}$ then can transform the vector of coefficients to $(1, 0, \ldots, 0)$, so there is a generic cubic normal form:

$$z_2 = z_1\bar{z}_1 + (\gamma + t_1) \cdot (z_1^2 + \bar{z}_1^2) + O(4), \tag{31}$$

and also a degenerate case where the quadratic and cubic parts do not depend on t.

Geometrically, the unfolding is about what one would expect: manifolds M_t near $M_{\mathbf{0}}$ will have Bishop invariant varying near γ, and (at least up to some degree) this variation can be normalized to depend linearly only on one t coordinate. For t close to $\mathbf{0}$, if $M_{\mathbf{0}}$ is elliptic, then so is M_t, and similarly if $M_{\mathbf{0}}$ is hyperbolic, then so is M_t. Since only the linear part of the transformation of the $\vec{w}$ coordinates contributed to the normalization, there is no difference in appearance between this cubic normal form under u-equivalence, and the normal form from Proposition 5.8, for n-manifolds in $\mathbb{C}^n$ under the larger transformation group (where $n = 2 + k$).

Example 5.12. The diophantine cases from Proposition 5.1 have unfoldings similar to those in the previous Example. In the degenerate case, where $\gamma = 1$, $C = 0$, the calculations of the previous Example work without changing the result. In the nondegenerate case, the subgroup (30) preserving the cubic part $h_2(z_1, \bar{z}_1, \mathbf{0}) = Q + (z_1 - \bar{z}_1)z_1\bar{z}_1 + O(4)$ is different, in particular, a_1 must equal 1, but the end result is the same, and the generic unfolding has normal form

$$z_2 = z_1\bar{z}_1 + (1 + t_1) \cdot (z_1^2 + \bar{z}_1^2) + (z_1 - \bar{z}_1)z_1\bar{z}_1 + O(4).$$

Example 5.13. In the generic elliptic case, the holomorphic invariants in the higher degree terms were found by [**MW**]; the implicit normal form is the polynomial defining equation

$$z_2 = z_1\bar{z}_1 + (\gamma + \delta x_2^s) \cdot (z_1^2 + \bar{z}_1^2),$$

where $0 < \gamma < \frac{1}{2}$, $\delta = \pm 1$, and the integer $s = 1, 2, 3, \ldots$ (or $s = \infty$ in the quadric case) determine the local biholomorphic equivalence class of a real analytic surface. When put into the graph form (8), the defining equation for M is

$$z_2 = z_1\bar{z}_1 + (\gamma + \hat{\delta}(z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2))) \cdot (z_1^2 + \bar{z}_1^2) + O(6), \tag{32}$$

where $\hat{\delta} = \delta$ if $s = 1$, and $\hat{\delta} = 0$ if $s > 1$ or $s = \infty$. A nondegenerate unfolding of this surface will be of the form (25), with

$$h_2 = Q + C + \hat{\delta}(z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)) \cdot (z_1^2 + \bar{z}_1^2) + F(z_1, \bar{z}_1, t) + O(5),$$

where $Q + C$ is as in (31), and the degree 4 terms depending on t are

$$\begin{aligned} F \;=\; & e_2^{30\alpha} z_1^3 t_\alpha + e_2^{21\alpha} z_1^2 \bar{z}_1 t_\alpha + e_2^{12\alpha} z_1 \bar{z}_1^2 t_\alpha + e_2^{03\alpha} \bar{z}_1^3 t_\alpha \\ & + e_2^{20\alpha\beta} z_1^2 t_\alpha t_\beta + e_2^{11\alpha\beta} z_1 \bar{z}_1 t_\alpha t_\beta + e_2^{02\alpha\beta} \bar{z}_1^2 t_\alpha t_\beta \\ & + e_2^{10\mathbf{K}} z_1 t^{\mathbf{K}} + e_2^{01\mathbf{K}} \bar{z}_1 t^{\mathbf{K}}. \end{aligned} \tag{33}$$

A transformation (27–28) of the form

$$\tilde{z}_1 = z_1 + p_1^{20\alpha} z_1^2 w_\alpha + p_1^{01\alpha} z_2 w_\alpha + p_1^{1\alpha\beta} z_1 w_\alpha w_\beta + p_1^{\mathbf{K}} w^{\mathbf{K}} \tag{34}$$

$$\begin{aligned} \tilde{z}_2 \;=\; & z_2 + p_2^{30\alpha} z_1^3 w_\alpha + p_2^{11\alpha} z_1 z_2 w_\alpha \\ & + p_2^{20\alpha\beta} z_1^2 w_\alpha w_\beta + p_2^{01\alpha\beta} z_2 w_\alpha w_\beta + p_2^{\mathbf{K}} z_1 w^{\mathbf{K}} \end{aligned} \tag{35}$$

$$\begin{aligned} \tilde{w}_1 &= w_1 + P_1^{\alpha\beta} w_\alpha w_\beta \\ \tilde{w}_j &= w_j,\ j = 2, \ldots, k, \end{aligned}$$

where $P_1^{\alpha\beta}$ are real coefficients and $p_1^{1\alpha\beta}$ are purely imaginary, can transform F to zero. Again, without stating all the details, the coefficients $p_1^{\mathbf{K}}$, $p_2^{\mathbf{K}}$ eliminate $e_2^{01\mathbf{K}}$ and $e_2^{10\mathbf{K}}$, the coefficients $p_2^{20\alpha\beta}$, $p_2^{01\alpha\beta}$ eliminate $e_2^{20\alpha\beta}$ and $e_2^{11\alpha\beta}$, and $P_1^{\alpha\beta}$, $p_1^{1\alpha\beta}$ cancel the real and imaginary parts of $e_2^{02\alpha\beta}$. Then $p_1^{20\alpha}$, $p_1^{01\alpha}$, $p_2^{30\alpha}$, $p_2^{11\alpha}$ can eliminate $e_2^{30\alpha}$, $e_2^{21\alpha}$, $e_2^{12\alpha}$, $e_2^{03\alpha}$.

The conclusion is that the generic k-parameter unfolding $\widehat{M}$ of a surface M with a generic elliptic point (32) has a normal form under the action of the unfolding subgroup:

$$h_2 = z_1\bar{z}_1 + (\gamma + t_1 + \hat{\delta}(z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2))) \cdot (z_1^2 + \bar{z}_1^2) + O(5), \tag{36}$$

which is the same, up to $O(5)$, as the implicit algebraic normal form of [**MW**] (Equation 5.4, adapted to our notation):

$$z_2 = z_1\bar{z}_1 + (\gamma + t_1 + \delta x_2^s) \cdot (z_1^2 + \bar{z}_1^2), \tag{37}$$

for a real $(2 + k)$-submanifold in $\mathbb{C}^{2+k}$ under the action of local biholomorphisms.

EXAMPLE 5.14. The next case to be considered is the $\gamma = 0$, $s = \infty$ elliptic normal form, where $M = M_{\mathbf{0}}$ has quadratic part $Q = z_1\bar{z}_1$ and, like the generic elliptic case, zero cubic part. Applying Lemma 5.11, the defining equation (25) becomes $h_2 = z_1\bar{z}_1 + C(z_1, \bar{z}_1, t) + O(4)$, where C is the same as (29). The transformations preserving this partial normal form are:

$$\begin{aligned} \tilde{z}_1 &= a_1 z_1 + a_2 z_2 - \frac{a_1\overline{a_2}}{\overline{a_1}} z_1^2 + p_1^{\alpha} z_1 w_\alpha + p_1^{\alpha\beta} w_\alpha w_\beta \\ \tilde{z}_2 &= |a_1|^2 z_1 + p_2^{1\alpha} z_1^2 w_\alpha + p_2^{2\alpha} z_2 w_\alpha + p_2^{1\alpha\beta} z_1 w_\alpha w_\beta \\ \tilde{w} &= \mathbf{R}\vec{w}, \end{aligned} \tag{38}$$

where a_1 is a nonzero complex scalar. As in Example 5.11, the coefficients $p_1^{\alpha\beta}$, $p_2^{1\alpha\beta}$, $p_2^{2\alpha}$ can eliminate the coefficients $e_2^{01\alpha\beta}$, $e_2^{10\alpha\beta}$, $e_2^{11\alpha}$.

The stabilizer of the new partial normal form is the above group with the conditions $p_2^{1\alpha\beta} = p_1^{\alpha\beta} = 0$, $p_2^{2\alpha} = a_1\overline{p_1^{\alpha}} + \overline{a_1}p_1^{\alpha}$, and the new defining equation after such a transformation with $\mathbf{R} = \mathbb{1}$ satisfies:

$$C(z_1, \bar{z}_1, t) = \left(\frac{p_2^{1\alpha}}{(a_1)^2} + \frac{\overline{a_1}}{a_1} e_2^{20\alpha} \right) z_1^2 t_\alpha + \frac{a_1}{\overline{a_1}} e_2^{02\alpha} \bar{z}_1^2 t_\alpha. \tag{39}$$

Note that none of the terms in (38) contributes any quantities of the form $\bar{z}_1^2 t_\alpha$, although each $z_1^2 t_\alpha$ can be assigned any coefficient by choice of $p_2^{1\alpha}$.

In the one-parameter ($k = 1$) case, the single coefficient e_2^{021} ($= e_2^{02\alpha}$ with $\alpha = 1$) can be rotated by a_1 and scaled by $\mathbf{R}_{1\times1}$, so for $e_2^{021} \neq 0$, a normal form for the 3-manifold $\widehat{M}$ is

$$\begin{aligned} z_2 &= z_1\bar{z}_1 + (z_1^2 + \bar{z}_1^2)t_1 + O(4) \\ s_1 &= 0. \end{aligned} \tag{40}$$

This is simply the $\gamma = 0$ case of (31), and geometrically, the circular paraboloid shape with $\gamma = 0$ deforms into an elliptical paraboloid with $\gamma = t_1$.

However, for $k \geq 2$, there are k complex coefficients $e_2^{02\alpha}$, but there is still only one complex scalar a_1. The real matrix $\mathbf{R}$ acts on the real and imaginary parts of the coefficients; it will generically take the vector $(e_2^{021}, e_2^{022}, \ldots, e_2^{02k})$ to $(1, i, 0, \ldots, 0)$, so a generic real valued normal form is

$$\begin{aligned} z_2 &= z_1\bar{z}_1 + (\bar{z}_1^2 + z_1^2)t_1 + i(\bar{z}_1^2 - z_1^2)t_2 + O(4) \\ s_\alpha &= 0. \end{aligned} \tag{41}$$

The normal form (40) can be obtained as a non-generic case, for example, if the $e_2^{02\alpha}$ are all real.

EXAMPLE 5.15. Here we consider the elliptic cusp singularity, where the defining equation of M is $z_2 = z_1\bar{z}_1 + z_1^3 + \bar{z}_1^3 + O(4)$. The calculations from the previous Example go through without much change. In particular, the subgroup (38) is the same, except that to preserve the form of the cubic terms $z_1^3 + \bar{z}_1^3$, the complex scalar a_1 must be a cube root of unity. For $k \geq 2$, the matrix $\mathbf{R}$ is enough to normalize the generic coefficients $e_2^{02\alpha}$ even with just $a_1 = 1$, so the generic normal form is similar to (41):

$$(42) \quad \begin{aligned} z_2 &= z_1\bar{z}_1 + z_1^3 + \bar{z}_1^3 + (\bar{z}_1^2 + z_1^2)t_1 + i(\bar{z}_1^2 - z_1^2)t_2 + O(4) \\ s_\alpha &= 0. \end{aligned}$$

However, for $k = 1$, a nonzero complex coefficient e_2^{021} can be scaled to the unit circle by $\mathbf{R}_{1\times 1}$, but can only be rotated by $\sqrt[3]{1}$ (the factor $\frac{a_1}{\bar{a}_1}$ from (39)). So, the cubic normal form for a generic one-parameter deformation has a "modulus," a continuous invariant under the group $\mathcal{U}_{2,2,1}$:

$$(43) \quad \begin{aligned} z_2 &= z_1\bar{z}_1 + z_1^3 + \bar{z}_1^3 + (e^{i\theta}\bar{z}_1^2 + e^{-i\theta}z_1^2)t_1 + O(4),\ 0 \leq \theta < \tfrac{2\pi}{3} \\ s_1 &= 0. \end{aligned}$$

EXAMPLE 5.16. For the higher cusp cases, with $\gamma = 0$ and $3 < s < \infty$ in the normal forms for M, the nearby manifolds M_t are generic elliptic, with a small but nonzero Bishop invariant, except in degenerate cases. In the $k = 1$ case, there will be a similar root of unity phenomenon in the stabilizer of the degree s normal form.

The generic normal form for the $s = 4$, $k > 1$ case is again similar to (41):

$$(44) \quad \begin{aligned} z_2 &= z_1\bar{z}_1 + z_1^4 + \bar{z}_1^4 + (\bar{z}_1^2 + z_1^2)t_1 + i(\bar{z}_1^2 - z_1^2)t_2 + O(4) \\ s_\alpha &= 0 \end{aligned}$$

where the $O(4)$ quantity may contain degree 4 terms depending on t. For $k = 1$, the quartic normal form for a generic one-parameter deformation has a modulus:

$$(45) \quad \begin{aligned} z_2 &= z_1\bar{z}_1 + z_1^4 + \bar{z}_1^4 + (e^{i\theta}\bar{z}_1^2 + e^{-i\theta}z_1^2)t_1 + O(5),\ 0 \leq \theta < \pi \\ s_1 &= 0. \end{aligned}$$

Without going through the details, the degree 4 terms depending on t, as in (33), can be eliminated in this $k = 1$ case.

EXAMPLE 5.17. The hyperbolic $\gamma = \infty$ fold singularity has quadratic part $Q = z_1^2 + \bar{z}_1^2$ and, like the generic elliptic case, zero cubic part. As mentioned earlier, a version of Lemma 5.10 works, so that the terms $e_2^{10\alpha}z_1t_\alpha + e_2^{01\alpha}\bar{z}_1t_\alpha$ can be eliminated without introducing any cubic terms in z_1, $\bar{z}_1$ only. The defining equation (25) becomes $h_2 = z_1\bar{z}_1 + C(z_1, \bar{z}_1, t) + O(4)$, where C is the same as (29). The transformations preserving this partial normal form are:

$$\begin{aligned} \tilde{z}_1 &= a_1 z_1 + p_1^{\alpha} z_1 w_\alpha + p_1^{\alpha\beta} w_\alpha w_\beta \\ \tilde{z}_2 &= \overline{a_1}^2 z_1 + ((a_1)^2 - \overline{a_1}^2) z_1^2 + p_2^{1\alpha} z_1^2 w_\alpha + p_2^{2\alpha} z_2 w_\alpha + p_2^{1\alpha\beta} z_1 w_\alpha w_\beta \\ \tilde{w} &= \mathbf{R}\vec{w}, \end{aligned} \tag{46}$$

where a_1 is a nonzero complex scalar. The coefficients $p_1^{\alpha\beta}$, $p_2^{1\alpha\beta}$, $p_2^{1\alpha}$, $p_2^{2\alpha}$ can eliminate the coefficients $e_2^{01\alpha\beta}$, $e_2^{10\alpha\beta}$, $e_2^{20\alpha}$, $e_2^{02\alpha}$.

The stabilizer of the new partial normal form is the above group with the conditions $p_2^{2\alpha} = 2\overline{a_1 p_1^{\alpha}}$, $p_2^{1\alpha} = 2a_1 p_1^{\alpha} - 2\overline{a_1 p_1^{\alpha}}$, and the new defining equation after such a transformation with $\mathbf{R} = \mathbb{1}$ satisfies:

$$C(z_1, \bar{z}_1, t) = \frac{\overline{a_1}}{a_1} e_2^{11\alpha} z_1 \bar{z}_1 t_\alpha.$$

Note that none of the terms in (46) contributes any quantities of the form $z_1\bar{z}_1 t_\alpha$.

In the one-parameter ($k = 1$) case, the single coefficient e_2^{021} ($= e_2^{02\alpha}$ with $\alpha = 1$) can be rotated by a_1 and scaled by $\mathbf{R}_{1\times 1}$, so for $e_2^{021} \neq 0$, a normal form for the 3-manifold $\widehat{M}$ is

$$\begin{aligned} z_2 &= z_1^2 + \bar{z}_1^2 + t_1 z_1 \bar{z}_1 + O(4) \\ s_1 &= 0. \end{aligned} \tag{47}$$

For $k \geq 2$, there are k complex coefficients $e_2^{02\alpha}$, but there is still only one complex scalar a_1. As in Example 5.14, the real matrix $\mathbf{R}$ acts on the real and imaginary parts of the coefficients, generically taking the vector $(e_2^{021}, e_2^{022}, \ldots, e_2^{02k})$ to $(1, i, 0, \ldots, 0)$, so a generic normal form is

$$\begin{aligned} z_2 &= \bar{z}_1^2 + z_1^2 + (t_1 + it_2) z_1 \bar{z}_1 + O(4) \\ s_\alpha &= 0. \end{aligned} \tag{48}$$

The normal form (47) can be obtained as a non-generic case, for example, if the $e_2^{02\alpha}$ are all real.

Example 5.18. The nondegenerate parabolic CR singularity is an important case; it is well-known that it is an unstable singularity for a surface $M = M_{\mathbf{0}}$ in $\mathbb{C}^2$, in the sense that, unlike the elliptic or hyperbolic properties from Example 5.11, one expects that surfaces M_t near M will generally not all have parabolic singularities. The following unfolding calculation makes this more precise.

For $M \subseteq \mathbb{C}^2$ defined by $z_2 = Q + C + O(4)$ with quadratic normal form $Q = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2)$, the transformations (9) of $\mathbb{C}^2$ stabilizing the quadratic normal form are $\tilde{z}_1 = a_1 z_1 + p_1$, $\tilde{z}_2 = (a_1)^2 z_2 + p_2$, with p_1 weight 2, p_2 weight 3, and a_1 real and nonzero. For $C = i(z_1 - \bar{z}_1)z_1\bar{z}_1$ as in Proposition 5.1, the stabilizer of the cubic normal form is $\tilde{z}_1 = z_1 + a_2 z_2 + p_1$, $\tilde{z}_2 = z_2 + p_2$, with $a_1 = 1$, a_2 purely imaginary, p_1 weight 3, and p_2 weight 4.

For $\widehat{M} \subseteq \mathbb{C}^{2+k}$ as in (26), a transformation (27–28) of the form $\tilde{z}_1 = a_1 z_1 + a^\alpha w_\alpha$, $\tilde{z}_2 = (a_1)^2 z_2$, $\tilde{w} = \vec{w}$ transforms the term $e_2^{01\alpha}\bar{z}_1 t_\alpha$ into $(a_1 e_2^{01\alpha} - a^\alpha - \overline{a^\alpha})\bar{z}_1 t_\alpha$. So, unlike every one of the previous Examples, only the real part of a^α contributes to the normalization. Canceling the real parts of $e_2^{01\alpha}$ leaves a vector of purely imaginary coefficients, which is either the zero vector, or can be normalized by $\mathbf{R}_{k\times k}$ to $-i\cdot(1, 0, \ldots, 0)$. Then, a transformation $\tilde{z}_2 = z_2 + p_2^\alpha z_1 w_\alpha$ gives a real valued quadratic part for the following normal form for $\widehat{M}$:

$$z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)t_1 + C(z_1, \bar{z}_1, t) + O(4), \tag{49}$$

where $C(z_1, \bar{z}_1, \mathbf{0}) = i(z_1 - \bar{z}_1)z_1\bar{z}_1$ is as in Proposition 5.1 and the other terms in C are as in (29). This normal form already resembles the normal form for a parabolic CR singularity of a real n-manifold in $\mathbb{C}^n$, as in Proposition 5.8.

The transformation group (27–28) contributing to the normalization of cubic terms, without introducing quadratic terms, is

$$\begin{aligned}
\tilde{z}_1 &= z_1 + a_2 z_2 + a^\alpha w_\alpha + p_1(z_1, z_2, \vec{w}) \\
\tilde{z}_2 &= z_2 + p_2^{1\alpha} w_1 w_\alpha + p_2(z_1, z_2, \vec{w}) \\
\tilde{w}_1 &= w_1 + P_1(\vec{w}) \\
\tilde{w}_j &= r_j^\alpha w_\alpha + P_j(\vec{w}),\ j = 2, \ldots, k
\end{aligned} \tag{50}$$

with a_2 and a^α purely imaginary and $p_2^{1\alpha} = 2ia^\alpha$.

If we use a particular transformation of the above form with $a_2 = 0$, $p_1 = p_1^\alpha z_1 w_\alpha$, $p_2 = p_2^\alpha z_2 w_\alpha$, $\tilde{w} = \vec{w}$, the coefficient of $e_2^{02\alpha}\bar{z}_1^2 t_\alpha$ is transformed to

$$e_2^{02\alpha} + \frac{1}{2}p_2^\alpha - \overline{p_1^\alpha} + ia^\alpha,$$

and the coefficient of $e_2^{11\alpha} z_1\bar{z}_1 t_\alpha$ is transformed to

$$e_2^{11\alpha} + p_2^\alpha - p_1^\alpha - \overline{p_1^\alpha} - 4ia^\alpha.$$

The a^α quantities come from transforming the cubic term $i(z_1 - \bar{z}_1)z_1\bar{z}_1$, so this step is where the nondegeneracy property of M is used. Setting both quantities to zero gives a system of two equations that has a solution: solving the first for p_2^α and plugging into the second gives

$$\frac{1}{2}(p_1^\alpha - \overline{p_1^\alpha}) + 3ia^\alpha - \frac{1}{2}e_2^{11\alpha} + e_2^{02\alpha} = 0,$$

where the imaginary part of p_1^α and the real quantity $3ia^\alpha$ are enough to cancel the given complex part. This cancellation, using the linear coefficients a^α, takes the normalization a little further than that of [**Webster**$_2$]; in the nondegenerate parabolic case of Proposition 5.8, the coefficients η_n^σ can be canceled by a transformation of $\mathbb{C}^n$ in $\mathcal{B}_{n,n}$.

The previous step cancels the $e_2^{02\alpha}\bar{z}_1^2t_\alpha$ and $e_2^{11\alpha}z_1\bar{z}_1t_\alpha$ terms, but also changes the coefficients of $\bar{z}_1t_\alpha t_\beta$ and other terms. We reassign the same e_2 labels to the new coefficients. Another particular transformation,

$$\begin{aligned}\tilde{z}_1 &= z_1 + p_1^{\alpha\beta}w_\alpha w_\beta\\ \tilde{z}_2 &= z_2 + p_2^{1\alpha\beta}z_1w_\alpha w_\beta\\ \tilde{w}_1 &= w_1 + P_1^{\alpha\beta}w_\alpha w_\beta\\ \tilde{w}_j &= w_j,\ j = 2,\ldots,k\end{aligned}$$

transforms the coefficient of $e_2^{01\alpha\beta}\bar{z}_1t_\alpha t_\beta$ to

$$e_2^{01\alpha\beta} - (p_1^{\alpha\beta} + \overline{p_1^{\alpha\beta}}) + iP_1^{\alpha\beta},$$

without reintroducing either of the previously normalized terms $z_1\bar{z}_1t_\alpha$, $\bar{z}_1^2t_\alpha$. The real part of $p_1^{\alpha\beta}$ and the real coefficients $P_1^{\alpha\beta}$ can eliminate all the $e_2^{01\alpha\beta}\bar{z}_1t_\alpha t_\beta$ terms. The coefficients $p_2^{1\alpha\beta}$ can eliminate the $e_2^{10\alpha\beta}z_1t_\alpha t_\beta$ terms and similarly, the remaining terms, not involving $\bar{z}_1$, can be eliminated by transformations of z_2, so the generic normal form for $\widehat{M}$ has cubic part not depending on t:

$$z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)t_1 + i(z_1 - \bar{z}_1)z_1\bar{z}_1 + O(4).$$

The last step with $P_1^{\alpha\beta}$ required the nondegeneracy of the quadratic normal form (49).

Although the case where the unfolding $\widehat{M}$ has a degenerate normal form, in which the quadratic terms do not depend on t, is not considered here, it is notable that the one-parameter unfolding resembling (31):

$$z_2 = z_1\bar{z}_1 + (\frac{1}{2} + t_1)\cdot(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)z_1\bar{z}_1 + O(4),$$

is not u-equivalent to a nondegenerate unfolding of the form (49), since by the above calculation, the cubic terms $(z_1^2 + \bar{z}_1^2)t_1$ can be eliminated without introducing any quadratic terms.

EXAMPLE 5.19. For $M \subseteq \mathbb{C}^2$ with a degenerate parabolic CR singularity, defined by

$$z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + \eta(z_1^2 + \bar{z}_1^2)z_1\bar{z}_1 + O(5),$$

the normal form has zero cubic part, and the quartic terms are $\eta(z_1^2+\bar{z}_1^2)z_1\bar{z}_1$, $\eta=\pm1,0$. The calculations for a normal form for $\widehat{M}$ proceed at first in the same way as in the previous Example, so the generic unfolding has a normal form with the same quadratic terms as (49). We will consider only the $\eta=\pm1$ cases.

The transformation group (27–28) contributing to the normalization of cubic terms, without introducing quadratic terms, is similar to (50):

$$\begin{aligned}
\tilde{z}_1 &= a_1z_1+a_2z_2+a^\alpha w_\alpha+p_1(z_1,z_2,\vec{w}) \\
\tilde{z}_2 &= z_2+p_2^{1\alpha}w_1w_\alpha+p_2(z_1,z_2,\vec{w}) \\
\tilde{w}_1 &= a_1w_1+P_1(\vec{w}) \\
\tilde{w}_j &= r_j^\alpha w_\alpha+P_j(\vec{w}),\ \ j=2,\dots,k
\end{aligned} \tag{51}$$

with $a_1=\pm1$, a_2 and a^α purely imaginary, and $p_2^{1\alpha}=2ia_1a^\alpha$.

If we apply the above transformation in its most general form except that $\tilde{w}_j=w_j$ for $j=2,\dots,k$, and label some terms $p_1^\alpha z_1w_\alpha$, $p_2^\alpha z_2w_\alpha$, the coefficient of $e_2^{02\alpha}\bar{z}_1^2t_\alpha$ is transformed in the $\alpha=1$ case to

$$a_1e_2^{021}+\frac{a_1}{2}p_2^1-\overline{p_1^1}-ia_2,$$

and for $\alpha=2,\dots,k$ to

$$e_2^{02\alpha}+\frac{1}{2}p_2^\alpha-a_1\overline{p_1^\alpha}.$$

The coefficient of $e_2^{11\alpha}z_1\bar{z}_1t_\alpha$ is transformed in the $\alpha=1$ case to

$$a_1e_2^{111}+a_1p_2^1-(p_1^1+\overline{p_1^1})-2ia_2,$$

and for $\alpha=2,\dots,k$ to

$$e_2^{11\alpha}+p_2^\alpha-a_1(p_1^\alpha+\overline{p_1^\alpha}).$$

Unlike the previous Example, the coefficients a^α do not contribute to these terms. Even including the imaginary coefficient a_2, the transformation group does not have enough degrees of freedom to cancel both the $e_2^{02\alpha}$ and $e_2^{11\alpha}$ terms simultaneously. We choose to cancel the $e_2^{11\alpha}$ coefficients and the imaginary parts of the $e_2^{02\alpha}$ coefficients (this is a different choice from the normalization of Proposition 5.8). Then the r_α^β block can normalize the real coefficients $e_2^{02\alpha}$ to $(0,1,0,\dots,0)$, generically if $k\geq2$. So, the generic k-parameter unfolding of a degenerate parabolic CR singularity has the following cubic normal form:

$$\begin{aligned}
z_2 &= z_1\bar{z}_1+(\frac{1}{2}+t_2)\cdot(z_1^2+\bar{z}_1^2)+i(z_1-\bar{z}_1)t_1 \\
&\quad \pm(z_1^2+\bar{z}_1^2)z_1\bar{z}_1+F(z_1,\bar{z}_1,t)+O(5),\quad k\geq2,
\end{aligned} \tag{52}$$

where the quartic terms $F = F(z_1, \bar{z}_1, t)$ depending on t are the same as (33).

For $k = 1$, or in a degenerate case where the $e_2^{02\alpha}$ terms depend only on t_1, the real coefficient cannot be re-scaled except by $a_1 = \pm 1$, so there is a real modulus. The remaining cubic coefficients can be normalized exactly as in the previous Example (assuming the nondegeneracy of the quadratic part so there is a $i(z_1 - \bar{z}_1)t_1$ term), so the generic 1-parameter unfolding (or a non-generic k-parameter unfolding), of a degenerate parabolic CR singularity has the following cubic normal form:

$$\begin{aligned} z_2 \;=\; & z_1\bar{z}_1 + (\frac{1}{2} + \varepsilon t_1) \cdot (z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)t_1 \\ & \pm (z_1^2 + \bar{z}_1^2)z_1\bar{z}_1 + F(z_1, \bar{z}_1, t) + O(5), \quad \varepsilon \geq 0. \end{aligned} \tag{53}$$

The modulus $\varepsilon \geq 0$ can be interpreted as a choice of line through the origin in the (t_1, t_2)-plane in the parameter space for (52).

Returning to the normal form (52), the quantity $F = F(z_1, \bar{z}_1, t)$ can in fact be normalized to zero by a transformation of the form (51). This elimination of F is similar to the result from Example 5.13, but the details of the lengthy calculation are different, and again only sketched here. Let $a_1 = 1$, $a_2 = 0$ in (51), let p_1 and p_2 be as in (34, 35) so they include (among others) terms labeled

$$p_1^{20\alpha} z_1^2 w_\alpha + p_1^{01\alpha} z_2 w_\alpha + p_1^{1\alpha\beta} z_1 w_\alpha w_\beta + p_1^{\mathbf{K}} w^{\mathbf{K}},$$

$$p_2^{11\alpha} z_1 z_2 w_\alpha + p_2^{01\alpha\beta} z_2 w_\alpha w_\beta,$$

and let $\tilde{w}_1 = w_1 + P_1^{\mathbf{K}} w^{\mathbf{K}}$, $\tilde{w}_2 = w_2 + P_2^{\alpha\beta} w_\alpha w_\beta$, $\tilde{w}_\alpha = w_\alpha$, $\alpha = 3, \ldots, k$. Then the system of equations from a comparison of coefficients is solvable: $p_1^{20\alpha}$ cancels the $\bar{z}_1^3 t_\alpha$ terms of F, $p_2^{11\alpha}$ cancels the $z_1\bar{z}_1^2 t_\alpha$ terms, $p_2^{01\alpha\beta}$ cancels the $\bar{z}_1^2 t_\alpha t_\beta$ terms, the real part of $p_1^{01\alpha}$ and the imaginary coefficients a^α cancel the $z_1^2\bar{z}_1 t_\alpha$ terms, the imaginary part of $p_1^{1\alpha\beta}$ and the real coefficients $P_2^{\alpha\beta}$ cancel the $z_1\bar{z}_1 t_\alpha t_\beta$ terms, and the real part of $p_1^{\mathbf{K}}$ and the real coefficients $P_1^{\mathbf{K}}$ cancel the $\bar{z}_1 t^{\mathbf{K}}$ terms. The other terms of F do not depend on $\bar{z}_1$ and can be canceled by corresponding terms remaining in p_2.

In the $k = 1$ case, $F(z_1, \bar{z}_1, t_1)$ can also be eliminated from (53) (for any $\varepsilon \geq 0$), but in this case the analogous computation uses the term $p_1^2 z_1 z_2$ from (21) with a real coefficient in place of $P_2^{\alpha\beta} = P_2^{11}$; the quantity $p_2(z_1, z_2)$ from (21) can compensate for the quartic terms in $z_1, \bar{z}_1$ this introduces.

EXAMPLE 5.20. As an example of a one-parameter family of surfaces which is so degenerate that it does not exhibit any of the unfoldings considered in Examples 5.11 – 5.19, but which is interesting from a global point of view as a case where the index sum formula (7) is nonzero, we consider the following map using homogeneous coordinates as a family of embeddings of $\mathbb{C}P^1$ in $\mathbb{C}P^2$, parametrized by $t_1 \in \mathbb{R}$.

$$\begin{aligned} G : \mathbb{C}P^1 \times \mathbb{R} &\to \mathbb{C}P^2 \\ ([z_0 : z_1], t_1) &\mapsto [z_0 \cdot (z_0\bar{z}_0 + z_1\bar{z}_1) \cdot (2z_0\bar{z}_0 + z_1\bar{z}_1 - z_0\bar{z}_1 - \bar{z}_0 z_1) \\ &\qquad : z_1 \cdot (z_0\bar{z}_0 + z_1\bar{z}_1) \cdot (2z_0\bar{z}_0 + z_1\bar{z}_1 - z_0\bar{z}_1 - \bar{z}_0 z_1) \\ &\qquad : t_1 z_0 \cdot (2z_0\bar{z}_0 + 3z_1\bar{z}_1) \cdot (z_0\bar{z}_0 - z_0\bar{z}_1 - \bar{z}_0 z_1)]. \end{aligned}$$

At $t_1 = 0$, this is a holomorphic embedding $[z_0 : z_1] \mapsto [z_0 : z_1 : 0]$ (so the image $G(\mathbb{C}P^1 \times \{0\})$ has a degenerate CR singularity at every point). For a fixed value of t_1, the restriction to one affine neighborhood in the domain $\mathbb{C}P^1$ has image contained in an affine neighborhood of $\mathbb{C}P^2$:

$$[1 : z_1] \mapsto \left[1 : z_1 : t_1 \cdot (2 + 3|z_1|^2) \cdot \left(\frac{1}{1 + |z_1|^2} - \frac{1}{1 + |z_1 - 1|^2} \right)\right].$$

The image fits inside $\mathbb{R}^3 \subseteq \mathbb{C}^2$, and for $t_1 \neq 0$, it has two elliptic points and no hyperbolic points. This is the entire image $G(\mathbb{C}P^1 \times \{t_1\})$, except for one point at infinity. A restriction to another pair of affine neighborhoods is:

$$[z_0 : 1] \mapsto \left[z_0 : 1 : \frac{t_1 z_0 \cdot (2z_0\bar{z}_0 + 3) \cdot (z_0\bar{z}_0 - z_0 - \bar{z}_0)}{(z_0\bar{z}_0 + 1) \cdot (2z_0\bar{z}_0 + 1 - z_0 - \bar{z}_0)}\right],$$

which for $t_1 \neq 0$ is in standard position (8), but not flattened inside $\mathbb{R}^3$. Observing the $z_0\bar{z}_0$ term in the numerator, the view in this neighborhood shows a third elliptic point at $[0 : 1 : 0]$. This example is consistent with the calculation of [**F**] §7 (see also [**IS**] App. IV), showing that there is a small perturbation of a complex projective line in $\mathbb{C}P^2$ into a real surface in general position, with index sum $\sum \operatorname{ind}(\mathbf{x}) = \int c_1(F, J) = 3$ (where (F, J) is the restriction of $T\mathbb{C}P^2$ to the surface), and exactly three positively oriented elliptic points.

5.3. Visualization.

From the global theory of immersed compact surfaces discussed in Section 2, one expects CR singularities disappearing (or appearing) in a deformation will generally cancel (or be created) in pairs: one elliptic and one hyperbolic, with the same orientation. Example 5.18 represents a local version of this phenomenon, and the normal form (49) for the unfolding flattens (at least up to $+O(3)$) to fit in $\mathbb{R}^3 \times \mathbb{R}$, so it can be illustrated by a series of pictures.

In fact, for most of the examples of unfoldings $\widehat{M}$ in the previous Subsection, we were able to choose normal forms so that for each t, the slice M_t is contained in $\mathbb{R}^3 \times \{t\}$, at least up to some degree — there may be higher degree terms with nonzero imaginary parts. Here, we truncate the defining equations to their lower-degree, holomorphically flattened normal forms, to view a graphical representation of the unfolding of $M \subseteq \mathbb{R}^3 \times \{\mathbf{0}\}$. In the following Figures, we visualize $\mathbb{R}^3 \subseteq \mathbb{C}^2$ as the x_1, y_1, x_2-subspace, where the x_1, y_1-plane is the z_1-axis and the horizontal planes parallel to it are also complex lines. Each of the normal forms for surfaces M in Proposition 5.1 is in standard position, so that the CR singularity is at the origin, and the x_1, y_1-plane is the complex tangent line. For some unfoldings $\widehat{M}$, a slice M_t could have a CR singularity at some point other than the origin of $\mathbb{C}^2 \times \{t\}$, or could have a complex tangent line other than the z_1-axis, although when M_t fits inside $\mathbb{R}^3 \times \{t\}$, the only possible complex tangent lines are horizontal planes parallel to the z_1-axis. So, in these pictures rendered by [**POV-Ray**], the CR singularities will be visible as horizontal tangents, i.e., the familiar critical points of the real height function in the x_2 direction.

EXAMPLE 5.21. In Figure 1, the elliptic CR singularity of a surface

$$z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2),\ 0 < \gamma < \frac{1}{2}$$

is shown as the vertex of an elliptic paraboloid, tangent to the z_1-axis at the origin. It meets complex lines (horizontal planes) in ellipses, or in a single point, at the CR singularity.

EXAMPLE 5.22. In Figure 2, the hyperbolic CR singularity of a surface

$$z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2),\ \frac{1}{2} < \gamma < \infty$$

is the saddle point, located at the crossing-point of the $\times$-shaped degenerate conic level set. The other level curves are hyperbolas. The

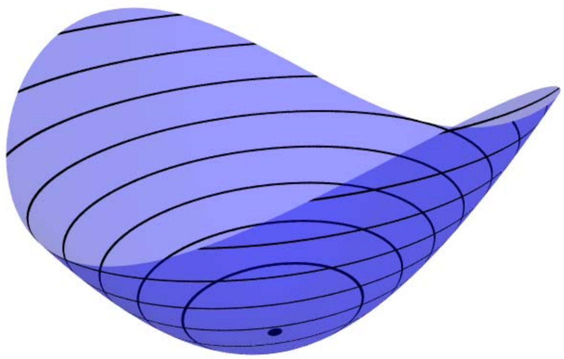

FIGURE 1. Elliptic CR singularity

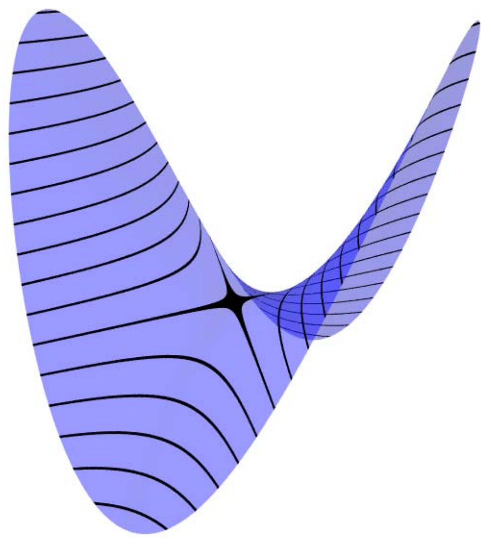

FIGURE 2. Hyperbolic CR singularity

angle at the singular point, formed by the intersection of the surface with the complex tangent line, is clearly a biholomorphic invariant.

Looking at the quadric models for generic elliptic and hyperbolic singularities, their non-trivial intersections with complex lines parallel to the x_1, y_1-plane are conics of a constant eccentricity. Varying the parameter t_1 near 0 in the generic unfolding (31) varies the eccentricity, and the angle in the hyperbolic case.

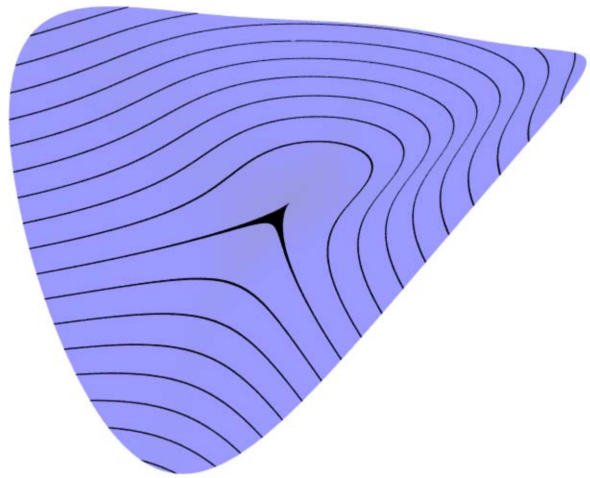

FIGURE 3. Nondegenerate parabolic CR singularity

EXAMPLE 5.23. In Figure 3, the nondegenerate parabolic CR singularity of the surface

$$z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)z_1\bar{z}_1$$

is located at the cusp of the $\succ$-shaped singular level set.

A nondegenerate unfolding $\widehat{M}$ of the above parabolic singularity, as in Example 5.18 with $k = 1$, is the cubic normal form:

$$z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)t_1 + i(z_1 - \bar{z}_1)z_1\bar{z}_1. \tag{54}$$

For $t = (t_1)$ with $t_1 < 0$, there is a pair of CR singularities in the deformed surface M_t, one elliptic and one hyperbolic, as shown in Figure 4, and for $t_1 > 0$, the surface M_t is totally real in $\mathbb{C}^2 \times \{t\}$, showing no horizontal tangents in Figure 5. The unfolding in Figures 3–5 is similar to Figure 1 of [**Callahan**], illustrating graphs of a family of smooth functions undergoing a "catastrophe" as the number of local minimum points varies with the parameter t.

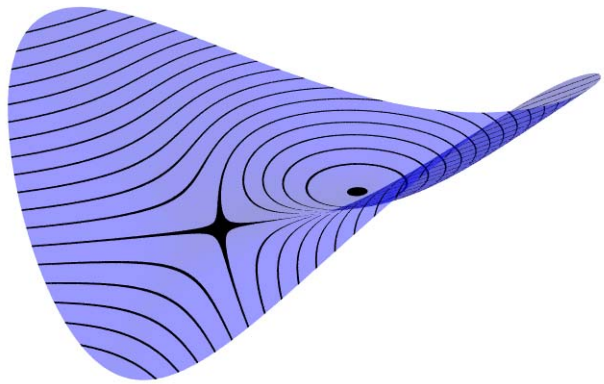

FIGURE 4. Deformation, $t_1 < 0$, of nondegenerate parabolic CR singularity into an elliptic/hyperbolic pair of CR singularities

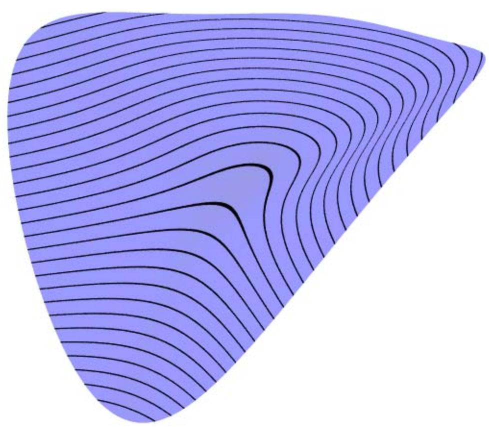

FIGURE 5. Totally real deformation, $t_1 > 0$, of nondegenerate parabolic CR singularity

To more precisely analyze the local geometry of the unfolding, we return to the complexification construction of Section 4. Corresponding to the smooth real variety $\widehat{M} \subseteq \mathbb{C}^{2+1}$ is a smooth complex variety $\widehat{M_c} \subseteq \mathbb{C}^{2(2+1)}$, parametrized by $\Sigma : \mathbb{C}^{2+1} \to \mathbb{C}^{2(2+1)}$. Let $\mathbb{C}^{2+1}$ have coordinates (z_1, ζ_1, ω_1). The image of $\widehat{M_c}$ under the projection $\pi : \mathbb{C}^{2(2+1)} \to \mathbb{C}^{2+1}$ is all of $\mathbb{C}^{2+1}$; the map $\pi \circ \Sigma$ as in (19):

$$\begin{aligned}\pi \circ \Sigma \quad &: \quad \mathbb{C}^{2+1} \to \mathbb{C}^{2+1} : (z_1, \zeta_1, \omega_1) \\ &\mapsto \quad (z_1, z_1\zeta_1 + \frac{1}{2}(z_1^2 + \zeta_1^2) + i(z_1 - \zeta_1)\omega_1 + i(z_1 - \zeta_1)z_1\zeta_1, \omega_1)^T\end{aligned}$$

is a two-to-one branched cover near $(\vec{0},\mathbf{0})$. At each point in the domain $\mathbb{C}^{2+1}$, the complex Jacobian of this polynomial map either will have full rank, 3, or will be singular, with rank 2. The singular locus in the domain is the complex affine quadric variety

$$\{z_1+\zeta_1-i\omega_1+iz_1^2-2iz_1\zeta_1=0\},$$

and, near the origin, this is also exactly the locus where $\pi\circ\Sigma$ is one-to-one.

As in (18), let $\boldsymbol{\delta}$ denote the inclusion of the totally real subspace

$$\{\zeta_1=\bar{z}_1,\omega_1=\overline{\omega_1}=t_1\}$$

in $\mathbb{C}^{2+1}$, so $\widehat{M}$ is the image of $\pi\circ\Sigma\circ\boldsymbol{\delta}$. The intersection of this real subspace with the singular locus is

$$\{(x_1+iy_1,x_1-iy_1,t_1):2x_1-2x_1y_1=0,t_1+x_1^2+3y_1^2=0\}.$$

The first condition factors as $2x_1(1-y_1)=0$, and if we are only considering points in $\widehat{M}$ near $(\vec{0},\mathbf{0})$, the solution set is $\{x_1=0,y_1=\pm\sqrt{-t_1/3}\}$. So, the candidates for CR singular points in $\widehat{M}$ near $(\vec{0},\mathbf{0})$ are of the form

$$\{(z_1,z_2,t_1)^T=(\pm i\sqrt{-t_1/3},\pm 4(-t_1/3)^{3/2},t_1)^T\}\subseteq\widehat{M}\subseteq\mathbb{C}^{2+1},$$

or in implicit form, this locus is the real twisted cubic curve $\{t_1+3y_1^2=0,x_2-4y_1^3=0\}$ in the y_1,x_2,t_1 real coordinate subspace, tangent to the y_1-axis.

In fact, this real curve is the CR singular locus N_1 of $\widehat{M}$ near $(\vec{0},\mathbf{0})$, as can be seen by fixing $t=(t_1)$ with $t_1<0$, and moving the slice M_t into standard position in $\mathbb{C}^2\times\{t\}$. Corresponding to one candidate point in the slice M_t, we first consider a translation of the form $\tilde{z}_1=z_1-i\sqrt{-t_1/3}$. Substituting $\tilde{z}_1+i\sqrt{-t_1/3}$ for z_1 in RHS of (54) (and dropping the tilde) gives:

$$4(-\frac{t_1}{3})^{\frac{3}{2}}+(1-4\sqrt{-\frac{t_1}{3}})z_1\bar{z}_1+(\frac{1}{2}+\sqrt{-\frac{t_1}{3}})(z_1^2+\bar{z}_1^2)+i(z_1-\bar{z}_1)z_1\bar{z}_1,$$

so translating and then rescaling z_2 gives the normal form:

$$z_2=z_1\bar{z}_1+\gamma_t(z_1^2+\bar{z}_1^2)+O(3),$$

where $\gamma_t=\frac{\frac{1}{2}+\sqrt{-t_1/3}}{1-4\sqrt{-t_1/3}}$. It can be concluded that the tangent plane to M_t at the original candidate point is a complex line parallel to the z_1-axis (as expected from Figure 4), and the CR singularity is hyperbolic for $\frac{-3}{16}<t_1<0$. The other substitution, $\tilde{z}_1-i\sqrt{-t_1/3}$,

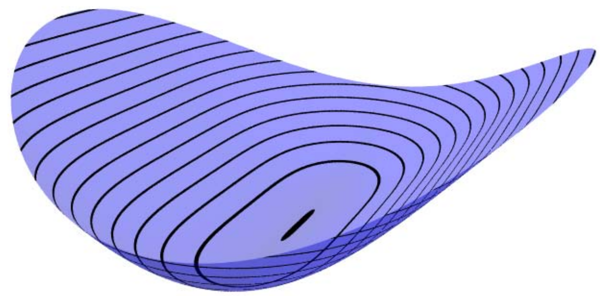

FIGURE 6. Degenerate parabolic CR singularity, $\eta = -1$

translating the other candidate point, leads to a similar calculation but with $\gamma_t = \frac{\frac{1}{2}-\sqrt{-t_1/3}}{1+4\sqrt{-t_1/3}}$, at the elliptic CR singularity.

The property that the complexified parametrization $\pi \circ \Sigma$ is a two-to-one ramified map (locally, near the parabolic point) is shared with the generic elliptic and hyperbolic points of real n-manifolds in $\mathbb{C}^n$, as considered by [**MW**]. The observation that the curve N_1 is tangent to the complex z_1-axis is an example of the intrinsic characterization of the parabolic point of the real 3-manifold $\widehat{M}$ in $\mathbb{C}^3$, as in Remark 5.9.

EXAMPLE 5.24. A surface with a degenerate parabolic CR singularity has a quartic normal form, as in Proposition 5.1:

$$(55)\ z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) + \eta(z_1^2 + \bar{z}_1^2)z_1\bar{z}_1 + O(5), \ \eta = +1, -1, 0.$$

We will consider the $\eta = \pm 1$ cases only, and drop the $O(5)$ part in the following illustrations.

In the $\eta = -1$ case, (55) simplifies to

$$x_2 = 2x_1^2 - 2x_1^4 + 2y_1^4,$$

the graph of which is a surface M^- with a local minimum at the origin in $\mathbb{R}^3$ (the $y_2 = 0$ subspace) in Figure 6.

The critical point of the height function is degenerate in the sense of Morse Theory, being flat in the y_1 direction.

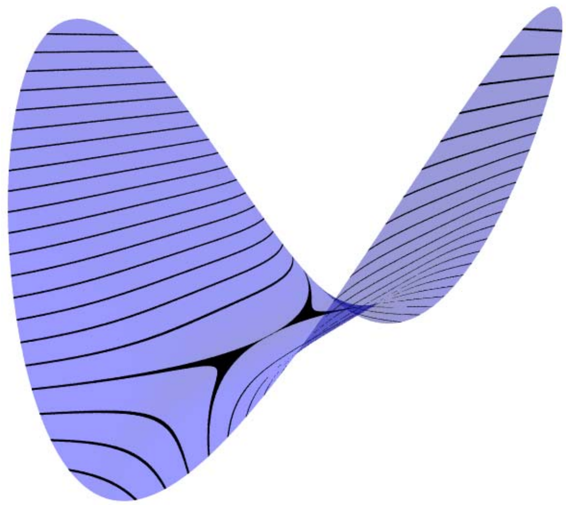

FIGURE 7. Degenerate parabolic CR singularity, $\eta = +1$

In the $\eta = +1$ case, (55) simplifies to

$$x_2 = 2x_1^2 + 2x_1^4 - 2y_1^4,$$

a surface M^+ which has a degenerate saddle point at the origin in $\mathbb{R}^3$ in Figure 7.

The level set of a nondegenerate saddle point has a $\times$ shape as in Figure 2, but the level set intersecting the critical point of the height function in Figure 7 is a self-tangent curve. The level sets of Figures 1–3, 6, and 7 resemble the sketches in [**Martinet**] §3.XII.5, of level curves of smooth functions of two real variables with critical points.

One expects from Morse Theory that smooth perturbations of the surface M^+ or M^- in $\mathbb{R}^3$ will replace one of these degenerate critical points by one or more nondegenerate critical points. The normal form calculation (52) from Example 5.19 shows that a generic 2-parameter unfolding of $M^\pm$ in $\mathbb{C}^2$ is u-equivalent (up to fourth degree, continuing to neglect $O(5)$) to the following $\widehat{M}^\pm \subseteq \mathbb{R}^3 \times \mathbb{R}^2 \subseteq \mathbb{C}^2 \times \mathbb{R}^2$, where each slice $M_t^\pm$ is visible in $\mathbb{R}^3 \times \{t\}$:

$$(56)\, z_2 \;=\; z_1\bar{z}_1 + (\frac{1}{2} + t_2) \cdot (z_1^2 + \bar{z}_1^2) + i(z_1 - \bar{z}_1)t_1 \pm (z_1^2 + \bar{z}_1^2)z_1\bar{z}_1$$

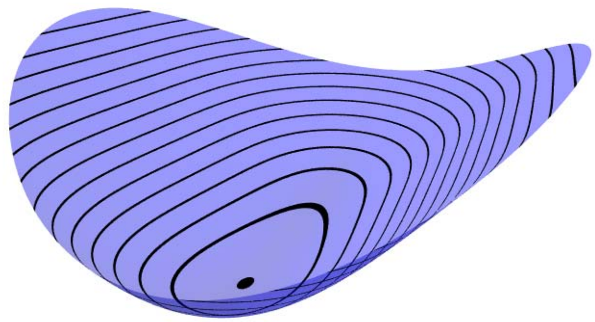

FIGURE 8. Deformation of the $\eta = -1$ degenerate parabolic CR singularity, $t_1 \neq 0$, $t_2 = 0$

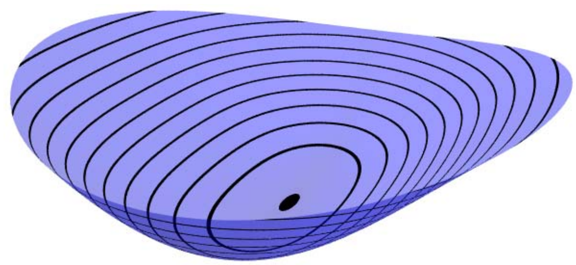

FIGURE 9. Deformation of the $\eta = -1$ degenerate parabolic CR singularity, $t_1 = 0$, $t_2 < 0$

In the $\eta = -1$ case, varying the first unfolding parameter t_1 while fixing the second, $t_2 = 0$, introduces a linear (in y_1) term in the real defining equation:

$$x_2 = 2x_1^2 - 2x_1^4 + 2y_1^4 - 2t_1 y_1.$$

The slices $M_t^- = M_{(t_1,0)}^-$ have a nondegenerate elliptic point away from the origin, as in Figure 8.

Continuing with $\eta = -1$, fixing the first unfolding parameter $t_1 = 0$ while choosing nonzero values for the second, t_2, introduces a quadratic (in x_1, y_1) term in the real defining equation:

$$x_2 = 2x_1^2 - 2x_1^4 + 2y_1^4 + 2t_2(x_1^2 - y_1^2).$$

For $t_2 < 0$, the slices $M_t^- = M_{(0,t_2)}^-$ have a nondegenerate elliptic point at the origin, as in Figure 9, but for $t_2 > 0$, there is a hyperbolic CR singularity at the origin and there are two elliptic points nearby, as in Figure 10.

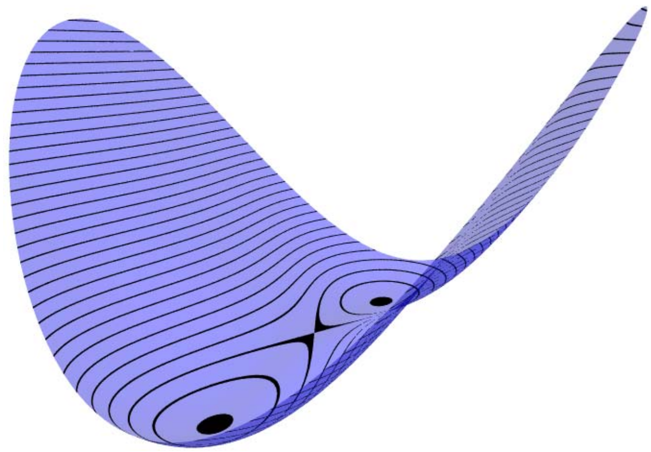

FIGURE 10. Deformation of the $\eta = -1$ degenerate parabolic CR singularity, $t_1 = 0$, $t_2 > 0$

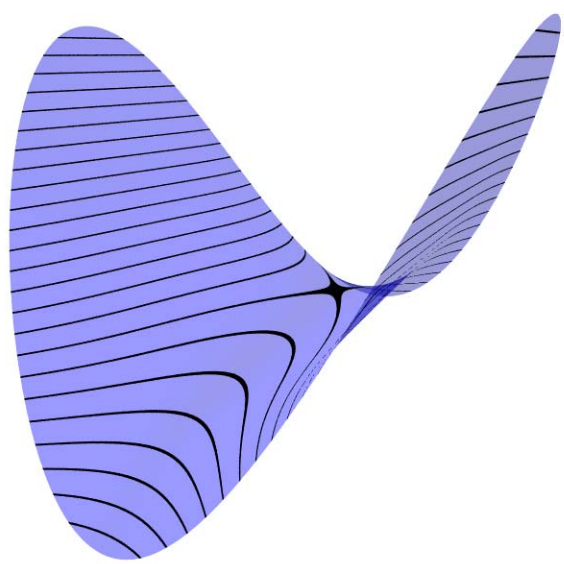

FIGURE 11. Deformation of the $\eta = +1$ degenerate parabolic CR singularity, $t_1 \neq 0$, $t_2 = 0$

In the $\eta = +1$ case, varying the first unfolding parameter t_1 while fixing the second, $t_2 = 0$, gives slices $M_t^+ = M_{(t_1,0)}^+$ with a nondegenerate hyperbolic point away from the origin, as in Figure 11.

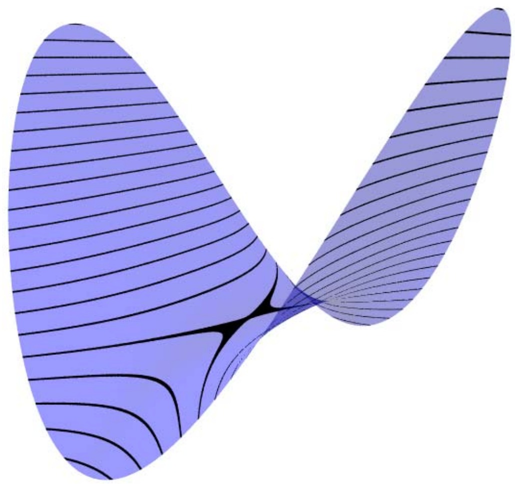

FIGURE 12. Deformation of the $\eta = +1$ degenerate parabolic CR singularity, $t_1 = 0$, $t_2 > 0$

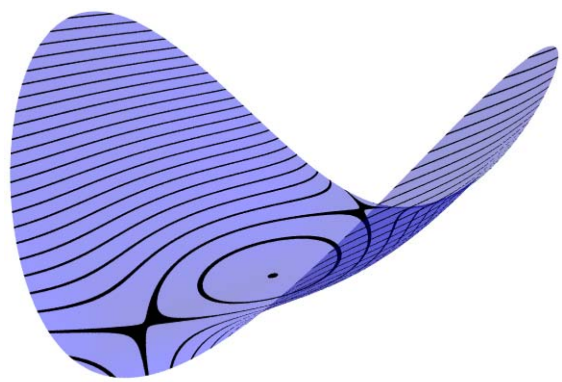

FIGURE 13. Deformation of the $\eta = +1$ degenerate parabolic CR singularity, $t_1 = 0$, $t_2 < 0$

Continuing with $\eta = +1$, fixing the first unfolding parameter $t_1 = 0$ while varying the second, t_2, gives slices $M_t^+ = M_{(0,t_2)}^+$ with one nondegenerate hyperbolic point at the origin for $t_2 > 0$, as in Figure 12, but for $t_2 < 0$, there is an elliptic CR singularity at the origin and two hyperbolic points nearby, as in Figure 13.

All these unfoldings of the degenerate and nondegenerate parabolic points are consistent with the conservation of topological index sum of real surfaces in $\mathbb{C}^2$ under perturbations as in Section 2. The unfolding of the nondegenerate parabolic point has a pair of points with indices $+1$ and -1 before they cancel and the surface becomes totally real. The slices in the unfolding of the degenerate parabolic case with $\eta = -1$ have index sum $+1$ near the origin, when there are 1 or 3 nondegenerate critical points; and similarly, the slices with nondegenerate singularities in the $\eta = +1$ case have a local sum of -1.

To more precisely analyze the local geometry of the unfoldings $\widehat{M}^{\pm}$, we turn again to the complexification construction of Section 4. Corresponding to the smooth real variety $\widehat{M}^{+} \subseteq \mathbb{C}^{2+2}$ is a smooth complex variety $\widehat{M}^{+}_{c} \subseteq \mathbb{C}^{2(2+2)}$, parametrized by $\Sigma : \mathbb{C}^{2+2} \to \mathbb{C}^{2(2+2)}$. Let $\mathbb{C}^{2+2}$ have coordinates $(z_1, \zeta_1, \omega_1, \omega_2)$. The image of $\widehat{M}^{+}_{c}$ under the projection $\pi : \mathbb{C}^{2(2+2)} \to \mathbb{C}^{2+2}$ is all of $\mathbb{C}^{2+2}$; the map $\pi \circ \Sigma$ as in (19):

$$\pi \circ \Sigma : \mathbb{C}^{2+2} \to \mathbb{C}^{2+2} : (z_1, \zeta_1, \omega_1, \omega_2) \mapsto$$
$$(z_1, z_1\zeta_1 + (\frac{1}{2} + \omega_2) \cdot (z_1^2 + \zeta_1^2) + i(z_1 - \zeta_1)\omega_1 + (z_1^2 + \zeta_1^2)z_1\zeta_1, \omega_1, \omega_2)^T$$

is a two-to-one branched cover near $(\vec{0}, \mathbf{0})$. (Globally, points in the target can have up to three inverse images, but points near the origin have at most two inverse images near the origin in the domain.) At each point in the domain $\mathbb{C}^{2+2}$, the complex Jacobian of this polynomial map either will have full rank, 4, or will be singular, with rank 3. The singular locus in the domain is the complex affine variety

$$\{z_1 + (\frac{1}{2} + \omega_2)2\zeta_1 - i\omega_1 + z_1^3 + 3z_1\zeta_1^2 = 0\}.$$

As in (18), let $\boldsymbol{\delta}$ denote the inclusion of the totally real subspace

$$\{\zeta_1 = \bar{z}_1, \omega_1 = \overline{\omega_1} = t_1, \omega_2 = \overline{\omega_2} = t_2\}$$

in $\mathbb{C}^{2+2}$, so $\widehat{M}^{+}$ is the image of $\pi \circ \Sigma \circ \boldsymbol{\delta}$. The intersection of this real subspace with the singular locus is

$$\{(x_1 + iy_1, x_1 - iy_1, t_1, t_2) : 2(1 + t_2)x_1 + 4x_1^3 = 0, 4y_1^3 + 2t_2y_1 + t_1 = 0\}.$$

The first condition factors as $2x_1(1 + t_2 + 2x_1^2) = 0$, and if we are only considering points in $\widehat{M}^{+}$ near $(\vec{0}, \mathbf{0})$, the solution set in the x_1, y_1-plane for each $t = (t_1, t_2)$ is a set of at most three points on the y_1-axis, as in Figures 7, 11–13. So, the candidates for CR singular points in $\widehat{M}$ near $(\vec{0}, \mathbf{0})$ are of the form

$$\{(z_1, z_2, t_1, t_2)^T = (iy_1, -2t_2y_1^2 - 2t_1y_1 - 2y_1^4, t_1, t_2)^T\} \subseteq \widehat{M}^{+} \subseteq \mathbb{C}^{2+2},$$

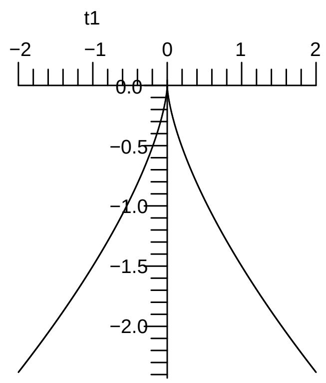

FIGURE 14. Discriminant locus in the t_1, t_2 parameter space for the unfolding $\widehat{M}^+$

where y_1 satisfies $4y_1^3 + 2t_2y_1 + t_1 = 0$. This locus is a surface contained in the real y_1, x_2, t_1, t_2 subspace, which is smooth near the origin, and its tangent plane at the origin is the y_1, t_2-plane.

The discriminant of the cubic equation $4y_1^3 + 2t_2y_1 + t_1 = 0$ is $-432t_1^2 - 128t_2^3$, and the zero locus of this curve is plotted in the t_1, t_2 parameter space in Figure 14.

When $t_1 = t_2 = 0$, the cubic equation has a triple root, and the slice $M_{\mathbf{0}}^+$ has an isolated degenerate parabolic CR singularity as in Figure 7. When $t = (t_1, t_2)$ is a point above the $\curlywedge$-shaped curve, so $t_2 > -\frac{3}{2}t_1^{2/3}$, the cubic has one real root and M_t^+ has an isolated hyperbolic CR singularity at $(z_1, z_2)^T = (iy_1, -2t_2y_1^2 - 2t_1y_1 - 2y_1^4)^T$ as in Figures 11 and 12. When t is below the curve, the cubic equation has three distinct roots and M_t^+ has two hyperbolic CR singularities and one elliptic CR singularity near $\vec{0}$. For points $t = (t_1, -\frac{3}{2}t_1^{2/3})$ on the curve, but not at the cusp at the origin, the cubic has two solutions, a simple root at $y_1 = -t_1^{1/3}$, and a double root at $y_1 = \frac{1}{2}t_1^{1/3}$. Near $\vec{0}$, M_t^+ has a hyperbolic CR singularity at $(z_1, z_2)^T = (-it_1^{1/3}, 3t_1^{4/3})^T$, and a nondegenerate parabolic CR singularity at $(z_1, z_2)^T = (\frac{i}{2}t_1^{1/3}, -\frac{3}{8}t_1^{4/3})^T$. This can be verified by a translation to standard position and then transformation to normal form as in the previous Example. In this case, the hyperbolic singularity of M_t^+ has $\gamma_t = \frac{1}{2} \cdot \frac{1+3t_1^{2/3}}{1-6t_1^{2/3}}, > \frac{1}{2}$ for t_1 close to but not equal to 0. The parabolic point of M_t^+ can be put, by a holomorphic (but nonlinear) transformation, into the form

$$z_2 = z_1\bar{z}_1 + \frac{1}{2}(z_1^2 + \bar{z}_1^2) - \frac{4it_1^{1/3}}{3t_1^{2/3} - 2}(z_1 - \bar{z}_1)z_1\bar{z}_1 + O(4),$$

so it is a nondegenerate parabolic point for t_1 close to but not equal to 0, as in Proposition 5.1.

Figure 14 resembles Figure 4 of [**Callahan**], and the overall geometry of this unfolding resembles the analogous catastrophe phenomenon described by [**Callahan**]. Considering $\widehat{M}^+$ as a four-dimensional submanifold of $\mathbb{C}^4$, it has a degenerate parabolic CR singularity at the origin, in the sense of Proposition 5.8. The CR singular locus N_1 is a totally real surface in $\widehat{M}^+$, and its tangent plane at the origin meets the complex tangent line in the y_1-axis, the parabolic line as in Remark 5.9. The locus of parabolic points

$$\{(z_1, z_2, w_1, w_2)^T = (\frac{i}{2}t_1^{1/3}, -\frac{3}{8}t_1^{4/3}, t_1, -\frac{3}{2}t_1^{2/3})^T\}$$

is a smooth curve, which could be re-parametrized as:

$$(\frac{i}{2}y_1, -\frac{3}{8}y_1^4, y_1^3, -\frac{3}{2}y_1^2)^T,$$

also tangent to the y_1-axis. The tangency of the parabolic locus to the parabolic line is an intrinsic characterization of the degenerate parabolic point of an n-manifold in $\mathbb{C}^n$ (Proposition 4.1 of [**Webster**$_2$]).

The geometry of the unfolding $\widehat{M}^-$ is analyzed by similar calculations. For $t = (t_1, t_2)$, the z_1 coordinates of the CR singularities of the slice M_t are iy_1, where y_1 is a solution of the cubic equation $-4y_1^3 + 2t_2y_1 + t_1 = 0$. The discriminant is $128t_2^3 - 432t_1^2$, so its zero locus is just an upside-down Figure 14. The slice M_t will have two elliptic points and one hyperbolic point as in Figure 10 for t above the $\curlyvee$-shaped curve, one elliptic point as in Figures 8, 9, for t below the curve, and one elliptic point and one nondegenerate parabolic point for $t \neq \mathbf{0}$ on the curve.

6. Real m-submanifolds in $\mathbb{C}^n$, $m < n$

We move to the case of higher dimension, or actually higher codimension, where $m < n$. In some of the results of this Section, we will also restrict our attention to the case $m = \frac{2}{3}(n+1)$, where M in general position will be totally real except for isolated points in N_1.

6.1. Normal forms.

We recall from [**Beloshapka**] and [$\mathbf{C}_1$], without re-working all of the details, the derivation of a normal form and the nondegeneracy conditions for the defining equations of M, in the stable dimension range $\frac{2}{3}(n+1) \le m < n$. The equations in standard position as in (8) are:

$$\begin{aligned} H_\sigma(z_1,\bar z_1,x) &= \alpha_\sigma z_1^2 + \beta_\sigma z_1\bar z_1 + \gamma_\sigma \bar z_1^2 + \left(\sum \delta_\sigma^{\sigma_1} z_1 x_{\sigma_1}\right) \\ &\quad + \left(\sum \epsilon_\sigma^{\sigma_1}\bar z_1 x_{\sigma_1}\right) + \left(\sum \theta_\sigma^{\sigma_1\sigma_2} x_{\sigma_1}x_{\sigma_2}\right) + E_\sigma(z_1,\bar z_1,x) \\ h_u(z_1,\bar z_1,x) &= \alpha_u z_1^2 + \beta_u z_1\bar z_1 + \gamma_u \bar z_1^2 + \left(\sum \delta_u^{\sigma_1} z_1 x_{\sigma_1}\right) \\ &\quad + \left(\sum \epsilon_u^{\sigma_1}\bar z_1 x_{\sigma_1}\right) + \left(\sum \theta_u^{\sigma_1\sigma_2} x_{\sigma_1}x_{\sigma_2}\right) + e_u(z_1,\bar z_1,x), \end{aligned} \tag{57}$$

where $E_\sigma = O(3)$ is real valued for $\sigma = 2, \ldots, m-1$, and $e_u = O(3)$ is complex valued for $u = m, \ldots, n$.

The first nondegeneracy condition is that the $(n-m+1)\times 2$ block of coefficients β_u, γ_u in the functions h_u satisfies:

$$\operatorname{rank}\begin{pmatrix} \beta_m & \gamma_m \\ \vdots & \vdots \\ \beta_n & \gamma_n \end{pmatrix} = 2. \tag{58}$$

In the nondegenerate case, there is a holomorphic transformation in a neighborhood of $\vec{0}$ in $\mathbb{C}^n$ taking M to the following partial normal form:

$$y_\sigma = H_\sigma(z_1,\bar z_1,x) = 0 + E_\sigma(z_1,\bar z_1,x) = O(3) \tag{59}$$

$$z_\tau = h_\tau(z_1,\bar z_1,x) = \left(\sum \epsilon_\tau^{\sigma_1}\bar z_1 x_{\sigma_1}\right) + e_\tau(z_1,\bar z_1,x) \tag{60}$$

$$\begin{aligned} z_{n-1} = h_{n-1}(z_1,\bar z_1,x) &= \bar z_1^2 + e_{n-1}(z_1,\bar z_1,x) \\ z_n = h_n(z_1,\bar z_1,x) &= z_1\bar z_1 + \left(\sum \epsilon_n^{\sigma_1}\bar z_1 x_{\sigma_1}\right) + e_n(z_1,\bar z_1,x), \end{aligned} \tag{61}$$

for $\tau = m\ldots, n-2$, or there are no h_τ expressions if $m = n-1$. (The quantities E_σ, e_u may have changed but are still $O(3)$.)

The real and imaginary parts of the coefficients $\epsilon_u^{\sigma_1}$ in (60) and (61), for $u = m, \ldots, n-2$ and $u = n$, on the terms $\bar z_1 x_{\sigma_1}$, $\sigma_1 = 2, \ldots, m-1$, form a real $2(n-m) \times (m-2)$ coefficient matrix, in this expression where the LHS is a column $(n-m)$-vector:

$$\left(\sum \epsilon_u^{\sigma_1} \bar{z}_1 x_{\sigma_1}\right)_{u=m,\dots,n-2,n} = \tag{62}$$

$$\begin{pmatrix} 1 & i \dots 0 & 0 \\ \vdots & & \vdots \\ 0 & 0 \dots 1 & i \end{pmatrix} \begin{pmatrix} \mathrm{Re}(\epsilon_m^2) & \mathrm{Re}(\epsilon_m^3) \dots & \mathrm{Re}(\epsilon_m^{m-1}) \\ \mathrm{Im}(\epsilon_m^2) & \mathrm{Im}(\epsilon_m^3) \dots & \mathrm{Im}(\epsilon_m^{m-1}) \\ \vdots & & \vdots \\ \mathrm{Re}(\epsilon_{n-2}^2) & \mathrm{Re}(\epsilon_{n-2}^3) \dots & \mathrm{Re}(\epsilon_{n-2}^{m-1}) \\ \mathrm{Im}(\epsilon_{n-2}^2) & \mathrm{Im}(\epsilon_{n-2}^3) \dots & \mathrm{Im}(\epsilon_{n-2}^{m-1}) \\ \mathrm{Re}(\epsilon_n^2) & \mathrm{Re}(\epsilon_n^3) \dots & \mathrm{Re}(\epsilon_n^{m-1}) \\ \mathrm{Im}(\epsilon_n^2) & \mathrm{Im}(\epsilon_n^3) \dots & \mathrm{Im}(\epsilon_n^{m-1}) \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \bar{z}_1.$$

The second nondegeneracy condition is that this matrix has rank $2(n-m)$. In the nondegenerate case, there is a linear transformation (the $\mathbf{R}$ block from the matrix $\mathbf{A}$ from (9)) taking this coefficient matrix to an echelon form, so the h_τ expressions (60), if any, become

$$(x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\bar{z}_1 + e_\tau(z_1, \bar{z}_1, x), \tag{63}$$

and the h_n expression (61) becomes

$$z_1\bar{z}_1 + x_2\bar{z}_1 + ix_3\bar{z}_1 + e_n(z_1, \bar{z}_1, x). \tag{64}$$

Proposition 6.1 ([$\mathbf{C}_6$]). *Given $\frac{2}{3}(n+1) \le m < n$, if M is a real analytic m-submanifold of $\mathbb{C}^n$ with a CR singularity satisfying both nondegeneracy conditions, then there exists a holomorphic coordinate change $\tilde{z} = \mathbf{A}\vec{z} + \vec{p}$ as in* (9)*, in a neighborhood of $\vec{0} \in \mathbb{C}^n$, transforming the equations* (59–61) *into the following real algebraic normal form:*

$$\begin{aligned} \tilde{y}_\sigma &= 0, \quad \sigma = 2, \dots, m-1 \\ \tilde{z}_\tau &= (\tilde{x}_{2(\tau-m+2)} + i\tilde{x}_{2(\tau-m+2)+1})\bar{\tilde{z}}_1, \quad \tau = m \dots, n-2 \\ \tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 \\ \tilde{z}_n &= (\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3)\bar{\tilde{z}}_1. \end{aligned} \tag{65}$$

∎

So, M can be transformed into the real algebraic variety (65), denoted $\widetilde{M}$, with the higher degree terms E_σ, e_u eliminated entirely in a neighborhood of the CR singularity, assuming only that M is real analytic and satisfies both second order nondegeneracy conditions at a point in N_1.

One could flatten $\widetilde{M}$ to fit inside $\mathbb{R}^{n+1} \subseteq \mathbb{C}^n$, by introducing some new quadratic terms to get

$$\begin{aligned} h_\tau(z_1, \bar{z}_1, x) &= (\bar{z}_1 + z_1)x_{2(\tau-m+2)} + i(\bar{z}_1 - z_1)x_{2(\tau-m+2)+1} \\ h_{n-1}(z_1, \bar{z}_1, x) &= z_1^2 + \bar{z}_1^2 \\ h_n(z_1, \bar{z}_1, x) &= z_1\bar{z}_1 + (\bar{z}_1 + z_1)x_2 + i(\bar{z}_1 - z_1)x_3, \end{aligned}$$

but this does not help as much with the visualization as the flattened normal forms from the previous Section.

So, M has the property of "algebraizability": having a real algebraic representative in its equivalence class. Some real analytic surfaces in $\mathbb{C}^2$ also have this property (Examples 5.4, 5.13), but unlike the case of surfaces in $\mathbb{C}^2$, for M as in Proposition 6.1, there is no continuous invariant. As remarked in [$\mathbf{C}_6$], the algebraic normal form (65) is analogous to, but different from, the simplest type of singularity for differentiable maps $\mathbb{C}^m \to \mathbb{C}^n$, where the components of the parametric map in normal form are monomials.

In the case $m = \frac{2}{3}(n+1)$, the CR singular point at $\vec{0}$ is an isolated point and the real coefficient matrix (62) is square, with size $2(n-m) \times 2(n-m)$. If the linear transformation $\mathbf{A}$ is required to preserve the orientation of $T_{\vec{0}}M$, then the nondegenerate normal form

$$h_n = z_1\bar{z}_1 + x_2\bar{z}_1 - ix_3\bar{z}_1 + e_n(z_1, \bar{z}_1, x)$$

is inequivalent to the formula (64) with the other sign. They are related by the holomorphic transformation $\tilde{z}_3 = -z_3$, but this switches the orientation of $T_{\vec{0}}M$. Recalling the topological description of CR singularities from Section 2, these two nondegenerate normal forms correspond to the differential-topological intersection index, $+1$ or -1, where the Gauss map $M \to G$ meets the oriented submanifold $\mathcal{D}_1 \setminus \mathcal{D}_2 \subseteq G$.

Continuing under the assumption $m = \frac{2}{3}(n+1)$, if the first nondegeneracy condition is satisfied but the second is not, then the real coefficient matrix in (62) has less than full rank. Assuming it has rank $2(n-m)-1$, a real linear transformation of the $x_2, \dots, x_{m-1}$ coordinates that puts the real matrix into echelon form can make the second column (corresponding to x_3) of the matrix zero, so the expressions (63) are the same but expression (61) becomes

$$h_n = z_1\bar{z}_1 + \epsilon_n^2\bar{z}_1x_2 + \left(\sum_{\ell=4}^{m-1} \epsilon_n^\ell \bar{z}_1 x_\ell\right) + O(3),$$

where ϵ_n^2 is a nonzero complex coefficient and for $\ell = 4, \dots, m-1$, ϵ_n^ℓ is purely imaginary. A complex linear transformation of the form

$$\tilde{z} = (a_1z_1, z_2, \dots, z_{m-1}, \overline{a_1}z_m, \dots, \overline{a_1}z_{n-2}, \overline{a_1}^2 z_{n-1}, |a_1|^2 z_n)^T$$

can normalize ϵ_n^2 to 1, transforming h_n into:

$$h_n = z_1\bar{z}_1 + \bar{z}_1 x_2 + \left(\sum_{\ell=4}^{m-1} \epsilon_n^\ell \bar{z}_1 x_\ell\right) + O(3),$$

where now the coefficients ϵ_n^ℓ are complex (using the same symbol even though the value of the coefficients may have changed). In the $(m,n) = (4,5)$ case, there are no terms besides $z_1\bar{z}_1 + \bar{z}_1 x_2$. Otherwise, the remaining terms satisfy the identity:

$$\begin{aligned}(66)\quad \sum_{\ell=4}^{m-1} \epsilon_n^\ell \bar{z}_1 x_\ell &= \sum_{j=2}^{\frac{m}{2}-1} (\epsilon_n^{2j} \bar{z}_1 x_{2j} + \epsilon_n^{2j+1} \bar{z}_1 x_{2j+1}) \\ &= \sum_{j=2}^{\frac{m}{2}-1} \Big((\mathrm{Im}(\epsilon_n^{2j+1}) + i\mathrm{Im}(\epsilon_n^{2j})) \cdot (x_{2j} + i x_{2j+1})\bar{z}_1 \\ &\qquad +(\mathrm{Re}(\epsilon_n^{2j}) - \mathrm{Im}(\epsilon_n^{2j+1}))\bar{z}_1 x_{2j} \\ &\qquad +(\mathrm{Re}(\epsilon_n^{2j+1}) + \mathrm{Im}(\epsilon_n^{2j}))\bar{z}_1 x_{2j+1}\Big)\,.\end{aligned}$$

The re-grouping allows the elimination of the $(x_{2j} + i x_{2j+1})\bar{z}_1$ terms by linear transformations of the form $\tilde{z}_n = z_n + \sum_{\tau=m}^{n-2} a_n^\tau z_\tau$, and the remaining $\bar{z}_1 x_\ell$ terms have real coefficients so they can be eliminated by a real linear transformation of the x_2 variable, $\tilde{z}_2 = z_2 + \sum_{\ell=4}^{m-1} r_2^\ell z_\ell$.

Note that it was at this point where we used the assumption $m = \frac{2}{3}(n+1)$. For $\frac{2}{3}(n+1) < m < n$, there would be terms $\bar{z}_1 x_\ell$, with $\ell = 4, \ldots, 2n-2m+1, 2n-2m+2, \ldots, m-1$, and only enough z_τ quantities available to cancel $\ell = 4, \ldots, 2n-2m+1$.

This leaves the following quadratic normal form for the defining equations of M near the origin:

$$\begin{aligned}(67)\qquad y_\sigma &= 0 + E_\sigma(z_1, \bar{z}_1, x) = O(3) \\ z_\tau &= (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1})\bar{z}_1 + e_\tau(z_1, \bar{z}_1, x) \\ z_{n-1} &= \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x), \\ z_n &= z_1\bar{z}_1 + \bar{z}_1 x_2 + e_n(z_1, \bar{z}_1, x).\end{aligned}$$

We will not consider degenerate CR singularities where the real coefficient matrix has rank less than $2(n-m)-1$, or where the first nondegeneracy condition fails. The remainder of this Section will deal with only the nondegenerate normal form (65) and the degenerate normal form (67).

Writing out all the cubic terms in m variables in the defining equations (57) would give a lengthy expression. To get a normal form for (67), using the transformation group (9) to eliminate as many terms as possible to simplify the expression, we take three shortcuts.

First, any cubic term in (57) not depending on $\bar{z}_1$, that is, having only $z_1, x_2, \ldots, x_{m-1}$ factors, can be eliminated by a holomorphic transformation of the form $\tilde{z}_j = z_j + p_j(z_1, z_2, \ldots, z_{m-1})$, $j = 2, \ldots, n$, which does not change any other cubic terms in the system of equations, with the possible exception of introducing a complex conjugate term (involving $\bar{z}_1$ and not z_1) in the H_σ expressions.

Second, we can take advantage of the fact that h_{n-1} and h_n expressions in the normal form (67) have quadratic part identical (up to re-numbered subscripts) to that of the nondegenerate normal form for real threefolds in $\mathbb{C}^4$, as in [$\mathbf{C}_5$]. It was shown there that all cubic terms of h_{n-1}, h_n in z_1, $\bar{z}_1$, x_2 can be eliminated by a holomorphic transformation $\tilde{z} = \vec{z} + \vec{p}(z_1, z_2, z_{n-1}, z_n)$.

Third, the E_σ, $\sigma = 4, \ldots, m-1$ and

$$h_\tau = (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1})\bar{z}_1 + O(3)$$

quantities have no analogue in [$\mathbf{C}_5$], but are identical to the expressions in [$\mathbf{C}_6$]. It was shown in the Proofs of Theorem 5.6 and Corollary 5.7 of [$\mathbf{C}_6$] that for each $\tau = m, \ldots, n-2$, all of the cubic terms of the three expressions h_τ, $E_{2(\tau-m+2)}$, and $E_{2(\tau-m+2)+1}$ can be eliminated by a holomorphic transformation of the form $\tilde{z}_\tau = z_\tau + p_\tau$, $\tilde{z}_{2(\tau-m+2)} = z_{2(\tau-m+2)} + p_{2(\tau-m+2)}$, $\tilde{z}_{2(\tau-m+2)+1} = z_{2(\tau-m+2)+1} + p_{2(\tau-m+2)+1}$, without contributing cubic terms to any of the other equations. An analogous argument applies in this case: the same calculation solving the system of three equations works, the only minor difference being in the quadratic part of h_n. It also follows from an analogy with the treatment of the E_6 equation from the Proof of Theorem 5.6 of [$\mathbf{C}_6$] that for each $\sigma = 2, 3$, the cubic terms of E_σ can be eliminated by a transformation of the form $\tilde{z}_\sigma = z_\sigma + p_\sigma$, where p_σ has weight 3 and does not contribute cubic terms to the other equations.

After these steps, the higher degree terms in (67) have a partial normal form:

$$
\begin{aligned}
E_\sigma &= O(4)\\
e_\tau &= O(4)\\
e_{n-1} &= \left(\sum_{j=3}^{m-1}(e_{n-1}^{1j}\bar{z}_1^2 x_j + e_{n-1}^{2j} z_1\bar{z}_1 x_j + e_{n-1}^{3j}\bar{z}_1 x_2 x_j\right.\\
&\qquad \left. +(\sum_{\ell=3}^{m-1} e_{n-1}^{4\ell j}\bar{z}_1 x_\ell)x_j)\right) + O(4)\\
e_n &= \left(\sum_{j=3}^{m-1}(e_n^{1j}\bar{z}_1^2 x_j + e_n^{2j} z_1\bar{z}_1 x_j + e_n^{3j}\bar{z}_1 x_2 x_j\right.\\
&\qquad \left. +(\sum_{\ell=3}^{m-1} e_n^{4\ell j}\bar{z}_1 x_\ell)x_j)\right) + O(4),
\end{aligned}
\tag{68}
$$

where in the above double sums, $e_{n-1}^{4\ell j}$ and $e_n^{4\ell j}$ are 0 if $\ell < j$.

A transformation of the form

$$
\begin{aligned}
\tilde{z}_1 &= z_1 + \left(\sum p_1^{2j} z_2 z_j\right) + \left(\sum p_1^{\ell j} z_\ell z_j\right),\\
\tilde{z}_{n-1} &= z_{n-1} + \left(\sum p_{n-1}^{1j} z_j z_{n-1}\right) + \left(\sum p_{n-1}^{2j} z_j z_n\right)
\end{aligned}
$$

can eliminate all the complex cubic coefficients from e_{n-1}.

A transformation of the form

$$
\begin{aligned}
\tilde{z}_1 &= z_1 + \sum p_1^{1j} z_1 z_j,\\
\tilde{z}_n &= z_n + \left(\sum p_n^{1j} z_j z_{n-1}\right) + \left(\sum p_n^{2j} z_j z_n\right)
\end{aligned}
$$

can eliminate the e_n^{1j}, e_n^{2j}, e_n^{3j} cubic coefficients from e_n. The p_1^{1j} coefficients re-introduce $\bar{z}_1^2 x_j$ terms in e_{n-1}, but they can be eliminated by another $\tilde{z}_{n-1} = z_{n-1} + \sum p_{n-1}^{1j} z_j z_{n-1}$ transformation without changing e_n. The partial sum $\sum_{\ell=4}^{m-1} e_n^{4\ell j}\bar{z}_1 x_\ell$ can be re-grouped exactly as in (66), so that for each $j = 3, \ldots, m-1$, terms of the form $e_n^{4\ell j}\bar{z}_1 x_\ell x_j$, with

$\ell = 4, \ldots, m-1$, can be eliminated by a transformation of the form

$$\begin{aligned} \tilde{z}_2 &= z_2 + \sum_{\ell=4}^{m-1} p_2^{\ell j} z_\ell z_j \\ \tilde{z}_n &= z_n + \sum_{\tau=m}^{n-2} p_n^{3\tau j} z_\tau z_j, \end{aligned}$$

where the coefficients $p_2^{\ell j}$ are real.

Here we are again using the assumption $m = \frac{2}{3}(n+1)$. For $\frac{2}{3}(n+1) < m < n$, there would be terms $\bar{z}_1 x_\ell x_j$, with $\ell = 4, \ldots, 2n-2m+1, 2n-2m+2, \ldots, m-1$, and only enough z_τ quantities available to cancel $\ell = 4, \ldots, 2n-2m+1$.

This leaves the cubic term $e_n^{433}\bar{z}_1 x_3^2$. A transformation of the form

$$\tilde{z} = (a_1 z_1, a_1 z_2 + p_2^{33} z_3^2, r_3^3 z_3, a_1 z_4, \ldots, a_1 z_{m-1}, (a_1)^2 z_m, \ldots, (a_1)^2 z_n)^T,$$

with a_1 and r_3^3 nonzero and real, and p_2^{33} real, results in the new defining equation

$$\tilde{z}_n = \tilde{z}_1 \bar{\tilde{z}}_1 + \bar{\tilde{z}}_1 \tilde{x}_2 + \left(\frac{a_1 e_n^{433} - p_2^{33}}{(r_3^3)^2} \right) \bar{\tilde{z}}_1 \tilde{x}_3^2 + O(4),$$

without introducing any other cubic terms or changing the quadratic part of the other equations. There are no other linear or nonlinear transformations that contribute to this term (the lengthy check is omitted). Using p_2^{33} to cancel the real part of the e_n^{433} coefficient, and then re-scaling by a_1, e_n^{433} can be normalized to either i or 0. We regard the 0 case as another degeneracy, and from this point only consider the following cubic normal form:

$$\begin{aligned} y_\sigma &= 0 + E_\sigma(z_1, \bar{z}_1, x) = O(4) \\ z_\tau &= (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1})\bar{z}_1 + e_\tau(z_1, \bar{z}_1, x) \\ z_{n-1} &= \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x), \\ z_n &= (z_1 + x_2 + i x_3^2)\bar{z}_1 + e_n(z_1, \bar{z}_1, x), \end{aligned} \tag{69}$$

where $E_2, \ldots, e_n$ have degree $d \geq 4$ and are real analytic.

Proposition 6.2. *Given $\frac{2}{3}(n+1) = m < n$, if M is a real analytic m-submanifold of $\mathbb{C}^n$ with a CR singularity that can be put into the cubic normal form* (69)*, then there exists a holomorphic coordinate change $\tilde{z} = \mathbf{A}\vec{z} + \vec{p}$ as in* (9)*, in a neighborhood of $\vec{0} \in \mathbb{C}^n$, transforming the defining equations into the following real algebraic normal form:*

$$\begin{aligned} \tilde{y}_\sigma &= 0, \quad \sigma = 2, \ldots, m-1 \\ \tilde{z}_\tau &= (\tilde{x}_{2(\tau-m+2)} + i\tilde{x}_{2(\tau-m+2)+1})\bar{\tilde{z}}_1, \quad \tau = m \ldots, n-2 \\ \tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 \\ \tilde{z}_n &= (\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3^2)\bar{\tilde{z}}_1. \end{aligned} \tag{70}$$

∎

This Proposition will follow as a corollary of a more general analytic normal form result, Main Theorem 6.5.

Observe that the the involution $\tilde{z}_3 = -z_3$, which reverses the orientation of the tangent plane $T_{\vec{0}}M$, leaves the normal form (69) invariant, and is a symmetry of the real variety in Proposition 6.2.

6.2. Unfolding CR singularities of m-submanifolds.

We continue to work with $\frac{2}{3}(n+1) \le m < n$. Recall that the nondegenerate CR singularities of M are stable in two senses — near such a point, any small real analytic perturbation of M is locally equivalent to a constant normal form, and when $m = \frac{2}{3}(n+1)$, the intersection index (± 1) is also constant under a small smooth perturbation. However, pairs of isolated points with opposite indices may cancel under a larger-scale homotopy. These three phenomena are all reflected in the analysis of the local normal forms for unfoldings — any unfolding of a stable CR singular point will be shown to be u-equivalent to a trivial (or "constant") unfolding, and a degenerate point will have an unfolding exhibiting a pair creation/annihilation. Unlike the unfoldings of surfaces considered in Section 5, we will find normal forms not only to arbitrarily high degree, but we will find the whole u-equivalence classes for two different normal forms for real analytic submanifolds under the group $\mathcal{U}_{m,n,k}$ of local biholomorphic transformations. The formal problem, finding a normal form for higher degree terms, is, like the calculations of Section 5, a matter of comparison of coefficients, to find series expressions for the nonlinear part $\vec{p}$ of transformations in $\mathcal{U}_{m,n,k}$ (11) in terms of the series expressions for the given defining equations (10). A normalizing transformation may not be unique, so our construction of $\vec{p}$ does more than just establish the existence of a formal series, it also makes some choices to avoid series that are divergent or that converge on insufficiently large sets.

THEOREM 6.3. *Given $\frac{2}{3}(n+1) \le m < n$, if M is a real analytic m-submanifold of $\mathbb{C}^n$ with a nondegenerate quadratic normal form as in the hypothesis of Proposition* 6.1*, and $\widehat{M}$ is any real analytic unfolding of M in $\mathbb{C}^{n+k}$, with defining equations in standard position*

$$\begin{aligned}
(71)\ y_\sigma &= E_\sigma(z_1, \bar{z}_1, x, t) = O(2) \\
z_\tau &= (x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\bar{z}_1 \\
&\quad + e_\tau^{10\alpha} t_\alpha z_1 + e_\tau^{01\alpha} t_\alpha \bar{z}_1 + e_\tau^{\alpha\beta} t_\alpha t_\beta + f_\tau^{\sigma_1\alpha} x_{\sigma_1} t_\alpha + e_\tau(z_1, \bar{z}_1, x, t) \\
z_{n-1} &= \bar{z}_1^2 + e_{n-1}^{10\alpha} t_\alpha z_1 + e_{n-1}^{01\alpha} t_\alpha \bar{z}_1 \\
&\quad + e_{n-1}^{\alpha\beta} t_\alpha t_\beta + f_{n-1}^{\sigma_1\alpha} x_{\sigma_1} t_\alpha + e_{n-1}(z_1, \bar{z}_1, x, t) \\
(72)\ z_n &= (z_1 + x_2 + ix_3)\bar{z}_1 \\
&\quad + e_n^{10\alpha} t_\alpha z_1 + e_n^{01\alpha} t_\alpha \bar{z}_1 + e_n^{\alpha\beta} t_\alpha t_\beta + f_n^{\sigma_1\alpha} x_{\sigma_1} t_\alpha + e_n(z_1, \bar{z}_1, x, t), \\
(73)\ s_\alpha &= 0,\ \alpha = 1, \ldots, k
\end{aligned}$$

then there exists a holomorphic coordinate change $(\tilde{z}, \tilde{w}) = \mathbf{A}(\vec{z}, \vec{w}) + \vec{p}(\vec{z}, \vec{w})$ in $\mathcal{U}_{m,n,k}$ (11)*, in a neighborhood of $(\vec{0}, \mathbf{0}) \in \mathbb{C}^{n+k}$, transforming the equations* (71–72) *into the real algebraic normal form* (65)*, which does not depend on t, and preserving the form of* (73)*: $\tilde{s}_\alpha = 0$.*

PROOF. The statement of the Theorem is that any real analytic unfolding $\widehat{M}$ of any real analytic, nondegenerate M is u-equivalent (in some small neighborhood of the origin in $\mathbb{C}^{n+k}$) to the trivial unfolding $\widetilde{M} \times \mathbb{R}^k$, where $\widetilde{M}$ is the real algebraic model from the conclusion of Proposition 6.1.

Considering the quadratic part first, as usual any terms in e_u not involving $\bar{z}_1$ can be eliminated by a transformation of the form $\tilde{z}_u = z_u + p_u(z_1, \ldots, z_{m-1}, \vec{w})$. Similarly, terms $z_1 t_\alpha$ can be eliminated from the real functions E_σ in (71) while simultaneously eliminating their complex conjugates. The terms $e_{n-1}^{01\alpha} t_\alpha \bar{z}_1$ can be eliminated by a linear transformation $\tilde{z}_1 = z_1 + a^\alpha w_\alpha$, as in the $\gamma = \infty$ version of Lemma 5.10.

The real and imaginary parts of the coefficients $e_\tau^{01\alpha}$ and $e_n^{01\alpha}$ can be eliminated by real linear transformations of the x variables, $\tilde{z}_\sigma = z_\sigma + r_\sigma^\alpha w_\alpha$, using real coefficients r_σ^α from the $\mathbf{R}_{(m-2)\times k}$ block of the matrix $\mathbf{A}$ (12). This is where the second nondegeneracy assumption on the quadratic normal form of M is used.

The result of this transformation is that the quadratic part of the defining equations of the unfolding $\widehat{M}$ does not depend on t. In fact, $\widehat{M}$ is a submanifold of real dimension $m+k$ in $\mathbb{C}^{n+k}$, satisfying $\frac{2}{3}((n+k)+1) < (m+k) < n+k$, and its quadratic part satisfies the nondegeneracy hypothesis of Proposition 3.3 of [$\mathbf{C}_6$], except for a merely notational difference: the real $t_1, \ldots, t_k$ variables in the above quantities appear in the place of the real $x_m, \ldots, x_{(m+k)-1}$ variables from the normal form of [$\mathbf{C}_6$], which occur in the higher degree terms but not the quadratic part. The conclusion of that Proposition is that there exists a local holomorphic transformation in $\mathcal{B}_{m,n+k}$ of a neighborhood of the origin in $\mathbb{C}^{n+k}$,

$$(74)\quad (\tilde{z}, \tilde{w}) = (\vec{z}, \vec{w}) + (p_1(\vec{z}, \vec{w}), \ldots, p_n(\vec{z}, \vec{w}), P_1(\vec{z}, \vec{w}), \ldots, P_k(\vec{z}, \vec{w}))^T,$$

taking $\widehat{M}$ to the claimed real algebraic normal form not depending on t and with $\tilde{s}_\alpha = \mathrm{Im}(\tilde{w}_\alpha) = 0$. However, that statement is not enough to establish u-equivalence, which requires that there exists such a transformation in the subgroup $\mathcal{U}_{m,n,k}$ (11).

An inspection of the Proof of Theorem 5.6 of [$\mathbf{C}_6$] will show that the transformation of the x_σ, $\sigma = 2(n+k) - 2(m+k) + 2, \ldots, m, \ldots, (m+k) - 1$, variables is of the form $x_\sigma = \mathrm{Re}(z_\sigma)$, $\tilde{x}_\sigma = \mathrm{Re}(\tilde{z}_\sigma)$, $\tilde{z}_\sigma = z_\sigma + p_\sigma$, such that the coefficients of p_σ are linear combinations of the real and imaginary parts of the coefficients from E_σ, where $y_\sigma = E_\sigma$ is one of the defining equations of the manifold (in [$\mathbf{C}_6$], this is worked out for $\sigma = 6$). In the case of $\widehat{M}$, y_σ, $\sigma = m, \ldots, (m+k)-1$, corresponds to $\mathrm{Im}(w_\alpha) = s_\alpha$, $\alpha = \sigma - (m-1) = 1, \ldots, k$, and E_σ corresponds to the constant function 0 from the RHS of (73). We can conclude that the component $\tilde{w}_\alpha = w_\alpha + P_\alpha(\vec{z}, \vec{w})$ of the transformation (74) is simply $\tilde{w}_\alpha = w_\alpha$. The transformation constructed in Theorem 5.6 and Corollary 5.7 of [$\mathbf{C}_6$] conveniently satisfies the reality conditions of (11) as a consequence of the assumption $\widehat{M} \subseteq \mathbb{C}^n \times \mathbb{R}^k \subseteq \mathbb{C}^{n+k}$, and the transformation of Proposition 3.3 of [$\mathbf{C}_6$] is the limit of composites of such transformations, so it also is in the subgroup $\mathcal{U}_{m,n,k}$. ∎

In [$\mathbf{C}_6$], an analogy was offered between the nondegenerate normal form result (quoted here previously as Proposition 6.1) and a result of Whitney, that the cross-cap parametrization $\mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \mapsto (u, uv, v^2)$ is a normal form for nondegenerate singularities of smooth maps under smooth coordinate changes. The normal form result of the above Theorem 6.3 could similarly be considered as analogous to the cross-cap map's property that any smooth unfolding is equivalent to a trivial unfolding (in a specific sense; see Part 4 of [**Martinet**]).

Next, for $m = \frac{2}{3}(n+1)$, let M have a degenerate CR singularity with cubic normal form (69), and let $\widehat{M}$ be a k-parameter unfolding of M in standard position. The defining equations of $\widehat{M}$ are the same as (71–72) from Theorem 6.3, except that the term $i\bar{z}_1 x_3$ in (72) is replaced by $i\bar{z}_1 x_3^2$. The elimination of all the quadratic terms involving t proceeds as in the above Proof, with the exception of the $e_n^{01\alpha}\bar{z}_1 t_\alpha$

terms. The real part of the complex coefficients $e_n^{01\alpha}$ can be eliminated by $\tilde{z}_2 = z_2 + r_2^\alpha w_\alpha$, but this leaves the transformed equation for z_n in the form

$$z_n = (z_1 + x_2 + ix_3^2)\bar{z}_1 + e_n^{01\alpha} t_\alpha \bar{z}_1 + e_n(z_1, \bar{z}_1, x, t),$$

with purely imaginary $e_n^{01\alpha}$. A real linear transformation of the t variables (the block $\mathbf{R}_{k\times k}$ of the matrix $\mathbf{A}$ from (12)) can normalize the coefficient vector $(e_n^{011}, \dots, e_n^{01k})$ to either $(i, 0, \dots, 0)$ or $(0, \dots, 0)$. We regard the zero case as a degenerate unfolding of M, and it is not considered further. A nondegenerate unfolding of M as in (69) has the following normal form:

$$\begin{array}{rrcl} (75) & y_\sigma & = & E_\sigma(z_1, \bar{z}_1, x, t) \\ & z_\tau & = & (x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\bar{z}_1 + e_\tau(z_1, \bar{z}_1, x, t) \\ & z_{n-1} & = & \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x, t) \\ & z_n & = & (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 + e_n(z_1, \bar{z}_1, x, t) \\ & s_\alpha & = & 0 \end{array}$$

where $E_2, \dots, e_n$ have degree 3, and all the cubic terms except $i\bar{z}_1 x_3^2$ depend on t.

Remark 6.4. Our first observation on the geometry of $\widehat{M}$ in the normal form (75) is that, as a real $(m+k)$-submanifold of $\mathbb{C}^{n+k}$, it has a nondegenerate CR singularity and is equivalent under a biholomorphic coordinate change in $\mathcal{B}_{m,n+k}$ near the origin of $\mathbb{C}^{n+k}$ to the real algebraic normal form $\widetilde{M} \times \mathbb{R}^k$ from the conclusion of Theorem 6.3. In particular, the complex linear transformation that switches x_3 and t_1, $\tilde{z}_3 = w_1$, $\tilde{w}_1 = z_3$, puts $\widehat{M}$ into the same quadratic normal form as (71–73), and then the cubic term $i\bar{\tilde{z}}_1 \tilde{t}_1^2$ can be eliminated as in Proposition 3.3 of [$\mathbf{C}_6$]. However, this $\widehat{M}$ is not u-equivalent to $\widetilde{M} \times \mathbb{R}^k$. Under the action of the subgroup $\mathcal{U}_{m,n,k}$ (11), which does not include any linear transformation switching x_3 and t_1, and which preserves the fiber $\mathbb{C}^n \times \{\mathbf{0}\}$, the slice $M = M_{\mathbf{0}}$ retains its degenerate normal form. This is a different phenomenon from Examples 5.23 and 5.24, where surfaces with parabolic points had unfoldings $\widehat{M}$ and $\widehat{M}^\pm$ that themselves had parabolic points as submanifolds of $\mathbb{C}^{2+k}$.

The following result is the Main Theorem, which states that any real analytic, nondegenerate unfolding (75) of M not only has the algebraizability property under u-equivalence, but, further, is u-equivalent to a unique real algebraic model. The slice $M_{\mathbf{0}}$ is simultaneously transformed into an algebraic normal form, so Proposition 6.2 follows as a corollary. The $m = 4$, $k = 1$, $d = 4$ case appeared in the Introduction.

Main Theorem 6.5. *Given* $\frac{2}{3}(n+1) = m < n$, *if* M *is a real analytic* m*-submanifold of* $\mathbb{C}^n$ *with a degenerate CR singularity having cubic normal form* (69), *and* $\widehat{M}$ *is a nondegenerate real analytic unfolding of* M *in* $\mathbb{C}^{n+k}$, *with defining equations in standard position given by* (75), *then there exists a holomorphic coordinate change* $(\tilde{z}, \tilde{w}) = \mathbf{A}(\vec{z}, \vec{w}) + \vec{p}(\vec{z}, \vec{w})$ *in* $\mathcal{U}_{m,n,k}$ (11), *in a neighborhood of* $(\vec{0}, \mathbf{0}) \in \mathbb{C}^{n+k}$, *transforming the equations* (75) *into the real algebraic normal form:*

$$
\begin{aligned}
\tilde{y}_\sigma &= 0 \\
\tilde{z}_\tau &= (\tilde{x}_{2(\tau-m+2)} + i\tilde{x}_{2(\tau-m+2)+1})\bar{\tilde{z}}_1 \\
\tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 \\
\tilde{z}_n &= (\tilde{z}_1 + \tilde{x}_2 + i\tilde{t}_1 + i\tilde{x}_3^2)\bar{\tilde{z}}_1 \\
\tilde{s}_\alpha &= 0.
\end{aligned}
\tag{76}
$$

Remark 6.6. The proof of the Main Theorem will be divided into several steps. The first step would be to show there exists a coordinate transformation $(\tilde{z}, \tilde{w}) = \mathbf{A}(\vec{z}, \vec{w}) + \vec{p}(\vec{z}, \vec{w})$ in $\mathcal{U}_{m,n,k}$ eliminating all the cubic terms from the normal form (75), except $i\bar{z}_1 x_3^2$, so that the quantities $E_2, \ldots, E_{m-1}$, $e_m, \ldots, e_n$ are all $O(4)$ in $z_1, \bar{z}_1, x, t$. However, rather than going through a lengthy but straightforward calculation to find a suitable invertible matrix $\mathbf{A}$ and polynomial $\vec{p}(\vec{z}, \vec{w})$ achieving such a transformation, we will skip ahead to the next step in an induction, and show that if $E_2, \ldots, e_n$ are $O(d)$ with $d \geq 4$, then there is a holomorphic transformation so that in the new coordinates, the higher degree terms of the defining functions have degree greater than d. The main part of that argument will be the solution of a certain system of linear equations, Theorem 7.6. After the Proof of that Theorem, it will be shown in Remark 7.7 how the calculation of that Proof can be modified in a small way to go back to the start of the induction at $d = 3$ and to establish this cubic normal form.

Then, showing that there exist a linear part $\mathbf{A}$ and a holomorphic function $\vec{p}(\vec{z}, \vec{w})$, so that $(\tilde{z}, \tilde{w}) = \mathbf{A}(\vec{z}, \vec{w}) + \vec{p}(\vec{z}, \vec{w})$ is in the subgroup $\mathcal{U}_{m,n,k}$ and takes the cubic normal form and eliminates all its higher degree terms to transform $\widehat{M}$ into the algebraic variety (76), involves finding $\vec{p}$ as a solution of a system of nonlinear functional equations, by iterating the solution of the linear system. The convergence argument is given in Section 7 in a series of steps, each stated as a Theorem.

6.3. The geometry of the algebraic unfolding.

Before proceeding to the Proof of Main Theorem 6.5, we conclude this Section by considering the geometry of the real algebraic variety given by (76), denoted $\widehat{M}$, which is a representative of the u-equivalence class for the nondegenerate unfolding $\widehat{M}$ of any real analytic M with cubic normal form (69). The example (1) from the Introduction is the $m=4$, $n=5$, $k=1$ case of $\widehat{M}$.

Recalling the complexification construction from Section 4, corresponding to the smooth real variety $\widehat{M} \subseteq \mathbb{C}^{n+k}$ is a smooth complex variety $\widehat{M}_c \subseteq \mathbb{C}^{2(n+k)}$, parametrized by $\Sigma : \mathbb{C}^{m+k} \to \mathbb{C}^{2(n+k)}$. Let $\mathbb{C}^{m+k}$ have coordinates

$$(z_1, \zeta_1, \xi_2, \ldots, \xi_{m-1}, \omega_1, \ldots, \omega_k) = (z_1, \zeta_1, \xi, \omega).$$

The image of $\widehat{M}_c$ under the projection $\pi : \mathbb{C}^{2(n+k)} \to \mathbb{C}^{n+k}$ is a singular complex variety, parametrized by $\pi \circ \Sigma$ as in (19):

$$\begin{array}{rcl} \pi \circ \Sigma : \mathbb{C}^{m+k} & \to & \mathbb{C}^{n+k} : \\ (z_1, \zeta_1, \xi, \omega) & \mapsto & (z_1, \xi_2, \ldots, \xi_{m-1}, \\ & & \quad \ldots, (\xi_{2(\tau-m+1)} + i\xi_{2(\tau-m+1)+1})\zeta_1, \ldots, \\ & & \quad \zeta_1^2, (z_1 + \xi_2 + i\omega_1 + i\xi_3^2)\zeta_1, \omega_1, \ldots, \omega_k)^T. \end{array}$$

At each point in the domain $\mathbb{C}^{m+k}$, the complex Jacobian of this polynomial map either will have full rank, $m+k$, or will be singular, with rank $m+k-1$. The singular locus in the domain is the affine variety

$$\{\xi_{2(\tau-m+1)} + i\xi_{2(\tau-m+1)+1} = 0, \zeta_1 = 0, z_1 + \xi_2 + i\omega_1 + i\xi_3^2 = 0\}.$$

As in (18), let $\boldsymbol{\delta}$ denote the inclusion of the totally real subspace

$$\{\zeta_1 = \bar{z}_1, \xi_2 = \overline{\xi_2} = x_2, \ldots, \omega_k = \overline{\omega_k} = t_k\}$$

in $\mathbb{C}^{m+k}$, so $\widehat{M}$ is the image of $\pi \circ \Sigma \circ \boldsymbol{\delta}$. The intersection of this real subspace with the singular locus is

$$\{x_4 = \ldots = x_{m-1} = 0, z_1 = \bar{z}_1 = 0, x_2 = 0, t_1 + x_3^2 = 0\},$$

so the candidates for CR singular points in $\widehat{M}$ are of the form

$$\{(0, 0, x_3, 0, \ldots, 0, t_1, t_2, \ldots, t_k)^T : t_1 + x_3^2 = 0\} \subseteq \widehat{M} \subseteq \mathbb{C}^{n+k},$$

and all the other points in $\widehat{M}$ are totally real. When $k=1$, this is a real parabola in $\widehat{M}$ contained in the real x_3, t_1 coordinate plane and tangent to the real x_3-axis at the origin; for $k > 1$, it is a k-dimensional real parabolic cylinder. As observed in Remark 6.4, $\widehat{M}$ considered as a real submanifold of $\mathbb{C}^{n+k}$ near the origin is locally biholomorphically equivalent to $\widetilde{M} \times \mathbb{R}^k$, so it has a totally real k-dimensional locus N_1 of nondegenerate CR singular points in a neighborhood of $(\vec{0}, \mathbf{0})$.

We continue studying the geometry of $\widehat{M}$ by considering the slices M_t for various t. The slice $M_{\mathbf{0}}$ is exactly the normal form variety of Proposition 6.2; as a submanifold of $\mathbb{C}^n$, it has a degenerate CR singular point at the origin and is totally real at every other point. For $t = (t_1, \ldots, t_n)$ with $t_1 > 0$, the slice M_t is totally real.

For t with $t_1 < 0$, the slice M_t has two candidates for CR singularities, at $(0, 0, x_3, 0, \ldots, 0)^T \in \mathbb{C}^n$, where $x_3 = \pm\sqrt{-t_1}$. Keeping in mind that t is fixed, so $t_1, \ldots, t_k$ are constants with t_1 negative, the equations for M_t in $\mathbb{C}^n$ are

$$\begin{aligned} y_\sigma &= 0 \\ z_\tau &= (x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\bar{z}_1 \\ z_{n-1} &= \bar{z}_1^2 \\ z_n &= (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1. \end{aligned} \tag{77}$$

This contains the origin but is not in standard position. Replacing x_3 with the quantity $x_3 - \sqrt{-t_1}$ is a translation that moves a CR singularity candidate point to the origin, and Equation (77) becomes

$$z_n = (z_1 + x_2 - 2i\sqrt{-t_1}x_3 + ix_3^2)\bar{z}_1,$$

which is in standard position, in the quadratic normal form (59–61), with $\sum \epsilon_n^{\sigma_1} x_{\sigma_1} = x_2 - 2i\sqrt{-t_1}x_3$ and $e_n = ix_3^2\bar{z}_1$.

Similarly, the other translation $x_3 \mapsto x_3 + \sqrt{-t_1}$ puts M_t into standard position with defining equation

$$z_n = (z_1 + x_2 + 2i\sqrt{-t_1}x_3 + ix_3^2)\bar{z}_1.$$

The conclusion is that both candidate points are in fact CR singularities, satisfying both nondegeneracy conditions, with complex tangent lines parallel to the z_1-axis, and with opposite indices, ± 1, corresponding to the sign of the coefficient of $ix_3\bar{z}_1$ in the quadratic normal form.

In the $k = 1$ case, we can consider t_1 as a time parameter increasing from negative to positive; the unfolding $\widehat{M}$ represents a cancellation, where M_t has two nondegenerate CR singularities of opposite index that approach each other, meet at the origin at $t_1 = 0$ so that $M_{\mathbf{0}}$ has a degenerate CR singularity, and then M_t is totally real afterward. By Main Theorem 6.5, the same phenomenon is expected to hold, locally in space and for a short interval of time, for any real analytic unfolding $\widehat{M}$ (satisfying a nondegeneracy condition as in (75)) of any real analytic $M_{\mathbf{0}}$ with a degenerate CR singularity having cubic normal form (69).

7. Rapid convergence Proof of the Main Theorem

7.1. A functional equation.

To show the existence of a normalizing transformation as claimed in Main Theorem 6.5, we will set up a system of nonlinear functional equations, so that any solution $\vec{p}$ of the system will define a normalizing transformation $(\tilde{z},\tilde{w})=(\vec{z},\vec{w})+\vec{p}$ in $\mathcal{U}_{m,n,k}$ (11). In addition to finding a formal power series solution, we will also have to show that the solution is convergent in some neighborhood of the origin. The method of proof is the rapid convergence technique, as used in [**Moser**], [$\mathbf{C}_4$], and [$\mathbf{C}_6$]. Rather than trying to solve the system of equations directly, we first find an approximate solution by solving a related system of linear equations. Iteration of this process gives a sequence of approximations that approach an exact solution. We are careful to construct the approximations so that their domains shrink slowly enough so their diameters are bounded below by a positive constant.

We start by considering the effect of a coordinate change (11) on the quadratic and cubic parts of the defining equations in normal form (75). The transformation $(\tilde{z},\tilde{w})=(\vec{z},\vec{w})+\vec{p}$ is formally invertible near $(\vec{0},\mathbf{0})$, and it may be useful to think of $(\tilde{z},\tilde{w})=(\vec{z},\vec{w})+\vec{p}$ as having identity linear part, although there may be some linear terms in $\vec{p}$, which could be included as entries in the above-diagonal blocks in the matrix $\mathbf{A}$ (12). In terms of $\tilde{z}$, $\tilde{w}$, $\vec{z}$, $\vec{w}$, consider the system of equations

$$
\begin{aligned}
(78)\quad 0 &= \operatorname{Im}(\tilde{z}_\sigma)=\operatorname{Im}(z_\sigma+p_\sigma(\vec{z},\vec{w}))\\
0 &= \tilde{z}_\tau-(\bar{\tilde{z}}_1\tilde{x}_{2(\tau-m+2)}+i\bar{\tilde{z}}_1\tilde{x}_{2(\tau-m+2)+1})\\
&= z_\tau+p_\tau(\vec{z},\vec{w})\\
&\quad -\overline{(z_1+p_1(\vec{z},\vec{w}))}\cdot\operatorname{Re}(z_{2(\tau-m+2)}+p_{2(\tau-m+2)}(\vec{z},\vec{w}))\\
&\quad -i\overline{(z_1+p_1(\vec{z},\vec{w}))}\cdot\operatorname{Re}(z_{2(\tau-m+2)+1}+p_{2(\tau-m+2)+1}(\vec{z},\vec{w}))\\
0 &= \tilde{z}_{n-1}-\bar{\tilde{z}}_1^2\\
&= z_{n-1}+p_{n-1}(\vec{z},\vec{w})-\overline{(z_1+p_1(\vec{z},\vec{w}))}^2\\
0 &= \tilde{z}_n-(\tilde{z}_1+\tilde{x}_2+i\tilde{t}_1+i\tilde{x}_3^2)\bar{\tilde{z}}_1\\
&= z_n+p_n(\vec{z},\vec{w})\\
&\quad -\overline{(z_1+p_1(\vec{z},\vec{w}))}\cdot(z_1+p_1(\vec{z},\vec{w}))\\
&\quad -\overline{(z_1+p_1(\vec{z},\vec{w}))}\cdot\operatorname{Re}(z_2+p_2(\vec{z},\vec{w}))\\
&\quad -i\overline{(z_1+p_1(\vec{z},\vec{w}))}\cdot\operatorname{Re}(w_1+P_1(\vec{w}))\\
&\quad -i\overline{(z_1+p_1(\vec{z},\vec{w}))}\cdot(\operatorname{Re}(z_3+p_3(\vec{z},\vec{w})))^2\\
0 &= \tilde{s}_\alpha=\operatorname{Im}(\tilde{w}_\alpha)=\operatorname{Im}(w_\alpha+P_\alpha(\vec{w})).
\end{aligned}
$$

In order to get (76) to be the defining equations for $\widehat{M}$ in the $(\tilde{z},\tilde{w})$ coordinates, the above equalities must hold for points $(\vec{z},\vec{w})$ on $\widehat{M}$ and near $(\vec{0},\mathbf{0})$. So, we can replace the $\vec{z}=(z_1,\ldots,z_n)^T$ and $\vec{w}=(w_1,\ldots,w_k)^T$ expressions in (78) by the defining functions (75):

$$
\begin{aligned}
\vec{z} &= (z_1, x_2 + iE_2(z_1,\bar{z}_1,x,t),\ldots,h_n(z_1,\bar{z}_1,x,t))^T, \\
\vec{w} &= t=(t_1,\ldots,t_k),
\end{aligned}
\tag{79}
$$

to get a system of equations where the RHS functions depend only on z_1, $\bar{z}_1$, x, t:

$$
\begin{aligned}
(80)\; 0 &= \mathrm{Im}(x_\sigma + iE_\sigma + p_\sigma(\vec{z},t)) = E_\sigma(z_1,\bar{z}_1,x,t) + \mathrm{Im}(p_\sigma(\vec{z},t)) \\
0 &= e_\tau(z_1,\bar{z}_1,x,t) + p_\tau(\vec{z},t) \\
&\quad -\overline{p_1(\vec{z},t)}\cdot(x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1}) \\
&\quad -\bar{z}_1\cdot(\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z},t)) + i\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z},t))) \\
&\quad -\overline{p_1(\vec{z},t)}\cdot(\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z},t)) + i\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z},t))) \\
0 &= e_{n-1}(z_1,\bar{z}_1,x,t) + p_{n-1}(\vec{z},t) - 2\bar{z}_1\overline{p_1(\vec{z},t)} - \overline{p_1(\vec{z},t)}^2 \\
0 &= e_n(z_1,\bar{z}_1,x,t) + p_n(\vec{z},t) \\
&\quad -\bar{z}_1\cdot(p_1(\vec{z},t) + \mathrm{Re}(p_2(\vec{z},t)) + iP_1(t) + 2ix_3\mathrm{Re}(p_3(\vec{z},t))) \\
(81)\quad &\quad -i\bar{z}_1\cdot(\mathrm{Re}(p_3(\vec{z},t)))^2 \\
&\quad -\overline{(p_1(\vec{z},t))}\cdot(z_1 + x_2 + it_1 + ix_3^2) \\
&\quad -\overline{(p_1(\vec{z},t))}\cdot(p_1(\vec{z},t) + \mathrm{Re}(p_2(\vec{z},t)) + iP_1(t)) \\
&\quad -\overline{(p_1(\vec{z},t))}\cdot(2ix_3\mathrm{Re}(p_3(\vec{z},t)) + i(\mathrm{Re}(p_3(\vec{z},t))^2)) \\
(82)\; 0 &= \mathrm{Im}(t_\alpha + P_\alpha(t)).
\end{aligned}
$$

The components of $\vec{e}=(E_2,\ldots,E_{m-1},e_m,\ldots,e_n)$ appear in (80–82) in two ways — as terms in the equations, and also in the $\vec{z}$ input (79) for each $p_j(\vec{z},t)$ in (80–82), $j=1,\ldots,n$. So, given $\vec{e}$, if we happen to have an exact solution $\vec{p}$ of the above system of functional equations, the conclusion of Main Theorem 6.5 holds and we are done. However, (80–82) is a nonlinear system in the unknown quantity $\vec{p}$, where in addition to the composition with the given defining functions (79), there are products of the components P_1, p_j, and $\overline{p_j}$.

If $k>1$, there are unknown quantities $P_2(t),\ldots,P_k(t)$, that appear only in line (82). The quantity $\mathrm{Im}(t_\alpha + P_\alpha(t))$ is identically zero for any P_α satisfying the reality condition of the subgroup $\mathcal{U}_{m,n,k}$, so we can choose $P_2=\ldots=P_k=0$ as the solution, leaving only the real quantity $P_1(t)$ as an unknown to be determined in terms of $\vec{e}$.

Suppose $\vec{e}$ has degree $d\geq 3$, then by inspection of the system (79–82), the components of a solution $\vec{p}$ have lower bounds on the weight:

$p_m, \dots, p_n$ have weight $\geq d$, $P_1, p_1, p_2, p_4, \dots, p_{m-1}$ have weight $\geq d-1$, and p_3 has weight $\geq d-2$. For $d=3$, p_3 may include some weight 1 linear terms, of the form

$$\tilde{z}_3 = z_3 + \left(\sum r_3^\alpha t_\alpha\right) + (\text{weight} \geq 2), \tag{83}$$

where r_3^α entries from the block $\mathbf{R}_{(m-2)\times k}$ from matrix $\mathbf{A}$ (12) will be needed (in Remark 7.7) to cancel the imaginary parts of cubic terms of the form $e_n^\alpha \bar{z}_1 x_3 t_\alpha$.

As a first step in solving for $\vec{p} = (p_1, \dots, p_n, P_1, 0, \dots, 0)^T$ in terms of $\vec{e}$, consider the following system of simpler equations:

$$\begin{aligned}
&(84)\quad 0 = E_\sigma(z_1, \bar{z}_1, x, t) + \mathrm{Im}(p_\sigma(\vec{z}, t))\\
&(85)\quad 0 = e_\tau(z_1, \bar{z}_1, x, t) + p_\tau(\vec{z}, t)\\
&\qquad\qquad -\overline{p_1(\vec{z}, t)} \cdot (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1})\\
&\qquad\qquad -\bar{z}_1 \cdot (\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z}, t)) + i\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z}, t)))\\
&(86)\quad 0 = e_{n-1}(z_1, \bar{z}_1, x, t) + p_{n-1}(\vec{z}, t) - 2\bar{z}_1 \overline{p_1(\vec{z}, t)}\\
&(87)\quad 0 = e_n(z_1, \bar{z}_1, x, t) + p_n(\vec{z}, t)\\
&\qquad\qquad -\bar{z}_1 \cdot (p_1(\vec{z}, t) + \mathrm{Re}(p_2(\vec{z}, t)) + iP_1(t) + 2ix_3\mathrm{Re}(p_3(\vec{z}, t)))\\
&\qquad\qquad -\overline{(p_1(\vec{z}, t))} \cdot (z_1 + x_2 + it_1 + ix_3^2),
\end{aligned}$$

where the $\vec{z}$ input for each p_j is:

$$\vec{z} = (z_1, x_2, \dots, x_{m-1}, (x_4 + ix_5)\bar{z}_1, \dots, \bar{z}_1^2, (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1)^T. \tag{88}$$

This simplifies $p_j(\vec{z}, t)$ by considering only the lower degree terms of the input (79) that appear in the algebraic normal form (76). Also, the products of P_1, p_j, $\overline{p_j}$ are dropped, so that these are (real) linear equations.

To see how the new equations are related to the original system, suppose $\vec{e}$ has degree $d \geq 3$, and that $\vec{p}$ is a solution of (84–88) so that p_3 has weight $\geq d-2$, $p_1, p_2, p_4, \dots, p_{m-1}$ have weight $\geq d-1$, and $p_m, \dots, p_n$ have weight $\geq d$. Evaluating the RHS of (80–82) with this solution for $\vec{p}$ results in expressions of degree $\geq 2d-2$, except for the $-i\bar{z}_1 \cdot (\mathrm{Re}(p_3(\vec{z}, t)))^2$ quantity (81), which has degree $2d-3$ if p_3 has weight $d-2$. Converting these expressions in z_1, $\bar{z}_1$, x, t to $\tilde{z}_1$, $\bar{\tilde{z}}_1$, $\tilde{x}$, $\tilde{t}$, and equating them to the $\tilde{z}$ expressions in (78) gives the higher degree terms of the new defining equations for $\widehat{M}$ in the $(\tilde{z}, \tilde{w})$ coordinate system. (It will be shown later (Theorem 7.13) that in fact for $(\vec{z}, \vec{w}) \in \widehat{M}$ close enough to $(\vec{0}, \mathbf{0})$, z_1, $\bar{z}_1$, x, t are real analytic functions of $\tilde{z}_1$, $\bar{\tilde{z}}_1$, $\tilde{x}$, $\tilde{t}$.) If $d \geq 4$, then $2d-3 > d$, so the expressions $E_2, \dots, e_n$ increase in degree, and while a solution $\vec{p}$ of the linearized equations is just an approximation to the solution of the original system, using such a $\vec{p}$ to define a coordinate transformation does have the effect of nearly doubling the order of vanishing of the $\vec{e}$ quantity.

7.2. A solution of a linearized equation.

The goal of this Subsection is to construct a solution $\vec{p}$ of the system of linear equations (84–88), given the higher degree terms of the defining equations, $\vec{e}$. The solution $\vec{p}$ constructed here will be an $(n+k)$-tuple of series in $\vec{z} = (z_1, \dots, z_n)^T$ and $\vec{w} = (w_1, \dots, w_k)^T$ with the following properties: the transformation $(\tilde{z}, \tilde{w}) = (\vec{z}, \vec{w}) + \vec{p}(\vec{z}, \vec{w})$ is in the subgroup $\mathcal{U}_{m,n,k}$ (11), the size of the domain of convergence of $\vec{p}$ in $\mathbb{C}^{n+k}$ is comparable in a certain sense to the size of the domain of $\vec{e}$, and also a suitable norm of $\vec{p}$ is bounded in terms of a suitable norm of $\vec{e}$.

Notation 7.1. For $\mathbf{r} = (r_1, \dots, r_N) \in \mathbb{R}^N$, with all $r_j > 0$, define a polydisc in $\mathbb{C}^N$ by

$$\mathbb{D}_{\mathbf{r}} = \{(Z_1, \dots, Z_N) : |Z_j| < r_j\}.$$

As special cases, let

$$D_r = \mathbb{D}_{(r,r,\dots,r)} \subseteq \mathbb{C}^{m+k}$$

and

$$\Delta_r = \mathbb{D}_{(r,\dots,r,2r^2,\dots,2r^2,r^2,3r^2+r^3,r,\dots,r)} \subseteq \mathbb{C}^{n+k},$$

where there are $m-1$ radius lengths r, followed by $n-m-1$ radius lengths $2r^2$, in the $z_m, \dots, z_{n-2}$ coordinate directions, and k radius lengths r in the $\vec{w}$ directions.

The initial assumption on the defining equations (75) is that

$$\vec{e}(z_1, \bar{z}_1, x, t) = (E_2, \dots, E_{m-1}, e_m, \dots, e_n)$$

is real analytic, so there is some $r > 0$ so that each component of $\vec{e}$ is the restriction to $\{\zeta = \bar{z}_1, x = \bar{x}, t = \bar{t}\}$ of a multivariable power series in (z_1, ζ, x, t) centered at the origin of $\mathbb{C}^{m+k}$, and with complex coefficients, which converges on a complex polydisc $D_r \subseteq \mathbb{C}^{m+k}$ (or, equivalently, a complex analytic function on D_r).

Notation 7.2. For a complex valued function $e(z_1, \zeta, x, t)$ of $m+k$ complex variables, which is defined on some set containing the polydisc D_r, define the norm

$$|e|_r = \sup_{(z_1,\zeta,x,t)\in D_r} |e(z_1, \zeta, x, t)|.$$

For an $(n-1)$-tuple $\vec{e} = (E_2, \dots, e_n)$, define

$$|\vec{e}|_r = |E_2|_r + \cdots + |e_n|_r.$$

For a complex valued function $p(z_1, \ldots, z_n, w_1, \ldots, w_k)$ of $n+k$ complex variables, which is defined on some set containing the polydisc Δ_r, define the norm

$$\|p\|_r = \sup_{(\vec{z}, \vec{w}) \in \Delta_r} |p(\vec{z}, \vec{w})|.$$

With this notation, we can further assume $r > 0$ is small enough so that $|\vec{e}(z_1, \zeta, x, t)|_r$ is finite. Given $\vec{e}$ with degree ≥ 4, the eventual goal is to find some $\tilde{r}$, $0 < \tilde{r} \leq r$, and a holomorphic map $\vec{p} : \Delta_{\tilde{r}} \to \mathbb{C}^{n+k}$, so that the transformation $(\tilde{z}, \tilde{w}) = (\vec{z}, \vec{w}) + \vec{p}(\vec{z}, \vec{w})$ is a biholomorphism with domain $\Delta_{\tilde{r}}$ taking $\widehat{M}$ to $\widehat{\widehat{M}}$ with the algebraic normal form (76). That is, if $(\vec{z}, \vec{w}) \in \widehat{M} \cap \Delta_{\tilde{r}}$, then $(\tilde{z}, \tilde{w})$ satisfies (76). However, for now we are only looking for $\vec{p}$ that is a solution of (84–88).

Some steps of the Proof of Theorem 7.6 will decompose series into subseries and their complex conjugates, where these preliminary Lemmas on the $|e|_r$ norm will be useful.

Lemma 7.3. *Given $0 < R < r$ and complex coefficients $f^{ab\mathbf{IK}}$, $g^{ab\mathbf{IK}}$, if*

$$\left| \sum f^{ab\mathbf{IK}} z_1^a \zeta^b x^{\mathbf{I}} t^{\mathbf{K}} \right|_r \leq K$$

and for complex x and t with $|x_\sigma| < r$, $|t_\alpha| < r$, $a, b = 0, 1, 2, 3, \ldots$,

$$\left| \sum_{\mathbf{I},\mathbf{K}} g^{ab\mathbf{IK}} x^{\mathbf{I}} t^{\mathbf{K}} \right| \leq \left| \sum_{\mathbf{I},\mathbf{K}} f^{ab\mathbf{IK}} x^{\mathbf{I}} t^{\mathbf{K}} \right|,$$

then

$$\left| \sum g^{ab\mathbf{IK}} z_1^a \zeta^b x^{\mathbf{I}} t^{\mathbf{K}} \right|_R \leq \frac{Kr^2}{(r-R)^2}.$$

∎

This is Lemma 5.3 of [$\mathbf{C}_6$], the Proof of which uses Cauchy's Estimate and the geometric series formula.

In the applications of the Lemma, for each pair (a, b), the coefficients $g^{ab\mathbf{IK}}$ will either be zero for all $\mathbf{I}$, $\mathbf{K}$, or equal to $f^{ab\mathbf{IK}}$ for all $\mathbf{I}$, $\mathbf{K}$, so the estimate in the hypothesis is satisfied.

Notation 7.4. On the complex vector space of formal power series, define the following real structure operator:

$$e = \sum e^{ab\mathbf{IK}} z_1^a \zeta^b x^{\mathbf{I}} t^{\mathbf{K}} \mapsto e' = \sum \overline{e^{ab\mathbf{IK}}} \zeta^a z_1^b x^{\mathbf{I}} t^{\mathbf{K}}.$$

The special case $h \mapsto \bar{h}$ of this operation appeared in Section 4.

Lemma 7.5. *Given $r > 0$, the restriction of the above map to the subspace $\{e : |e|_r < \infty\}$ is an isometry.* ∎

This is Lemma 5.5 of [$\mathbf{C}_6$].

Of course, this map is a representation of complex conjugation: given a series $e(z_1, \bar{z}_1, x, t)$ for real x and t, which complexifies to $e = e(z_1, \zeta, x, t)$ for $(z_1, \zeta, x, t) \in D_r$ for the purposes of finding its norm as in Notation 7.2, expanding $\overline{e(z_1, \bar{z}_1, x, t)}$ as a series in $(z_1, \bar{z}_1, x, t)$ and then complexifying gives $e' = e'(z_1, \zeta, x, t)$.

THEOREM 7.6. *For each $m = 4, 6, 8, \ldots$, and n satisfying $\frac{2}{3}(n+1) = m < n$, there are nonzero polynomials $C_m^1(.,.,.,.)$ and $C_m^2(.,.,.,.)$ with real, nonnegative coefficients such that for any $0 < R < r$, and any $\vec{e}(z_1, \zeta, x, t)$ convergent on D_r with $|\vec{e}|_r < \infty$ and degree $d \geq 4$, there exists*

$$\vec{p}(\vec{z}, \vec{w}) = (p_1, \ldots, p_n, P_1, \ldots, P_k)^T$$

which is convergent on Δ_r, and solves the linear system of equations (84–88), and satisfies: $P_2 = \ldots = P_k = 0$, $P_1(t) = \overline{P_1(t)}$ for real t, and such that

$$\begin{aligned} &\max\{\|p_1\|_R, \ldots, \|p_n\|_R, \|P_1\|_R\} \\ \leq \quad &\left(C_m^1(r, R, \frac{1}{r}, \frac{1}{R}) + \frac{C_m^2(r, R, \frac{1}{r}, \frac{1}{R})}{(r-R)^2} \right) |\vec{e}|_r. \end{aligned}$$

PROOF. First, notice that if $\vec{p}(z_1, \ldots, z_n, w_1, \ldots, w_k)$ is a formal series solution of (84–88), it does not follow that $\vec{p}$ is convergent at any point (other than the origin). For example, with any component p_j, the series expressions $p_j(\vec{z}, \vec{w})$ and

$$(89) \qquad p_j(\vec{z}, \vec{w}) + ((z_1 + z_2 + iw_1 + iz_3^2)^2 z_{n-1} - z_n^2) \cdot \mathrm{Q}(\vec{z}, \vec{w})$$

are formally the same when restricted to $\vec{z}$ as in (88) and $\vec{w} = t$, for any (possibly divergent) series Q. So, if one formal solution $\vec{p}$ exists, then there exist infinitely many divergent solutions. There may also exist formal series solutions that are convergent only on some neighborhood of the origin much smaller than that claimed in the Theorem.

Continuing with the abbreviations $x = x_2, \ldots, x_{m-1}$, $t = t_1, \ldots, t_k$, and also using $z = z_2, \ldots, z_{m-1}$, and multi-index notation $z^{\mathbf{I}}$ and $w^{\mathbf{K}} = w_1^{\mathbf{k}_1} \cdots w_k^{\mathbf{k}_k}$, the following choice of normalization will simplify the construction of the solution $\vec{p}$ satisfying the claimed convergence and bounds:

$$\begin{aligned} p_1(\vec{z}, \vec{w}) &= p_1(z_1, z, z_{n-1}, \vec{w}) \\ p_j(\vec{z}, \vec{w}) &= p_j^E(z_1, z, z_{n-1}, \vec{w}) + z_n p_j^O(z_1, z, z_{n-1}, \vec{w}), \ j = 2, \ldots, n-1 \\ p_n(\vec{z}, \vec{w}) &= p_n^E(z_1, z, z_{n-1}, \vec{w}) + z_n p_n^O(z_1, z, z_{n-1}, \vec{w}) \\ &\quad + p_n^L(z, z_m, \ldots, z_{n-2}, \vec{w}). \end{aligned}$$

Note that p_n^L is the only term in the above expressions that depends on $z_m, \ldots, z_{n-2}$, and we will further assume that the quantity p_n^L is linear in $z_m, \ldots, z_{n-2}$:

$$p_n^L(z, z_m \ldots, z_{n-2}, \vec{w}) = \sum_{\tau=m}^{n-2} p_{n,\tau}^L(z, \vec{w}) \cdot z_\tau.$$

The first component p_1 does not depend on z_n, and the remaining components, p_j, have some terms not depending on z_n, labeled p_j^E, and other terms which have exactly one linear factor of z_n. The p_j^E and p_j^O terminology corresponds to even and odd powers of $\bar{z}_1$ which appear after the substitution of (88) into $\vec{p}$. The choice that $\vec{p}$ has at most linear terms in $z_\tau = (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1})\bar{z}_1$ and $z_n = (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1$ is made to avoid high powers of the non-monomial quantities, since any multinomial coefficients in the series expansion of $\vec{p}(\vec{z}, \vec{w})$ could be large enough to affect the size of the domain of convergence.

The exact formulas for the claimed polynomials C_m^1 and C_m^2 are not important, it will be enough that they depend only on m and not on d or $\vec{e}$, and that $(r-R)^{-2}$ is the only (nonzero) power of $r - R$. These properties will follow from the choices made in the construction of $\vec{p}$, and from estimates calculated in the course of this Proof.

We begin with (86), the e_{n-1} equation of system (84–88), and apply the normalization condition on $\vec{p}$ to get the linear equation:

$$\begin{aligned} 0 \;=\; & e_{n-1}(z_1, \bar{z}_1, x, t) + p_{n-1}^E(z_1, x, \bar{z}_1^2, t) \\ & + (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 p_{n-1}^O(z_1, x, \bar{z}_1^2, t) \\ & - 2\bar{z}_1 p_1(z_1, x, \bar{z}_1^2, t). \end{aligned} \tag{90}$$

If the series expansion of e_{n-1} had only even powers of $\bar{z}_1$, then it would be a simple matter to compare the coefficients of $e_{n-1}(z_1, \bar{z}_1, x, t)$ and $p_{n-1}^E(z_1, x, \bar{z}_1^2, t)$, and get a solution of the equation with $p_{n-1}^O = p_1 = 0$. The odd powers of $\bar{z}_1$ in e_{n-1} make the p_{n-1}^O and p_1 quantities necessary to solve the equation.

The consideration of the terms of the components of the given quantity $\vec{e}$ which are even or odd in $\bar{z}_1$ follows the general plan for comparison of coefficients from [**Beloshapka**], [$\mathbf{C}_1$], [$\mathbf{C}_4$] in other normal form problems for CR singularities, and from analogous calculations in [$\mathbf{Whitney}_1$] for singularities of differentiable maps. We will also use some subseries decompositions, rearrangements, and add-and-subtract tricks, as in [$\mathbf{C}_5$], [$\mathbf{C}_6$], to get the given terms of $\vec{e}$ to correspond to the available terms of $\vec{p}$ in (90). In fact, e_{n-1} will be decomposed into more subseries than necessary just to solve Equation (90); the plan is to organize the solution p_1 into pieces that will be convenient for the later solution of Equation (87).

First, decompose e_{n-1} into even and odd parts $e_{n-1,A}$, $e_{n-1,B}$, $e_{n-1,C}$:

$$
\begin{aligned}
e_{n-1} &= \sum e_{n-1}^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} = e_{n-1,A} + e_{n-1,B} + e_{n-1,C}, \\
e_{n-1,A} &= \sum_{b \text{ even}} e_{n-1}^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
e_{n-1,B} &= \sum_{a \text{ even, } b \text{ odd}} e_{n-1}^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
e_{n-1,C} &= \sum_{a,\, b \text{ odd}} e_{n-1}^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}}.
\end{aligned} \tag{91}
$$

Since these subseries can be found by averaging, there are the estimates:

$$
\begin{aligned}
& |e_{n-1,A}|_r \\
= & \left| \frac{1}{2}(e_{n-1}(z_1,\zeta,x) + e_{n-1}(z_1,-\zeta,x)) \right|_r \le |e_{n-1}|_r, \\
& |e_{n-1,B}|_r \\
= & \left| \frac{e_{n-1}(z_1,\zeta,x) - e_{n-1}(z_1,-\zeta,x) + e_{n-1}(-z_1,\zeta,x) - e_{n-1}(-z_1,-\zeta,x)}{4} \right|_r \\
\le & |e_{n-1}|_r, \\
& |e_{n-1,C}|_r \\
= & \left| \frac{e_{n-1}(z_1,\zeta,x) - e_{n-1}(z_1,-\zeta,x) - e_{n-1}(-z_1,\zeta,x) + e_{n-1}(-z_1,-\zeta,x)}{4} \right|_r \\
\le & |e_{n-1}|_r.
\end{aligned}
$$

Since the x_2 and x_3 variables have a distinguished role in the normal form, we will index them separately and use the multi-index $\mathbf{J}$ to denote $x^{\mathbf{J}} = x_4^{j_4} \cdots x_{m-1}^{j_{m-1}}$. In the $(m,n) = (4,5)$ case, there are no such variables, but to avoid changing notation for this case, we can just assume $\mathbf{J} = \mathbf{0}$ and $x^{\mathbf{J}} = 1$ in the calculations that follow.

The $e_{n-1,B}$ subseries is further decomposed:

$$
\begin{aligned}
(92)\quad e_{n-1,B} &= e_{n-1,D}+e_{n-1,E}+e_{n-1,F}+e_{n-1,G}+e_{n-1,H}+e_{n-1,I},\\
e_{n-1,D} &= \sum_{a\text{ even},\ b\text{ odd},\ d>0} e_{n-1}^{abcd\mathbf{JK}} z_1^a\bar z_1^b x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n-1,E} &= \sum_{a\text{ even},\ b\text{ odd},\ b>1} e_{n-1}^{abc0\mathbf{JK}} z_1^a\bar z_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n-1,F} &= \sum_{a\text{ even},\ a>0} e_{n-1}^{a1c0\mathbf{JK}} z_1^a\bar z_1 x_2^c x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n-1,G} &= \sum_{c>0} e_{n-1}^{01c0\mathbf{JK}} \bar z_1 x_2^c x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n-1,H} &= \sum_{\mathbf{J}\neq\mathbf{0}} e_{n-1}^{0100\mathbf{JK}} \bar z_1 x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n-1,I} &= \sum e_{n-1}^{0100\mathbf{0K}} \bar z_1 t^{\mathbf{K}}.
\end{aligned}
$$

(The exponent and summation index ^{d} is not to be confused with the symbol d used to denote the degree of $\vec{e}$.)

Each of these subseries can be estimated in terms of $|e_{n-1,B}|_r$:

$$
\begin{aligned}
|e_{n-1,D}|_r &= |e_{n-1,B}(z_1,\zeta,x,t)-e_{n-1,B}(z_1,\zeta,x_2,0,x_4,\ldots,x_{m-1},t)|_r\\
&\leq 2|e_{n-1,B}|_r.
\end{aligned}
$$

Let $f_{n-1,A}$ denote the formal series

$$
f_{n-1,A}(z_1,\zeta,x,t)=\frac{e_{n-1,B}(z_1,\zeta,x_2,0,x_4,\ldots,x_{m-1},t)}{\zeta},
$$

which extends to a holomorphic function $f_{n-1,A}$ on D_r since $e_{n-1,B}$ is odd in ζ. Using the notation $D_r^*=D_r\setminus\{\zeta=0\}$, the maximum principle, and the Schwarz Lemma ([**A**]),

$$
\begin{aligned}
|f_{n-1,A}|_r &= \sup_{(z,\zeta,x,t)\in D_r^*}\left|\frac{e_{n-1,B}(z_1,\zeta,x_2,0,x_4,\ldots,x_{m-1},t)}{\zeta}\right|\\
&\leq \sup_{(z,\zeta,x,t)\in D_r^*}\frac{\frac{|\zeta|}{r}\sup_{|\zeta|<r}|e_{n-1,B}(z_1,\zeta,x_2,0,x_4,\ldots,x_{m-1},t)|}{|\zeta|}\\
&\leq \frac{1}{r}|e_{n-1,B}|_r.
\end{aligned}
$$

From

$$
e_{n-1,E}(z_1,\zeta,x,t)=\zeta\cdot(f_{n-1,A}(z_1,\zeta,x,t)-f_{n-1,A}(z_1,0,x,t)),
$$

we get the estimate $|e_{n-1,E}|_r \leq 2|e_{n-1,B}|_r$. Similarly, from

$$e_{n-1,F}(z_1, \zeta, x, t) = \zeta \cdot (f_{n-1,A}(z_1, 0, x, t) - f_{n-1,A}(0, 0, x, t)),$$

we get the estimate $|e_{n-1,F}|_r \leq 2|e_{n-1,B}|_r$, from

$$\begin{aligned} & e_{n-1,G}(z_1, \zeta, x, t) \\ = \ & \zeta \cdot (f_{n-1,A}(0, 0, x_2, \ldots, x_{m-1}, t) - f_{n-1,A}(0, 0, 0, 0, x_4, \ldots, x_{m-1}, t)), \end{aligned}$$

we get the estimate $|e_{n-1,G}|_r \leq 2|e_{n-1,B}|_r$, from

$$\begin{aligned} & e_{n-1,H}(z_1, \zeta, x, t) \\ = \ & \zeta \cdot (f_{n-1,A}(0, 0, 0, 0, x_4, \ldots, x_{m-1}, t) - f_{n-1,A}(0, 0, 0, 0, 0, \ldots, 0, t)), \end{aligned}$$

we get the estimate $|e_{n-1,H}|_r \leq 2|e_{n-1,B}|_r$, and from

$$e_{n-1,I}(z_1, \zeta, x, t) = \zeta f_{n-1,A}(0, 0, 0, 0, 0, \ldots, 0, t),$$

we get the estimate $|e_{n-1,I}|_r \leq |e_{n-1,B}|_r$.

The $e_{n-1,C}$ subseries also decomposes into parts:

$$\begin{aligned} e_{n-1,C} &= e_{n-1,J} + e_{n-1,K}, \\ e_{n-1,J} &= \sum_{a,\, b \text{ odd},\, d>0} e_{n-1}^{abcd\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \\ e_{n-1,K} &= \sum_{a,\, b \text{ odd}} e_{n-1}^{abc0\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}}, \end{aligned} \tag{93}$$

with each subseries estimated in terms of $|e_{n-1,C}|_r$:

$$\begin{aligned} |e_{n-1,J}|_r &= |e_{n-1,C}(z_1, \zeta, x, t) - e_{n-1,C}(z_1, \zeta, x_2, 0, x_4, \ldots, x_{m-1}, t)|_r \\ &\leq 2|e_{n-1,C}|_r, \\ |e_{n-1,K}|_r &= |e_{n-1,C}(z_1, \zeta, x_2, 0, x_4, \ldots, x_{m-1}, t)|_r \\ &\leq |e_{n-1,C}|_r. \end{aligned}$$

These two series are further rearranged:

$$\begin{aligned}
e_{n-1,J} &= e_{n-1,L} - e_{n-1,M},\\
e_{n-1,L} &= \sum_{a,\, b \text{ odd},\, d>0} e_{n-1}^{abcd\mathbf{JK}} z_1^{a-1}\bar{z}_1^{b-1} x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \cdot (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1\\
e_{n-1,M} &= \sum_{a,\, b \text{ odd},\, d>0} e_{n-1}^{abcd\mathbf{JK}} z_1^{a-1}\bar{z}_1^{b} x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \cdot (x_2 + it_1 + ix_3^2),\\
e_{n-1,K} &= e_{n-1,N} - e_{n-1,O} - e_{n-1,P} - e_{n-1,Q}\\
&\quad - e_{n-1,R} - e_{n-1,S} - e_{n-1,T} - e_{n-1,U},\\
e_{n-1,N} &= \sum_{a,\, b \text{ odd}} e_{n-1}^{abc0\mathbf{JK}} z_1^{a-1}\bar{z}_1^{b-1} x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1\\
e_{n-1,O} &= \sum_{a,\, b \text{ odd}} e_{n-1}^{abc0\mathbf{JK}} z_1^{a-1}\bar{z}_1^{b} x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (ix_3^2)\\
e_{n-1,P} &= \sum_{a,\, b \text{ odd},\, b>1} e_{n-1}^{abc0\mathbf{JK}} z_1^{a-1}\bar{z}_1^{b} x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (x_2 + it_1)\\
e_{n-1,Q} &= \sum_{a \text{ odd},\, a>1} e_{n-1}^{a1c0\mathbf{JK}} z_1^{a-1}\bar{z}_1 x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (x_2 + it_1)\\
e_{n-1,R} &= \sum e_{n-1}^{11c0\mathbf{JK}} \bar{z}_1 x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (x_2)\\
e_{n-1,S} &= \sum_{c>0} e_{n-1}^{11c0\mathbf{JK}} \bar{z}_1 x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (it_1)\\
e_{n-1,T} &= \sum_{\mathbf{J}\neq\mathbf{0}} e_{n-1}^{1100\mathbf{JK}} \bar{z}_1 x^{\mathbf{J}} t^{\mathbf{K}} \cdot (it_1)\\
e_{n-1,U} &= \sum e_{n-1}^{11000\mathbf{K}} \bar{z}_1 t^{\mathbf{K}} \cdot (it_1).
\end{aligned}$$

The Schwarz Lemma gives an estimate for $e_{n-1,M}$, and similarly for $e_{n-1,O}$, where this time D_r^* denotes $D_r \setminus \{z_1 = 0\}$:

$$\begin{aligned}
e_{n-1,M} &= \frac{x_2 + it_1 + ix_3^2}{z_1} e_{n-1,J}(z_1, \bar{z}_1, x, t)\\
\implies |e_{n-1,M}|_r &= \sup_{D_r^*} \left| (x_2 + it_1 + ix_3^2) \frac{e_{n-1,J}(z_1, \zeta, x, t)}{z_1} \right|\\
&\leq (2r + r^2) \sup_{D_r^*} \frac{\frac{|z_1|}{r} \sup_{|z_1|<r} |e_{n-1,J}(z_1, \zeta, x, t)|}{|z_1|}\\
&\leq (2 + r)|e_{n-1,J}|_r.
\end{aligned}$$

$$
\begin{aligned}
e_{n-1,O} &= \frac{ix_3^2}{z_1} e_{n-1,K}(z_1, \bar{z}_1, x, t) \\
\Longrightarrow\ |e_{n-1,O}|_r &= \sup_{D_r^*} \left| (ix_3^2) \frac{e_{n-1,K}(z_1, \zeta, x, t)}{z_1} \right| \\
&\leq r^2 \sup_{D_r^*} \frac{\frac{|z_1|}{r} \sup\limits_{|z_1|<r} |e_{n-1,K}(z_1, \zeta, x, t)|}{|z_1|} \leq r|e_{n-1,K}|_r.
\end{aligned}
$$

For estimates on the remaining terms, we introduce the expression

$$
f_{n-1,B} = \frac{e_{n-1,K}(z_1, \zeta, x, t)}{z_1 \zeta},
$$

which can be estimated using two applications of the Schwarz Lemma:

$$
\begin{aligned}
(94)\quad |f_{n-1,B}|_r &= \sup_{(z_1,\zeta,x,t)\in D_r^{**}} \left| \frac{e_{n-1,K}(z_1, \zeta, x, t)}{z_1 \zeta} \right| \\
&\leq \sup_{(z_1,\zeta,x,t)\in D_r^{**}} \frac{\frac{|z_1|}{r} \sup\limits_{|z_1|<r} |e_{n-1,K}(z_1, \zeta, x, t)|}{|z_1||\zeta|} \\
&\leq \frac{1}{r} \sup_{(z_1,\zeta,x,t)\in D_r^{**}} \frac{\sup\limits_{|z_1|<r} \left| \frac{|\zeta|}{r} \sup\limits_{|\zeta|<r} |e_{n-1,K}(z_1, \zeta, x, t)| \right|}{|\zeta|} \\
&\leq \frac{1}{r^2} |e_{n-1,K}|_r.
\end{aligned}
$$

In some of the above steps, we restricted to the open subset $D_r^{**} = D_r \setminus (\{z_1 = 0\} \cup \{\zeta = 0\})$, again to avoid division by 0 but not affecting the sup by the maximum principle.

$$
\begin{aligned}
e_{n-1,P} &= (x_2 + it_1)\zeta \cdot (f_{n-1,B}(z_1, \zeta, x, t) - f_{n-1,B}(z_1, 0, x, t)) \implies \\
|e_{n-1,P}|_r &\leq 2r^2(|f_{n-1,B}|_r + |f_{n-1,B}|_r) \leq 4|e_{n-1,K}|_r, \\
e_{n-1,Q} &= (x_2 + it_1)\zeta \cdot (f_{n-1,B}(z_1, 0, x, t) - f_{n-1,B}(0, 0, x, t)) \implies \\
|e_{n-1,Q}|_r &\leq 2r^2(|f_{n-1,B}|_r + |f_{n-1,B}|_r) \leq 4|e_{n-1,K}|_r, \\
e_{n-1,R} &= x_2\zeta f_{n-1,B}(0, 0, x, t) \implies \\
|e_{n-1,R}|_r &\leq r^2|f_{n-1,B}|_r \leq |e_{n-1,K}|_r, \\
e_{n-1,S} &= (it_1)\zeta \cdot (f_{n-1,B}(0, 0, x, t) \\
&\qquad -f_{n-1,B}(0, 0, 0, 0, x_4, \ldots, x_{m-1}, t)) \implies \\
|e_{n-1,S}|_r &\leq r^2(|f_{n-1,B}|_r + |f_{n-1,B}|_r) \leq 2|e_{n-1,K}|_r, \\
e_{n-1,T} &= (it_1)\zeta \cdot (f_{n-1,B}(0, 0, 0, 0, x_4, \ldots, x_{m-1}, t) \\
&\qquad -f_{n-1,B}(0, 0, \mathbf{0}, t)) \implies \\
|e_{n-1,T}|_r &\leq r^2(|f_{n-1,B}|_r + |f_{n-1,B}|_r) \leq 2|e_{n-1,K}|_r, \\
e_{n-1,U} &= (it_1)\zeta f_{n-1,B}(0, 0, \mathbf{0}, t) \implies \\
|e_{n-1,U}|_r &\leq r^2|f_{n-1,B}|_r \leq |e_{n-1,K}|_r.
\end{aligned}
$$

We also make a more specific choice of normalization of $p_1(\vec{z}, \vec{w})$ as follows:

$$
\begin{aligned}
p_1 &= p_{1A} + p_{1B} + p_{1C} + p_{1D} + p_{1E} + p_{1F} + p_{1G} \\
p_{1A} &= \sum_{\alpha \text{ even},\ \mu > 0} p_{1A}^{\alpha\lambda\mu\mathbf{J}\beta\mathbf{K}} z_1^\alpha z_2^\lambda z_3^\mu z^{\mathbf{J}} z_{n-1}^\beta w^{\mathbf{K}} \\
p_{1B} &= \sum_{\alpha \text{ even},\ \alpha > 0} p_{1B}^{\alpha\lambda 0\mathbf{J}\beta\mathbf{K}} z_1^\alpha z_2^\lambda z^{\mathbf{J}} z_{n-1}^\beta w^{\mathbf{K}} \\
p_{1C} &= \sum_{\beta > 0} p_{1C}^{0\lambda 0\mathbf{J}\beta\mathbf{K}} z_2^\lambda z^{\mathbf{J}} z_{n-1}^\beta w^{\mathbf{K}} \\
p_{1D} &= \sum_{\lambda > 0} p_{1D}^{0\lambda 0\mathbf{J}0\mathbf{K}} z_2^\lambda z^{\mathbf{J}} w^{\mathbf{K}} \\
p_{1E} &= \sum_{\mathbf{J} \neq \mathbf{0}} p_{1E}^{000\mathbf{J}0\mathbf{K}} z^{\mathbf{J}} w^{\mathbf{K}} \\
p_{1F} &= \sum p_{1F}^{000\mathbf{0}0\mathbf{K}} w^{\mathbf{K}} \\
p_{1G} &= \sum_{\alpha \text{ odd}} p_{1G}^{\alpha\lambda 0\mathbf{J}\beta\mathbf{K}} z_1^\alpha z_2^\lambda z^{\mathbf{J}} z_{n-1}^\beta w^{\mathbf{K}}.
\end{aligned}
$$

Now we can re-group the RHS of (90) to get the equation:

$$
\begin{aligned}
0 \quad = \quad & e_{n-1,A} + p^E_{n-1} - 2\bar{z}_1\overline{p_{1G}} \\
& + e_{n-1,L} + e_{n-1,N} + (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 p^O_{n-1} \\
& + e_{n-1,D} - e_{n-1,M} - e_{n-1,O} - 2\bar{z}_1\overline{p_{1A}} \\
& + e_{n-1,E} - e_{n-1,P} - 2\bar{z}_1\overline{p_{1B}} \\
& + e_{n-1,F} - e_{n-1,Q} - 2\bar{z}_1\overline{p_{1C}} \\
& + e_{n-1,G} - e_{n-1,R} - e_{n-1,S} - 2\bar{z}_1\overline{p_{1D}} \\
& + e_{n-1,H} - e_{n-1,T} - 2\bar{z}_1\overline{p_{1E}} \\
& + e_{n-1,I} - e_{n-1,U} - 2\bar{z}_1\overline{p_{1F}},
\end{aligned} \tag{95}
$$

with like terms in each line, so that setting each line equal to zero gives, by inspection, a unique solution for p^O_{n-1} and $p_{1A}, \ldots, p_{1F}$. The quantity p_{1G} will be found later and then p^E_{n-1} can be determined.

We record the estimates for the solved terms, beginning with p^O_{n-1}:

$$
\begin{aligned}
p^O_{n-1}(z_1, x, \bar{z}_1^2, t) &= -\frac{e_{n-1,L} + e_{n-1,N}}{(z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1} \\
&= -\frac{e_{n-1,J} + e_{n-1,K}}{z_1\bar{z}_1} = -\frac{e_{n-1,C}}{z_1\bar{z}_1} \\
\Longrightarrow p^O_{n-1}(z_1, x, \zeta^2, t) &= -\frac{e_{n-1,C}(z_1, \zeta, x, t)}{z_1\zeta} \\
\Longrightarrow ||p^O_{n-1}||_r &= \sup_{|z_1|<r, |x_\sigma|<r, |\zeta^2|<r^2, |t_\alpha|<r} |p^O_{n-1}(z_1, x, \zeta^2, t)| \\
&= \sup_{|z_1|<r, |x_\sigma|<r, |\zeta|<r, |t_\alpha|<r} \left| -\frac{e_{n-1,C}(z_1, \zeta, x, t)}{z_1\zeta} \right| \\
&\leq \frac{1}{r^2}|e_{n-1,C}|_r \qquad (96) \\
\Longrightarrow \|z_n p^O_{n-1}\|_r &\leq \|z_n\|_r \|p^O_{n-1}\|_r \leq (3 + r)|e_{n-1}|_r.
\end{aligned}
$$

The above step (96) uses the Schwarz Lemma twice, in steps similar to (94). The following calculation uses the Schwarz Lemma again, and also Lemma 7.5, and later calculations with similar steps may use these

Lemmas without mention.

$$
\begin{aligned}
p_{1A}(z_1,x,\bar{z}_1^2,t) &= \overline{\left(\frac{e_{n-1,D}-e_{n-1,M}-e_{n-1,O}}{2\bar{z}_1}\right)} \\
\implies p_{1A}(z_1,x,\zeta^2,t) &= \frac{e'_{n-1,D}-e'_{n-1,M}-e'_{n-1,O}}{2z_1} \\
\implies \|p_{1A}\|_r &\leq \frac{1}{2r}(|e'_{n-1,D}|_r+|e'_{n-1,M}|_r+|e'_{n-1,O}|_r) \\
&\leq \frac{1}{2r}(2|e_{n-1,B}|_r+(2+r)|e_{n-1,J}|_r+r|e_{n-1,K}|_r) \\
&\leq (\frac{3}{r}+\frac{3}{2})|e_{n-1}|_r. \\
p_{1B}(z_1,x,\bar{z}_1^2,t) &= \overline{\left(\frac{e_{n-1,E}-e_{n-1,P}}{2\bar{z}_1}\right)} \\
\implies p_{1B}(z_1,x,\zeta^2,t) &= \frac{e'_{n-1,E}-e'_{n-1,P}}{2z_1} \\
\implies \|p_{1B}\|_r &\leq \frac{1}{2r}(|e'_{n-1,E}|_r+|e'_{n-1,P}|_r) \\
&\leq \frac{1}{2r}(2|e_{n-1,B}|_r+4|e_{n-1,K}|_r)\leq\frac{3}{r}|e_{n-1}|_r. \\
p_{1C}(z_1,x,\bar{z}_1^2,t) &= \overline{\left(\frac{e_{n-1,F}-e_{n-1,Q}}{2\bar{z}_1}\right)} \\
\implies p_{1C}(z_1,x,\zeta^2,t) &= \frac{e'_{n-1,F}-e'_{n-1,Q}}{2z_1} \\
\implies \|p_{1C}\|_r &\leq \frac{1}{2r}(|e'_{n-1,F}|_r+|e'_{n-1,Q}|_r) \\
&\leq \frac{1}{2r}(2|e_{n-1,B}|_r+4|e_{n-1,K}|_r)\leq\frac{3}{r}|e_{n-1}|_r. \\
p_{1D}(z_1,x,\bar{z}_1^2,t) &= \overline{\left(\frac{e_{n-1,G}-e_{n-1,R}-e_{n-1,S}}{2\bar{z}_1}\right)} \\
\implies p_{1D}(z_1,x,\zeta^2,t) &= \frac{e'_{n-1,G}-e'_{n-1,R}-e'_{n-1,S}}{2z_1} \\
\implies \|p_{1D}\|_r &\leq \frac{1}{2r}(|e'_{n-1,G}|_r+|e'_{n-1,R}|_r+|e'_{n-1,S}|_r) \\
&\leq \frac{1}{2r}(2|e_{n-1,B}|_r+|e_{n-1,K}|_r+2|e_{n-1,K}|_r) \\
&\leq \frac{5}{2r}|e_{n-1}|_r.
\end{aligned}
$$

$$
\begin{aligned}
p_{1E}(z_1, x, \bar{z}_1^2, t) &= \overline{\left(\frac{e_{n-1,H} - e_{n-1,T}}{2\bar{z}_1}\right)} \\
\implies p_{1E}(z_1, x, \zeta^2, t) &= \frac{e'_{n-1,H} - e'_{n-1,T}}{2z_1} \\
\implies \|p_{1E}\|_r &\leq \frac{1}{2r}(|e'_{n-1,H}|_r + |e'_{n-1,T}|_r) \\
&\leq \frac{1}{2r}(2|e_{n-1,B}|_r + 2|e_{n-1,K}|_r) \leq \frac{2}{r}|e_{n-1}|_r. \\
p_{1F}(z_1, x, \bar{z}_1^2, t) &= \overline{\left(\frac{e_{n-1,I} - e_{n-1,U}}{2\bar{z}_1}\right)} \\
\implies p_{1F}(z_1, x, \zeta^2, t) &= \frac{e'_{n-1,I} - e'_{n-1,U}}{2z_1} \\
\implies \|p_{1F}\|_r &\leq \frac{1}{2r}(|e'_{n-1,I}|_r + |e'_{n-1,U}|_r) \\
&\leq \frac{1}{2r}(|e_{n-1,B}|_r + |e_{n-1,K}|_r) \leq \frac{1}{r}|e_{n-1}|_r.
\end{aligned}
$$

The norm of p_1 cannot be estimated until we find p_{1G}, but the other terms can be combined to get:

$$
\begin{aligned}
(97) \quad & p_{1A} + \ldots + p_{1F} \\
= \quad & \frac{1}{2z_1}\left(e'_{n-1,D} + \ldots + e'_{n-1,I} - e'_{n-1,M} - (e'_{n-1,O} + \ldots + e'_{n-1,U})\right) \\
= \quad & \frac{1}{2z_1}\left(e'_{n-1,B} - e'_{n-1,M} - \left(\frac{x_2 + it_1 + ix_3^2}{z_1} e_{n-1,K}\right)'\right) \\
\implies \quad & \|p_{1A} + \ldots + p_{1F}\|_r \\
\leq \quad & \frac{1}{2r}|e_{n-1}|_r + \frac{1}{2r}(2+r)2|e_{n-1}|_r + \frac{1}{2r}\frac{2r + r^2}{r}|e_{n-1}|_r \\
= \quad & (\frac{7}{2r} + \frac{3}{2})|e_{n-1}|_r.
\end{aligned}
$$

The next step is to consider Equation (87), and to solve for the unknowns p_n, P_1, p_{1G}, p_2, and p_3. The unknown quantities p_2, p_3 also appear in the $\sigma = 2, 3$ cases of (84), so we will need to consider a system of three equations:

$$
\begin{aligned}
0 &= E_2(z_1, \bar{z}_1, x, t) + \mathrm{Im}(p_2(\vec{z}, t)) \\
0 &= E_3(z_1, \bar{z}_1, x, t) + \mathrm{Im}(p_3(\vec{z}, t)) \\
0 &= e_n(z_1, \bar{z}_1, x, t) + p_n^E(z_1, x, \bar{z}_1^2, t) \\
&\quad +(z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 p_n^O(z_1, x, \bar{z}_1^2, t) + p_n^L(\vec{z}, t) \\
&\quad -\bar{z}_1 \cdot (p_1(\vec{z}, t) + \mathrm{Re}(p_2(\vec{z}, t)) + iP_1(t) + 2ix_3\mathrm{Re}(p_3(\vec{z}, t))) \\
&\quad -\overline{(p_1(\vec{z}, t))} \cdot (z_1 + x_2 + it_1 + ix_3^2).
\end{aligned} \tag{98}
$$

Finding the term p_{1G} will finish the solution for p_1 and then p_{n-1}^E can be determined in (95).

Starting with the RHS of (84), the following decomposition of E_σ will be used for all $\sigma = 2, \ldots, m-1$:

$$
\begin{aligned}
E_\sigma &= E_{\sigma,A} + e_{\sigma,B} + \overline{e_{\sigma,B}} + e_{\sigma,C} + \overline{e_{\sigma,C}} + E_{\sigma,D}, \\
E_{\sigma,A} &= \sum_{a,\, b \text{ even}} E_\sigma^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
e_{\sigma,B} &= \sum_{a>b,\, a \text{ odd},\, b \text{ even}} E_\sigma^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
e_{\sigma,C} &= \sum_{a>b,\, a \text{ even},\, b \text{ odd}} E_\sigma^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
E_{\sigma,D} &= \sum_{a,\, b \text{ odd}} E_\sigma^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
&= e_{\sigma,E} - e_{\sigma,F}, \\
e_{\sigma,E} &= (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 \sum_{a,\, b \text{ odd}} E_\sigma^{ab\mathbf{IK}} z_1^{a-1} \bar{z}_1^{b-1} x^{\mathbf{I}} t^{\mathbf{K}} \\
e_{\sigma,F} &= (x_2 + it_1 + ix_3^2) \sum_{a,\, b \text{ odd}} E_\sigma^{ab\mathbf{IK}} z_1^{a-1} \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}}.
\end{aligned}
$$

The $E_{\sigma,A}$ piece is simply an even part, so $|E_{\sigma,A}|_r \le |E_\sigma|_r$, and similarly for the odd part, $|E_{\sigma,D}|_r \le |E_\sigma|_r$. The other two subseries satisfy the estimate from Lemma 7.3: $|e_{\sigma,B}|_R \le \frac{r^2}{(r-R)^2}|E_\sigma|_r$ and $|e_{\sigma,C}|_R \le \frac{r^2}{(r-R)^2}|E_\sigma|_r$. We re-group some of these subseries:

$$
\begin{aligned}
m_\sigma(z_1, \bar{z}_1, x) &= \overline{e_{\sigma,B}} + e_{\sigma,C} - e_{\sigma,F} = \sum_{a \text{ even},\, b \text{ odd}} m_\sigma^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
f_\sigma(z_1, \bar{z}_1, x) &= E_{\sigma,A} + e_{\sigma,B} + \overline{e_{\sigma,C}} + \overline{m_\sigma} = \sum_{b \text{ even}} f_\sigma^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}},
\end{aligned}
$$

So, for each σ, there is a decomposition of the real valued series E_σ of the form

$$E_\sigma = f_\sigma + e_{\sigma,E} + m_\sigma - \overline{m_\sigma}, \tag{99}$$

where f_σ is even in $\bar{z}_1$ and m_σ is even in z_1 and odd in $\bar{z}_1$. This decomposition is similar to that of E_4 in [$\mathbf{C}_6$].

The estimates follow from Lemma 7.5 and the Schwarz Lemma:

$$\begin{aligned}
|e_{\sigma,E}|_r &= \left|\frac{z_1 + x_2 + it_1 + ix_3^2}{z_1} E_{\sigma,D}\right|_r \\
&\leq (3+r)|E_\sigma|_r, \\
|m_\sigma|_R &= |e'_{\sigma,B} + e_{\sigma,C} - e_{\sigma,F}|_R \leq |e_{\sigma,B}|_R + |e_{\sigma,C}|_R + |e_{\sigma,F}|_R \\
&\leq 2\frac{r^2}{(r-R)^2}|E_\sigma|_r + |x_2 + it_1 + ix_3^2|_R \left|\frac{E_{\sigma,D}}{z_1}\right|_R \\
&\leq \left(\frac{2r^2}{(r-R)^2} + 2 + R\right)|E_\sigma|_r, \\
|f_\sigma|_R &= |E_{\sigma,A} + 2e_{\sigma,B} + 2e'_{\sigma,C} - e'_{\sigma,F}|_R \\
&\leq \left(\frac{4r^2}{(r-R)^2} + 3 + R\right)|E_\sigma|_r.
\end{aligned}$$

For $\sigma = 2$ only, we decompose $m_2 = m_{2A} + m_{2B}$:

$$\begin{aligned}
m_{2A}(z_1, \bar{z}_1, x, t) &= \sum_{a \text{ even},\, b \text{ odd},\, d>0} m_2^{abcd\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \\
m_{2A}(z_1, \zeta, x, t) &= m_2(z_1, \zeta, x_2, x_3, x_4, \ldots, x_{m-1}, t) \\
&\quad - m_2(z_1, \zeta, x_2, 0, x_4, \ldots, x_{m-1}, t) \\
\implies |m_{2A}|_R &\leq 2|m_2|_R, \\
m_{2B}(z_1, \bar{z}_1, x, t) &= \sum_{a \text{ even},\, b \text{ odd}} m_2^{abc0\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}}, \\
m_{2B}(z_1, \zeta, x, t) &= m_2(z_1, \zeta, x_2, 0, x_4, \ldots, x_{m-1}, t) \\
\implies |m_{2B}|_R &\leq |m_2|_R.
\end{aligned}$$

Recalling that the quantities $p_{1A}, \ldots, p_{1F}$ have already been determined in terms of e_{n-1}, we re-write Equation (98) with the known terms first (100–101), and introduce some new known quantities by adding and subtracting (102–103) some of the terms from the decomposition of E_2 and E_3:

$$
\begin{aligned}
(100)\quad 0 &= e_n(z_1,\bar{z}_1,x,t)-\bar{z}_1\cdot(p_{1A}+\ldots+p_{1F}) \\
(101)\quad &\quad -(z_1+x_2+it_1+ix_3^2)\cdot\overline{(p_{1A}+\ldots+p_{1F})} \\
(102)\quad &\quad +\bar{z}_1\cdot(im_2-i\overline{m_2}-2x_3m_3+2x_3\overline{m_3}) \\
(103)\quad &\quad -i\bar{z}_1m_2+i\bar{z}_1\overline{m_{2A}}+i\bar{z}_1\overline{m_{2B}}+2\bar{z}_1x_3m_3-2\bar{z}_1x_3\overline{m_3} \\
&\quad +p_n^E(z_1,x,\bar{z}_1^2,t) \\
&\quad +(z_1+x_2+it_1+ix_3^2)\bar{z}_1p_n^O(z_1,x,\bar{z}_1^2,t)+p_n^L(\vec{z},t) \\
&\quad -\bar{z}_1\cdot(p_{1G}(\vec{z},t)+\mathrm{Re}(p_2(\vec{z},t))+iP_1(t)+2ix_3\mathrm{Re}(p_3(\vec{z},t))) \\
&\quad -\overline{(p_{1G}(\vec{z},t))}\cdot(z_1+x_2+it_1+ix_3^2).
\end{aligned}
$$

An analogous add-and-subtract trick was used in [$\mathbf{C}_6$]; here it will eventually lead to a cancellation that allows for the solution of p_2 and p_3 in terms of e_n, e_{n-1}, E_2, and E_3.

To get all the unknown terms to correspond to the given terms, we continue with a series decomposition of e_n:

As in (91), decompose e_n into even and odd parts $e_{n,A}$, $e_{n,B}$, $e_{n,C}$:

$$
\begin{aligned}
e_n &= \sum e_n^{ab\mathbf{IK}}z_1^a\bar{z}_1^bx^{\mathbf{I}}t^{\mathbf{K}}=e_{n,A}+e_{n,B}+e_{n,C}, \\
e_{n,A} &= \sum_{b\text{ even}} e_n^{ab\mathbf{IK}}z_1^a\bar{z}_1^bx^{\mathbf{I}}t^{\mathbf{K}} \\
e_{n,B} &= \sum_{a\text{ even},\, b\text{ odd}} e_n^{ab\mathbf{IK}}z_1^a\bar{z}_1^bx^{\mathbf{I}}t^{\mathbf{K}} \\
e_{n,C} &= \sum_{a,\, b\text{ odd}} e_n^{ab\mathbf{IK}}z_1^a\bar{z}_1^bx^{\mathbf{I}}t^{\mathbf{K}}.
\end{aligned}
$$

As with e_{n-1}, each of these subseries has norm bounded by $|e_n|_r$. The $e_{n,B}$ subseries is further decomposed:

$$\begin{aligned}
e_{n,B} &= e_{n,D} + e_{n,E} + e_{n,F} + e_{n,G} + e_{n,H} + e_{n,I},\\
e_{n,D} &= \sum_{a \text{ even},\, b \text{ odd},\, d>0} e_n^{abcd\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n,E} &= \sum_{a \text{ even},\, b \text{ odd},\, a>0} e_n^{abc0\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n,F} &= \sum_{b \text{ odd},\, b>1} e_n^{0bc0\mathbf{JK}} \bar{z}_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n,G} &= \sum_{c>0} e_n^{01c0\mathbf{JK}} \bar{z}_1 x_2^c x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n,H} &= \sum_{\mathbf{J}\neq\mathbf{0}} e_n^{0100\mathbf{JK}} \bar{z}_1 x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n,I} &= \sum e_n^{01000\mathbf{K}} \bar{z}_1 t^{\mathbf{K}}.
\end{aligned}$$

This is not exactly the same as the $e_{n-1,B}$ decomposition (92); the $e_{n,E}$ and $e_{n,F}$ terms are defined differently. However, the estimates are similar: $|e_{n,D}|_r \le 2|e_{n,B}|_r \le 2|e_n|_r$, $|e_{n,E}|_r \le 2|e_{n,B}|_r \le 2|e_n|_r$, and using the Schwarz Lemma, $|e_{n,F}|_r \le 2|e_{n,B}|_r \le 2|e_n|_r$, $|e_{n,G}|_r \le 2|e_{n,B}|_r \le 2|e_n|_r$, $|e_{n,H}|_r \le 2|e_{n,B}|_r \le 2|e_n|_r$, and $|e_{n,I}| \le |e_{n,B}|_r \le |e_n|_r$.

The $e_{n,C}$ subseries also decomposes into parts:

$$\begin{aligned}
e_{n,C} &= e_{n,J} + e_{n,K},\\
e_{n,J} &= \sum_{a,\, b \text{ odd},\, d>0} e_n^{abcd\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}}\\
e_{n,K} &= \sum_{a,\, b \text{ odd}} e_n^{abc0\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}},
\end{aligned}$$

and this is just like the $e_{n-1,C}$ decomposition (93), with similar estimates: $|e_{n,J}|_r \le 2|e_{n,C}|_r \le 2|e_n|_r$, $|e_{n,K}|_r \le |e_{n,C}|_r \le |e_n|_r$.

To start collecting some like terms in the known quantities, let:

$$\begin{aligned}
f_{n,A} &= e_{n,J} + i\bar{z}_1\overline{m_{2A}} - 2\bar{z}_1 x_3 \overline{m_3} \\
&= \sum_{a,\, b \text{ odd},\, d>0} f_{n,A}^{abcd\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} = f_{n,B} - f_{n,C}, \\
f_{n,B} &= \sum_{a,\, b \text{ odd},\, d>0} f_{n,A}^{abcd\mathbf{JK}} z_1^{a-1} \bar{z}_1^{b-1} x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \cdot (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 \\
f_{n,C} &= \sum_{a,\, b \text{ odd},\, d>0} f_{n,A}^{abcd\mathbf{JK}} z_1^{a-1} \bar{z}_1^{b} x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \cdot (x_2 + it_1 + ix_3^2).
\end{aligned}$$

Let:

$$f_{n,E}(\bar{z}_1, x, t) = e_{n,F} - \bar{z}_1 p_{1C} = \sum_{b \text{ odd},\, b>1} f_{n,E}^{0bc0\mathbf{JK}} \bar{z}_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}},$$

and let $g_{n,A}(\zeta, x, t) = \frac{1}{\zeta} f_{n,E}(\zeta, x, t)$, so that

$$f_{n,E}(\bar{z}_1, x, t) = \bar{z}_1 \cdot (g_{n,A} + \overline{g_{n,A}}) - \bar{z}_1 \overline{g_{n,A}}.$$

Let:

$$\begin{aligned}
f_{n,D} &= e_{n,E} - \bar{z}_1 p_{1B} - \bar{z}_1 \overline{g_{n,A}} \\
&= \sum_{a \text{ even},\, b \text{ odd},\, a>0} f_{n,D}^{abc0\mathbf{JK}} z_1^a \bar{z}_1^b x_2^c x^{\mathbf{J}} t^{\mathbf{K}} = f_{n,F} - f_{n,G} - f_{n,H}, \\
f_{n,F} &= \sum_{a \text{ even},\, b \text{ odd},\, a>0} f_{n,D}^{abc0\mathbf{JK}} z_1^{a-1} \bar{z}_1^{b-1} x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 \\
f_{n,G} &= \sum_{a \text{ even},\, b \text{ odd},\, a>0} f_{n,D}^{abc0\mathbf{JK}} z_1^{a-1} \bar{z}_1^{b} x_2^c x^{\mathbf{J}} t^{\mathbf{K}} \cdot (x_2 + it_1) \\
f_{n,H} &= \sum_{a \text{ even},\, b \text{ odd},\, a>0} f_{n,D}^{abc0\mathbf{JK}} z_1^{a-1} \bar{z}_1^{b} x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \cdot (ix_3^2) = f_{n,I} - f_{n,J}, \\
f_{n,I} &= \sum_{a \text{ even},\, b \text{ odd},\, a>0} f_{n,D}^{abc0\mathbf{JK}} z_1^{a-2} \bar{z}_1^{b-1} x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \cdot (ix_3^2) \cdot (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 \\
f_{n,J} &= \sum_{a \text{ even},\, b \text{ odd},\, a>0} f_{n,D}^{abc0\mathbf{JK}} z_1^{a-2} \bar{z}_1^{b} x_2^c x_3^d x^{\mathbf{J}} t^{\mathbf{K}} \cdot (ix_3^2) \cdot (x_2 + it_1 + ix_3^2),
\end{aligned}$$

so $f_{n,D} = f_{n,F} - f_{n,G} - f_{n,I} + f_{n,J}$.

Using the Schwarz Lemma and Lemma 7.5 we get the estimate:

$$\begin{aligned}
f_{n,G}(z_1,\zeta,x,t) &= \frac{(e_{n,E}-\zeta p_{1B}-\zeta g'_{n,A})\cdot(x_2+it_1)}{z_1}\\
&= \left(\frac{e_{n,E}}{z_1}-\frac{\zeta p_{1B}}{z_1}-\frac{\zeta e'_{n,F}}{z_1^2}+\frac{\zeta p'_{1C}}{z_1}\right)\cdot(x_2+it_1)\\
\implies |f_{n,G}|_r &\leq 2|e_{n,E}|_r+2r|p_{1B}|_r+2|e_{n,F}|_r+2r|p_{1C}|_r\\
&\leq 4|e_n|_r+12|e_{n-1}|_r.
\end{aligned}$$

Let:

$$f_{n,K}(\bar z_1,x,t)=e_{n,G}-\bar z_1 p_{1D}=\sum_{c>0} f_{n,K}^{01c0\mathbf{JK}}\bar z_1 x_2^c x^{\mathbf{J}}t^{\mathbf{K}},$$

and let $g_{n,B}(x,t)=\frac{1}{\zeta}f_{n,K}(\zeta,x,t)$, so that

$$f_{n,K}(\bar z_1,x,t)=\frac{1}{2}\bar z_1\cdot(g_{n,B}+\overline{g_{n,B}})+\frac{1}{2}\bar z_1\cdot(g_{n,B}-\overline{g_{n,B}}).$$

Further, re-arrange the second quantity:

$$\begin{aligned}
&\frac{1}{2}\bar z_1\cdot(g_{n,B}-\overline{g_{n,B}})\\
&= \sum_{c>0} i\mathrm{Im}(f_{n,K}^{01c0\mathbf{JK}})\bar z_1 x_2^c x^{\mathbf{J}}t^{\mathbf{K}}=g_{n,C}-g_{n,D}-g_{n,E},\\
g_{n,C} &= \sum_{c>0} i\mathrm{Im}(f_{n,K}^{01c0\mathbf{JK}})x_2^{c-1}x^{\mathbf{J}}t^{\mathbf{K}}\cdot(z_1+x_2+it_1+ix_3^2)\bar z_1\\
g_{n,D} &= \sum_{c>0} i\mathrm{Im}(f_{n,K}^{01c0\mathbf{JK}})x_2^{c-1}x^{\mathbf{J}}t^{\mathbf{K}}z_1\bar z_1\\
g_{n,E} &= \sum_{c>0} i\mathrm{Im}(f_{n,K}^{01c0\mathbf{JK}})x_2^{c-1}x^{\mathbf{J}}t^{\mathbf{K}}\cdot(it_1+ix_3^2)\bar z_1.
\end{aligned}$$

Note that this last re-arrangement is different from that of $e_{n-1,J}$, $e_{n-1,K}$, $E_{\sigma,D}$, $f_{n,A}$, $f_{n,D}$, and $f_{n,H}$. (A similar step appeared in [$\mathbf{C}_5$].)

Using the Schwarz Lemma and Lemma 7.5 we get the estimate:

$$\begin{aligned}
g_{n,D}(z_1,\zeta,x,t) &= \frac{\zeta}{2}\cdot(g_{n,B}-g'_{n,B})\cdot\frac{z_1}{x_2}\\
&= \frac{z_1e_{n,G}-z_1\zeta p_{1D}-\zeta e'_{n,G}+z_1\zeta p'_{1D}}{2x_2}\\
\implies |g_{n,D}|_r &\leq |e_{n,G}|_r+r|p_{1D}|_r\leq 2|e_n|_r+\frac{5}{2}|e_{n-1}|_r.
\end{aligned}$$

Let:

$$f_{n,L}(\bar z_1,x_4,\ldots,x_{m-1},t)=e_{n,H}-\bar z_1 p_{1E}=\sum_{\mathbf{J}\neq\mathbf{0}} f_{n,L}^{0100\mathbf{JK}}\bar z_1 x^{\mathbf{J}}t^{\mathbf{K}}.$$

Recall that in the $(m,n)=(4,5)$ case, there are no such terms. For $m>4$, in a manner analogous to (66), this can be re-arranged as:

$$\begin{aligned}
f_{n,L} &= \sum_{\ell=4}^{m-1} f_{n,L,\ell}(\bar{z}_1,x,t),\\
f_{n,L,4} &= \sum_{j_4>0} f_{n,L}^{0100\mathbf{JK}}\bar{z}_1 x_4^{j_4}x_5^{j_5}\cdots x_{m-1}^{j_{m-1}}t^{\mathbf{K}}\\
&= f_{n,L}(\bar{z}_1,x_4,\ldots,x_{m-1},t)-f_{n,L}(\bar{z}_1,0,x_5,\ldots,x_{m-1},t),\ldots,\\
f_{n,L,\ell} &= \sum_{j_\ell>0} f_{n,L}^{0100\mathbf{JK}}\bar{z}_1 x_\ell^{j_\ell}\cdots x_{m-1}^{j_{m-1}}t^{\mathbf{K}}\\
&= f_{n,L}(\bar{z}_1,0,\ldots,0,x_\ell,\ldots,x_{m-1},t)\\
&\quad -f_{n,L}(\bar{z}_1,0,\ldots,0,0,x_{\ell+1},\ldots,x_{m-1},t),\ldots,\\
f_{n,L,m-1} &= \sum_{j_{m-1}>0} f_{n,L}^{0100\mathbf{JK}}\bar{z}_1 x_{m-1}^{j_{m-1}}t^{\mathbf{K}}\\
&= f_{n,L}(\bar{z}_1,0,\ldots,0,x_{m-1},t).
\end{aligned}$$

For each ℓ,

$$\begin{aligned}
|f_{n,L,\ell}(\zeta,x,t)|_r &\leq 2|f_{n,L}|_r = 2|e_{n,H}-\zeta p_{1E}|_r\\
&\leq 2(2|e_{n,B}|_r + r\cdot\frac{2}{r}|e_{n-1}|_r)\leq 4|e_n|_r+4|e_{n-1}|_r.
\end{aligned}$$

Of course, there is a smaller estimate for the last quantity, $f_{n,L,m-1}$, but it will be simpler later to treat them all the same.

For $\ell=4,\ldots,m-1$, let $g_{n,F,\ell}(x,t)=\frac{1}{\zeta x_\ell}f_{n,L,\ell}(\zeta,x,t)$, so that:

$$\begin{aligned}
f_{n,L} &= \sum_{j=2}^{\frac{m}{2}-1}(x_{2j}g_{n,F,2j}(x,t)+x_{2j+1}g_{n,F,2j+1}(x,t))\bar{z}_1\\
&= \sum_{j=2}^{\frac{m}{2}-1}\Big((\mathrm{Im}(g_{n,F,2j+1})+i\mathrm{Im}(g_{n,F,2j}))\cdot(x_{2j}+ix_{2j+1})\bar{z}_1\\
&\quad +(\mathrm{Re}(g_{n,F,2j})-\mathrm{Im}(g_{n,F,2j+1}))\,x_{2j}\bar{z}_1\\
&\quad +(\mathrm{Re}(g_{n,F,2j+1})+\mathrm{Im}(g_{n,F,2j}))\,x_{2j+1}\bar{z}_1\Big).
\end{aligned}$$

Finally, let:

$$f_{n,M}(\bar{z}_1,t)=e_{n,I}-\bar{z}_1p_{1F}=\sum f_{n,M}^{01000\mathbf{0K}}\bar{z}_1t^{\mathbf{K}},$$

and let $g_{n,G}(t)=\frac{1}{\zeta}f_{n,M}(\zeta,t)$, so that

$$f_{n,M}(\bar{z}_1,t)=\frac{1}{2}\bar{z}_1\cdot(g_{n,G}+\overline{g_{n,G}})+\frac{1}{2}\bar{z}_1\cdot(g_{n,G}-\overline{g_{n,G}}).$$

To solve (100–103), it is sufficient to collect together groups of like terms from RHS, known and unknown, and to set each group equal to 0, so that we can solve for the unknowns one at a time. Grouping together quantities with even powers of $\bar{z}_1$ gives:

$$0 = e_{n,A} - i\bar{z}_1 m_2 + 2\bar{z}_1 x_3 m_3 - (z_1 + x_2 + it_1 + ix_3^2)\cdot\overline{(p_{1A} + \ldots + p_{1F})} + p_n^E.$$

This determines p_n^E, and we get the estimate

$$\begin{aligned}
\|p_n^E\|_R &\leq |e_{n,A}|_R + R|m_2|_R + 2R^2|m_3|_R \\
&\quad + |(z_1 + x_2 + it_1 + ix_3^2)\cdot(p'_{1A} + \ldots + p'_{1F})|_R \\
&\leq |e_n|_r + \left(\frac{2r^2}{(r-R)^2} + 2 + R\right)(R|E_2|_r + 2R^2|E_3|_r) \\
&\quad + (3+R)(\frac{7}{2} + \frac{3}{2}R)|e_{n-1}|_r.
\end{aligned}$$

Collecting quantities which are odd in both z_1 and $\bar{z}_1$, and do not depend on x_3,

$$0 = e_{n,K} + i\bar{z}_1\overline{m_{2B}} - f_{n,G} - g_{n,D} - \bar{z}_1 p_{1G},$$

which determines p_{1G}:

$$p_{1G}(z_1, x, \zeta^2, t) = \frac{e_{n,K} - f_{n,G} - g_{n,D}}{\zeta} + im'_{2B}.$$

We get the estimate:

$$\begin{aligned}
(104)\quad \|p_{1G}\|_R &\leq \frac{1}{r}|e_{n,K}|_r + \frac{1}{r}|f_{n,G}|_r + \frac{1}{r}|g_{n,D}|_r + |m'_{2B}|_R \\
&\leq \frac{7}{r}|e_n|_r + \frac{29}{2r}|e_{n-1}|_r + \left(\frac{2r^2}{(r-R)^2} + 2 + R\right)|E_2|_r.
\end{aligned}$$

Collecting quantities with $(z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1$ as a factor,

$$\begin{aligned}
0 &= f_{n,B} + f_{n,F} - f_{n,I} + g_{n,C} - \overline{p_{1G}}\cdot(z_1 + x_2 + it_1 + ix_3^2) \\
&\quad + p_n^O\cdot(z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1.
\end{aligned}$$

This determines p_n^O,

$$\begin{aligned}
p_n^O(z_1, x, \zeta^2, t) &= -\frac{f_{n,B} + f_{n,F} - f_{n,I} + g_{n,C}}{(z_1 + x_2 + it_1 + ix_3^2)\zeta} + \frac{p'_{1G}}{\zeta} \\
&= -\frac{f_{n,A} + f_{n,D} - f_{n,H}}{z_1\zeta} - \frac{g_{n,C}}{(z_1 + x_2 + it_1 + ix_3^2)\zeta} \\
&\quad + \frac{e'_{n,K} - f'_{n,G} - g'_{n,D}}{z_1\zeta} - \frac{im_{2B}}{\zeta}.
\end{aligned}$$

Since

$$-\frac{g_{n,C}}{(z_1+x_2+it_1+ix_3^2)\zeta} = -\frac{g_{n,B}-g'_{n,B}}{2x_2} = \frac{g'_{n,D}}{z_1\zeta},$$

these terms cancel in the above expression. (This minor simplification is not crucial to the Proof.) Expanding this expression, we get an estimate:

$$\begin{aligned}
p_n^O &= -\frac{e_{n,J}}{z_1\zeta} - \frac{im'_{2A}}{z_1} + \frac{2x_3m'_3}{z_1} \\
&\quad -\frac{e_{n,E}}{z_1\zeta} + \frac{p_{1B}}{z_1} + \frac{e'_{n,F}}{z_1^2} - \frac{p'_{1C}}{z_1} \\
&\quad +\frac{ix_3^2e_{n,E}}{z_1^2\zeta} - \frac{ix_3^2p_{1B}}{z_1^2} - \frac{ix_3^2e'_{n,F}}{z_1^3} + \frac{ix_3^2p'_{1C}}{z_1^2} \\
&\quad +\frac{e'_{n,K}}{z_1\zeta} - \frac{f'_{n,G}}{z_1\zeta} - \frac{im_{2B}}{\zeta} \\
\implies \|z_np_n^O\|_R &\leq (3+R)\cdot[(11+4R)|e_n|_r + (18+6R)|e_{n-1}|_r \\
&\quad + \left(\frac{2r^2}{(r-R)^2}+2+R\right)(3R|E_2|_r + 2R^2|E_3|_r)].
\end{aligned}$$

Collecting quantities with $(x_{2j}+ix_{2j+1})\bar{z}_1$ as a factor,

$$\begin{aligned}
(105)\quad 0 &= \sum_{j=2}^{\frac{m}{2}-1}\big((\mathrm{Im}(g_{n,F,2j+1}) + i\mathrm{Im}(g_{n,F,2j}))\cdot(x_{2j}+ix_{2j+1})\bar{z}_1\big) \\
&\quad + \sum_{\tau=m}^{n-2} p_{n,\tau}^L(x,t)\cdot(x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\bar{z}_1.
\end{aligned}$$

The corresponding terms in the sums match, with $j=\tau-m+2$, again using the assumption $m=\frac{2}{3}(n+1)$, as in (68). This determines each $p_{n,\tau}^L$ and $p_n^L = \sum_{\tau=m}^{n-2} z_\tau p_{n,\tau}^L(z,\vec{w})$:

$$\begin{aligned}
p_{n,\tau}^L(x,t) &= \frac{if_{n,L,2(\tau-m+2)+1}}{2\zeta x_{2(\tau-m+2)+1}} - \frac{if'_{n,L,2(\tau-m+2)+1}}{2z_1x_{2(\tau-m+2)+1}} \\
&\quad -\frac{f_{n,L,2(\tau-m+2)}}{2\zeta x_{2(\tau-m+2)}} + \frac{f'_{n,L,2(\tau-m+2)}}{2z_1x_{2(\tau-m+2)}} \\
\|z_\tau p_{n,\tau}^L\|_r &\leq 2r^2\left(2\frac{|f_{n,L,2(\tau-m+2)+1}|_r}{2r^2} + 2\frac{|f_{n,L,2(\tau-m+2)}|_r}{2r^2}\right) \\
&\leq 16|e_n|_r + 16|e_{n-1}|_r \\
\implies \|p_n^L\|_r &\leq 8(m-4)(|e_n|_r + |e_{n-1}|_r).
\end{aligned}$$

Collecting some terms linear in $\bar{z}_1$ and otherwise depending only on t,

$$0 = \frac{1}{2}\bar{z}_1 \cdot (g_{n,G} - \overline{g_{n,G}}) - i\bar{z}_1 P_1(t). \tag{106}$$

This determines $P_1(t)$, and the solution satisfies the reality condition:

$$\begin{aligned} P_1(t) &= \frac{1}{2i}(g_{n,G} - \overline{g_{n,G}}) \\ P_1(\vec{w}) &= \frac{1}{2i}\left(\frac{e_{n,I}}{\zeta} - p_{1F} - \frac{e'_{n,I}}{z_1} + p'_{1F}\right) \\ \|P_1\|_r &\leq \frac{1}{r}|e_n|_r + \frac{1}{r}|e_{n-1}|_r. \end{aligned}$$

Collecting all the remaining terms,

$$\begin{aligned} 0 = \; & e_{n,D} - f_{n,C} + \bar{z}_1 \cdot (g_{n,A} + \overline{g_{n,A}}) + f_{n,J} + \frac{1}{2}\bar{z}_1 \cdot (g_{n,B} + \overline{g_{n,B}}) - g_{n,E} \\ & + \sum_{j=2}^{\frac{m}{2}-1} \big((\mathrm{Re}(g_{n,F,2j}) - \mathrm{Im}(g_{n,F,2j+1}))\, x_{2j}\bar{z}_1 \\ & + (\mathrm{Re}(g_{n,F,2j+1}) + \mathrm{Im}(g_{n,F,2j}))\, x_{2j+1}\bar{z}_1\big) \\ & + \frac{1}{2}\bar{z}_1 \cdot (g_{n,G} + \overline{g_{n,G}}) + \bar{z}_1 \cdot (im_2 - i\overline{m_2} - 2x_3m_3 + 2x_3\overline{m_3}) \\ & - \bar{z}_1 \cdot (\mathrm{Re}(p_2(\vec{z}, t)) + 2ix_3\mathrm{Re}(p_3(\vec{z}, t))). \end{aligned}$$

The only remaining unknowns are p_2 and p_3, so we move them to the LHS and divide by $\bar{z}_1$:

$$\begin{aligned} \mathrm{Re}(p_2) + 2ix_3\mathrm{Re}(p_3) = \; & \frac{e_{n,D} - f_{n,C} + f_{n,J} - g_{n,E}}{\bar{z}_1} \\ & + g_{n,A} + \overline{g_{n,A}} + \frac{1}{2}(g_{n,B} + \overline{g_{n,B}}) \\ & + \sum_{j=2}^{\frac{m}{2}-1} \big((\mathrm{Re}(g_{n,F,2j}) - \mathrm{Im}(g_{n,F,2j+1}))\, x_{2j} \\ & + (\mathrm{Re}(g_{n,F,2j+1}) + \mathrm{Im}(g_{n,F,2j}))\, x_{2j+1}\big) \\ & + \frac{1}{2}(g_{n,G} + \overline{g_{n,G}}) \\ & + im_2 - i\overline{m_2} - 2x_3m_3 + 2x_3\overline{m_3}. \end{aligned} \tag{107}$$

By construction, several of the above terms, including $\frac{g_{n,E}}{\bar{z}_1}$, or pairs of terms, are real valued, and the rest are divisible by x_3, so setting the imaginary parts equal gives

$$\begin{aligned}(108)\qquad \mathrm{Re}(p_3) &= \frac{i\mathrm{Im}\left(\frac{e_{n,D}-f_{n,C}+f_{n,J}}{\bar{z}_1}\right) - 2x_3m_3 + 2x_3\overline{m_3}}{2ix_3} \\ &= \mathrm{Im}\left(\frac{e_{n,D}-f_{n,C}+f_{n,J}}{2\bar{z}_1x_3}\right) + im_3 - i\overline{m_3}.\end{aligned}$$

From (98) and (99), we have

$$\mathrm{Im}(p_3) = -(f_3 + e_{3,E} + m_3 - \overline{m_3}),$$

and combining $\mathrm{Re}(p_3) + i\mathrm{Im}(p_3)$, the m_3 terms cancel — this cancellation is a key step, as mentioned previously when the m_2, m_3 terms were introduced in (102–103):

$$\begin{aligned}p_3 &= \frac{e_{n,D}}{4i\zeta x_3} - \frac{e'_{n,D}}{4iz_1x_3} - \frac{f_{n,C}}{4i\zeta x_3} + \frac{f'_{n,C}}{4iz_1x_3} + \frac{f_{n,J}}{4i\zeta x_3} - \frac{f'_{n,J}}{4iz_1x_3} - if_3 - ie_{3,E} \\ &= p_3^E(z_1,x,\zeta^2,t) + (z_1+x_2+it_1+ix_3^2)\zeta p_3^O(z_1,x,\zeta^2,t).\end{aligned}$$

We get estimates on some of these terms using the Schwarz Lemma, and on their conjugates by Lemma 7.5. The $e_{3,E}$ term determines p_3^O, the rest determine p_3^E.

$$\begin{aligned}(109)\qquad p_3^O &= \frac{-ie_{3,E}}{(z_1+x_2+it_1+ix_3^2)\zeta} = \frac{-iE_{3,D}}{z_1\zeta} \\ \implies \|z_np_3^O\|_r &\leq (3+r)|E_3|_r.\end{aligned}$$

$$\begin{aligned}\left|\frac{e_{n,D}}{4i\zeta x_3}\right|_R &\leq \frac{2|e_n|_r}{4R^2}, \\ \frac{f_{n,C}}{4i\zeta x_3} &= \frac{\frac{x_2+it_1+ix_3^2}{z_1}f_{n,A}}{4i\zeta x_3} \\ &= (x_2+it_1+ix_3^2)\cdot\left(\frac{e_{n,J}}{4iz_1\zeta x_3} + \frac{m'_{2A}}{4z_1x_3} - \frac{m'_3}{2iz_1}\right) \\ \implies \left|\frac{f_{n,C}}{4i\zeta x_3}\right|_R &\leq (2R+R^2)\left(\frac{2|e_n|_r}{4R^3}\right. \\ &\quad \left. + \left(\frac{2r^2}{(r-R)^2} + 2 + R\right)\left(\frac{2|E_2|_r}{4R^2} + \frac{|E_3|_r}{2R}\right)\right),\end{aligned}$$

$$
\begin{aligned}
\frac{f_{n,J}}{4i\zeta x_3} &= \frac{\frac{(x_2+it_1+ix_3^2)}{z_1} f_{n,H}}{4i\zeta x_3} = \frac{x_2+it_1+ix_3^2}{4i\zeta z_1 x_3}\cdot\frac{ix_3^2}{z_1} f_{n,D} \\
&= (x_2+it_1+ix_3^2)x_3\cdot\left(\frac{e_{n,E}}{4z_1^2\zeta}-\frac{p_{1B}}{4z_1^2}-\frac{e'_{n,F}}{4z_1^3}+\frac{p'_{1C}}{4z_1^2}\right) \\
\Longrightarrow \left|\frac{f_{n,J}}{4i\zeta x_3}\right|_R &\leq (2R^2+R^3)\left(\frac{|e_n|_r}{R^3}+\frac{\frac{6}{r}|e_{n-1}|_r}{4R^2}\right) \\
\Longrightarrow \|p_3^E\|_R &\leq \frac{|e_n|_r}{R^2}+(2+R)\frac{|e_n|_r}{R^2} \\
&\quad +(2+R)\left(\frac{2r^2}{(r-R)^2}+2+R\right)\left(\frac{|E_2|_r}{R}+|E_3|_r\right) \\
&\quad +(2+R)\left(\frac{2|e_n|_r}{R}+\frac{3|e_{n-1}|_r}{r}\right) \\
&\quad +\left(\frac{4r^2}{(r-R)^2}+3+R\right)|E_3|_r.
\end{aligned}
$$

Returning to (107), setting the real parts of LHS and RHS equal gives

$$
\begin{aligned}
\mathrm{Re}(p_2) &= \mathrm{Re}\left(\frac{e_{n,D}-f_{n,C}+f_{n,J}}{\bar{z}_1}\right)-\frac{g_{n,E}}{\bar{z}_1} \\
&\quad +g_{n,A}+\overline{g_{n,A}}+\frac{1}{2}(g_{n,B}+\overline{g_{n,B}}) \\
&\quad +\sum_{j=2}^{\frac{m}{2}-1}\Big(\left(\mathrm{Re}(g_{n,F,2j})-\mathrm{Im}(g_{n,F,2j+1})\right)x_{2j} \\
&\qquad +\left(\mathrm{Re}(g_{n,F,2j+1})+\mathrm{Im}(g_{n,F,2j})\right)x_{2j+1}\Big) \\
&\quad +\frac{1}{2}(g_{n,G}+\overline{g_{n,G}}) \\
&\quad +im_2-i\overline{m_2}.
\end{aligned}
$$

From (98) and (99), we have

$$
\mathrm{Im}(p_2) = -(f_2+e_{2,E}+m_2-\overline{m_2}),
$$

and combining $\mathrm{Re}(p_2) + i\mathrm{Im}(p_2)$, the m_2 terms cancel, as before with p_3 and m_3.

$$
\begin{aligned}
p_2 \;=\; & \frac{e_{n,D}}{2\zeta} + \frac{e'_{n,D}}{2z_1} - \frac{f_{n,C}}{2\zeta} - \frac{f'_{n,C}}{2z_1} + \frac{f_{n,J}}{2\zeta} + \frac{f'_{n,J}}{2z_1} \\
& - \frac{g_{n,E}}{\zeta} + g_{n,A} + g'_{n,A} + \frac{1}{2}(g_{n,B} + g'_{n,B}) \\
& + \sum_{j=2}^{\frac{m}{2}-1} \Bigg(\frac{f_{n,L,2j}}{2\zeta} + \frac{f'_{n,L,2j}}{2z_1} - \frac{f_{n,L,2j+1}x_{2j}}{2i\zeta x_{2j+1}} + \frac{f'_{n,L,2j+1}x_{2j}}{2iz_1 x_{2j+1}} \\
& \quad + \frac{f_{n,L,2j+1}}{2\zeta} + \frac{f'_{n,L,2j+1}}{2z_1} + \frac{f_{n,L,2j}x_{2j+1}}{2i\zeta x_{2j}} - \frac{f'_{n,L,2j}x_{2j+1}}{2iz_1 x_{2j}} \Bigg) \\
& + \frac{1}{2}(g_{n,G} + g'_{n,G}) - if_2 - ie_{2,E} \\
=\; & p_2^E(z_1, x, \zeta^2, t) + (z_1 + x_2 + it_1 + ix_3^2)\zeta p_2^O(z_1, x, \zeta^2, t).
\end{aligned}
$$

Again similarly to the above calculation for p_3, the $e_{2,E}$ term determines p_2^O as in (109), the rest determine p_2^E.

$$
\begin{aligned}
p_2^O &= \frac{-ie_{2,E}}{(z_1 + x_2 + it_1 + ix_3^2)\zeta} = \frac{-iE_{2,D}}{z_1\zeta} \\
\implies \|z_n p_2^O\|_r &\leq (3 + r)|E_2|_r.
\end{aligned}
$$

The estimates for the $e_{n,D}$, $f_{n,C}$, $f_{n,J}$, $f_{n,L,\ell}$ and f_2 terms are similar to some previous calculations.

$$
\begin{aligned}
\frac{g_{n,E}}{\zeta} &= \left(\frac{e_{n,G}}{2\zeta x_2} - \frac{p_{1D}}{2x_2} - \frac{e'_{n,G}}{2z_1 x_2} + \frac{p'_{1D}}{2x_2} \right) \cdot (it_1 + ix_3^2) \\
\implies \left| \frac{g_{n,E}}{\zeta} \right|_R &\leq 2(\frac{1}{R} + 1)|e_n|_r + \frac{5}{2r}(1 + R)|e_{n-1}|_r,
\end{aligned}
$$

$$
\begin{aligned}
g_{n,A} &= \frac{e_{n,F}}{\zeta} - p_{1C} \implies |g_{n,A}|_R \le \frac{2|e_n|_r}{R} + \frac{3}{r}|e_{n-1}|_r, \\
g_{n,B} &= \frac{e_{n,G}}{\zeta} - p_{1D} \implies |g_{n,B}|_R \le \frac{2|e_n|_r}{R} + \frac{5}{2r}|e_{n-1}|_r, \\
g_{n,G} &= \frac{e_{n,I}}{\zeta} - p_{1F} \implies |g_{n,G}|_R \le \frac{|e_n|_r}{R} + \frac{1}{r}|e_{n-1}|_r \\
\implies \|p_2^E\|_R &\le \frac{2|e_n|_r}{R^2} + (4+2R)\frac{|e_n|_r}{R} \\
&\quad + (4+2R)\left(\frac{2r^2}{(r-R)^2} + 2 + R\right)(|E_2|_r + R|E_3|_r) \\
&\quad + (4R+2R^2)\left(\frac{2|e_n|_r}{R} + \frac{3|e_{n-1}|_r}{r}\right) \\
&\quad + 2(\frac{1}{R}+1)|e_n|_r + \frac{5}{2r}(1+R)|e_{n-1}|_r \\
&\quad + \frac{4}{R}|e_n|_r + \frac{6}{r}|e_{n-1}|_r + \frac{2}{R}|e_n|_r + \frac{5}{2r}|e_{n-1}|_r \\
&\quad + 8(m-4)(|e_n|_r + |e_{n-1}|_r) + \frac{1}{R}|e_n|_r + \frac{1}{r}|e_{n-1}|_r \\
&\quad + \left(\frac{4r^2}{(r-R)^2} + 3 + R\right)|E_2|_r.
\end{aligned}
$$

This completes the solution of the system of equations (98). From (97) and (104), we can conclude that $p_1 = p_{1A} + \ldots + p_{1F} + p_{1G}$ has been determined in terms of e_{n-1}, e_n, and E_2, so that

$$
\|p_1\|_R \le \left(\frac{18}{r} + \frac{3}{2}\right)|e_{n-1}|_r + \frac{7}{r}|e_n|_r + \left(\frac{2r^2}{(r-R)^2} + 2 + R\right)|E_2|_r.
$$

From (95),

$$
\begin{aligned}
p_{n-1}^E &= -e_{n-1,A} + 2\bar{z}_1\overline{p_{1G}} \implies \\
\|p_{n-1}^E\|_R &\le 30|e_{n-1}|_r + 14|e_n|_r + 2R\left(\frac{2r^2}{(r-R)^2} + 2 + R\right)|E_2|_r.
\end{aligned}
$$

If $(m,n) = (4,5)$, we are done, all the equations have been solved. Otherwise, the remaining equations from (84–88) can be solved in blocks of three. The details of the calculation are similar to those in Theorem 5.6 of [$\mathbf{C}_6$], and simpler than the previous subsystem (98). For

$\tau = m, \ldots, n-2$, the three equations to be solved are:

$$
\begin{aligned}
(110)\quad 0 &= E_{2(\tau-m+2)}(z_1, \bar{z}_1, x, t) + \mathrm{Im}(p_{2(\tau-m+2)}(\vec{z}, t)) \\
0 &= E_{2(\tau-m+2)+1}(z_1, \bar{z}_1, x, t) + \mathrm{Im}(p_{2(\tau-m+2)+1}(\vec{z}, t)) \\
(111)\quad 0 &= e_\tau(z_1, \bar{z}_1, x, t) + p_\tau(\vec{z}, t) \\
&\quad -\overline{p_1(\vec{z}, t)} \cdot (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1}) \\
&\quad -\bar{z}_1 \cdot (\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z}, t)) + i\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z}, t))).
\end{aligned}
$$

We start by collecting some known quantities. Let

$$
f_\tau(z_1, \bar{z}_1, x, t) = e_\tau(z_1, \bar{z}_1, x, t) - \overline{p_1(\vec{z}, t)} \cdot (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1}),
$$

so that

$$
\begin{aligned}
|f_\tau|_R \;\leq\; & |e_\tau|_R + 2R\left(\left(\frac{18}{r} + \frac{3}{2}\right)|e_{n-1}|_r \right. \\
& \left. + \frac{7}{r}|e_n|_r + \left(\frac{2r^2}{(r-R)^2} + 2 + R\right)|E_2|_r\right),
\end{aligned}
$$

and as in (91), decompose it into even and odd parts:

$$
\begin{aligned}
f_\tau &= \sum f_\tau^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} = f_{\tau,A} + f_{\tau,B} + f_{\tau,C}, \\
f_{\tau,A} &= \sum_{b \text{ even}} f_\tau^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
f_{\tau,B} &= \sum_{a,\, b \text{ odd}} f_\tau^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} \\
f_{\tau,C} &= \sum_{a \text{ even},\, b \text{ odd}} f_\tau^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}}.
\end{aligned}
$$

The same bound holds for the even and odd parts: $|f_{\tau,A}|_R \leq |f_\tau|_R$, $|f_{\tau,B}|_R \leq |f_\tau|_R$, and $|f_{\tau,B}|_R \leq |f_\tau|_R$.

Then, adding and subtracting some known quantities to (111), as in (102–103), and using the normalized form of p_τ, gives:

$$
\begin{aligned}
0 = \;& f_{\tau,A} + f_{\tau,B} + f_{\tau,C} \\
& + p_\tau^E(z_1, x, \bar{z}_1^2, t) + (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 p_2^O(z_1, x, \bar{z}_1^2, t) \\
& + \bar{z}_1 \cdot (i m_{2(\tau-m+2)} - i\overline{m_{2(\tau-m+2)}} - m_{2(\tau-m+2)+1} + \overline{m_{2(\tau-m+2)+1}}) \\
& - i\bar{z}_1 m_{2(\tau-m+2)} + i\bar{z}_1 \overline{m_{2(\tau-m+2)}} + \bar{z}_1 m_{2(\tau-m+2)+1} - \bar{z}_1 \overline{m_{2(\tau-m+2)+1}} \\
& - \bar{z}_1 \cdot (\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z}, t)) + i\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z}, t))).
\end{aligned}
$$

Let:

$$
\begin{aligned}
g_{\tau,A} &= f_{\tau,B} - i\bar{z}_1\overline{m_{2(\tau-m+2)}} + \bar{z}_1\overline{m_{2(\tau-m+2)+1}} \\
&= \sum_{a,\, b \text{ odd}} g_{\tau,A}^{ab\mathbf{IK}} z_1^a \bar{z}_1^b x^{\mathbf{I}} t^{\mathbf{K}} = g_{\tau,B} - g_{\tau,C}, \\
g_{\tau,B} &= \sum_{a,\, b \text{ odd}} g_{\tau,A}^{ab\mathbf{IK}} z_1^{a-1} \bar{z}_1^{b-1} x^{\mathbf{I}} t^{\mathbf{K}} \cdot (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 \\
g_{\tau,C} &= \sum_{a,\, b \text{ odd}} g_{\tau,A}^{ab\mathbf{IK}} z_1^{a-1} \bar{z}_1^{b} x^{\mathbf{I}} t^{\mathbf{K}} \cdot (x_2 + it_1 + ix_3^2),
\end{aligned}
$$

so

$$
\begin{aligned}
|g_{\tau,C}|_R &= \left| \frac{x_2 + it_1 + ix_3^2}{z_1} g_{\tau,A} \right|_R \\
&\leq (2+R)[|f_\tau|_R \\
&\quad + R\left(\frac{2r^2}{(r-R)^2} + 2 + R\right)(|E_{2(\tau-m+2)}|_r + |E_{2(\tau-m+2)+1}|_r)].
\end{aligned}
$$

Then (111) can be solved by comparing like terms. Grouping together quantities with even powers of $\bar{z}_1$ gives:

$$
0 = f_{\tau,A} - i\bar{z}_1 m_{2(\tau-m+2)} + \bar{z}_1 m_{2(\tau-m+2)+1} + p_\tau^E,
$$

which determines p_τ^E, and

$$
\begin{aligned}
& \|p_\tau^E\|_R \\
\leq\ & |f_\tau|_R + R\left(\frac{2r^2}{(r-R)^2} + 2 + R\right)(|E_{2(\tau-m+2)}|_r + |E_{2(\tau-m+2)+1}|_r).
\end{aligned}
$$

Another group is

$$
0 = g_{\tau,B} + (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 p_\tau^O(z_1, x, \bar{z}_1^2, t),
$$

which determines p_τ^O, and

$$
\begin{aligned}
p_\tau^O &= \frac{-g_{\tau,A}}{z_1\bar{z}_1} = -\left(\frac{f_{\tau,B}}{z_1\bar{z}_1} - i\frac{\overline{m_{2(\tau-m+2)}}}{z_1} + \frac{\overline{m_{2(\tau-m+2)+1}}}{z_1}\right) \\
\Longrightarrow\ & \|z_n p_\tau^O\|_R \\
\leq\ & (3R^2 + R^3)\left(\frac{1}{R^2}|f_\tau|_R\right. \\
&\quad \left. + \frac{1}{R}\left(\frac{2r^2}{(r-R)^2} + 2 + R\right)(|E_{2(\tau-m+2)}|_r + |E_{2(\tau-m+2)+1}|_r)\right).
\end{aligned}
$$

The remaining terms are:

$$\begin{aligned} 0 &= f_{\tau,C} - g_{\tau,C} \\ &\quad + \bar{z}_1 \cdot (i m_{2(\tau-m+2)} - i\overline{m_{2(\tau-m+2)}} - m_{2(\tau-m+2)+1} + \overline{m_{2(\tau-m+2)+1}}) \\ &\quad - \bar{z}_1 \cdot (\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z},t)) + i\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z},t))). \end{aligned}$$

Dividing by $\bar{z}_1$ and considering the real and imaginary parts,

$$\begin{aligned} &\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z},t)) \\ = &\ \frac{f_{\tau,C} - g_{\tau,C}}{2\bar{z}_1} + \frac{\overline{f_{\tau,C}} - \overline{g_{\tau,C}}}{2z_1} + i m_{2(\tau-m+2)} - i\overline{m_{2(\tau-m+2)}}, \\ &\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z},t)) \\ = &\ \frac{f_{\tau,C} - g_{\tau,C}}{2i\bar{z}_1} - \frac{\overline{f_{\tau,C}} - \overline{g_{\tau,C}}}{2iz_1} + i m_{2(\tau-m+2)+1} - i\overline{m_{2(\tau-m+2)+1}}. \end{aligned}$$

From (110) and (99), we have

$$\begin{aligned} &\mathrm{Im}(p_{2(\tau-m+2)}(\vec{z},t)) \\ = &\ -(f_{2(\tau-m+2)} + e_{2(\tau-m+2),E} + m_{2(\tau-m+2)} - \overline{m_{2(\tau-m+2)}}), \\ &\mathrm{Im}(p_{2(\tau-m+2)+1}(\vec{z},t)) \\ = &\ -(f_{2(\tau-m+2)+1} + e_{2(\tau-m+2)+1,E} + m_{2(\tau-m+2)+1} - \overline{m_{2(\tau-m+2)+1}}). \end{aligned}$$

Combining the real and imaginary parts of $p_{2(\tau-m+2)}$ and $p_{2(\tau-m+2)+1}$, the m terms cancel, as before with p_2 and p_3, to give:

$$\begin{aligned} &p_{2(\tau-m+2)} \\ = &\ p^E_{2(\tau-m+2)}(z_1, x, \zeta^2, t) \\ &\ +(z_1 + x_2 + it_1 + ix_3^2)\zeta p^O_{2(\tau-m+2)}(z_1, x, \zeta^2, t) \\ = &\ \frac{f_{\tau,C} - g_{\tau,C}}{2\zeta} + \frac{f'_{\tau,C} - g'_{\tau,C}}{2z_1} - i f_{2(\tau-m+2)} - i e_{2(\tau-m+2),E}, \\ &p_{2(\tau-m+2)+1} \\ = &\ p^E_{2(\tau-m+2)+1}(z_1, x, \zeta^2, t) \\ &\ +(z_1 + x_2 + it_1 + ix_3^2)\zeta p^O_{2(\tau-m+2)+1}(z_1, x, \zeta^2, t) \\ = &\ \frac{f_{\tau,C} - g_{\tau,C}}{2i\zeta} - \frac{f'_{\tau,C} - g'_{\tau,C}}{2iz_1} - i f_{2(\tau-m+2)+1} - i e_{2(\tau-m+2)+1,E}. \end{aligned}$$

Similar estimates hold in both cases, $\sigma = 2(\tau-m+2), 2(\tau-m+2)+1$; the $e_{\sigma,E}$ term determines p^O_σ, and the rest of the terms determine p^E_σ. As in (109),

$$\|z_n p^O_\sigma\|_r \ \leq \ (3+r)|E_\sigma|_r,$$

and

$$\|p_\sigma^E\|_R \;\leq\; \frac{1}{R}|f_\tau|_r + \frac{1}{R}|g_{\tau,C}|_R + \left(\frac{4r^2}{(r-R)^2} + 3 + R\right)|E_\sigma|_r.$$

The claimed estimate from the statement of the Theorem follows; it is possible, but not necessary, to explicitly calculate the polynomial expressions C_m^1 and C_m^2. ∎

REMARK 7.7. The Proof of the above Theorem can be modified to apply to the degree $d = 3$ case, as discussed in Remark 6.6, to establish the existence of a holomorphic coordinate change that eliminates all cubic terms depending on t in (75), leaving only the cubic term $i\bar{z}_1 x_3^2$, and possibly changing the $O(4)$ terms. In the nonlinear system of equations (80–82), if $d = 3$, then the solution p_3 may have linear terms of the form $\sum r_3^\alpha t_\alpha$, as remarked in (83), or as follows from the construction of the above Proof, where terms of the form $\bar{z}_1 x_3 t_\alpha$ in $e_{n,D}$ could contribute such real valued linear terms to p_3, from (108). In this case, the term $-i\bar{z}_1 \cdot (\mathrm{Re}(p_3(\vec{z},t)))^2$ from line (81) can have degree 3 terms, and so it cannot be included among the higher-degree terms that are neglected to arrive at the linear system (84–88). The only needed modification to the above Proof is that the degree 3 part of the nonlinear, but purely imaginary, quantity $-i\bar{z}_1 \cdot (\mathrm{Re}(p_3(\vec{z},t)))^2$ is inserted into Equation (106), to get:

$$0 \;=\; \frac{1}{2}\bar{z}_1 \cdot (g_{n,G} - \overline{g_{n,G}}) - i\bar{z}_1 \cdot \left(\sum r_3^\alpha t_\alpha\right)^2 - i\bar{z}_1 P_1(t). \tag{112}$$

Then, while p_3 eliminates terms in e_n of the form $\bar{z}_1 x_3 t_\alpha$, there still exists a weight 2 solution $P_1(t)$ of Equation (112) that eliminates the terms of the form $\bar{z}_1 t_\alpha t_\beta$.

COROLLARY 7.8. *There is a constant* $c_1 > 0$ *(depending only on* m*), such that for any* $\vec{p}$ *and* $\vec{e}$ *as in Theorem 7.6, and any radius lengths* ρ, r *with* $\frac{1}{2} < \rho < r \leq 1$*, the following hold:*

$$\max\{\|P_1\|_\rho, \|p_1\|_\rho, \ldots, \|p_n\|_\rho\} \leq \frac{c_1|\vec{e}|_r}{(r-\rho)^2}$$

$$\max_{j=1,\ldots,n}\left\{\left\|\frac{dP_1}{dz_j}\right\|_\rho + \sum_{\ell=1}^n \left\|\frac{dp_\ell}{dz_j}\right\|_\rho\right\} \leq \frac{c_1|\vec{e}|_r}{(r-\rho)^3},$$

$$\max_{\alpha=1,\ldots,k}\left\{\left\|\frac{dP_1}{dw_\alpha}\right\|_\rho + \sum_{\ell=1}^n \left\|\frac{dp_\ell}{dw_\alpha}\right\|_\rho\right\} \leq \frac{c_1|\vec{e}|_r}{(r-\rho)^3}.$$

PROOF. Let $R = \frac{1}{2}(\rho + r)$. The bound on each p_j follows from $\|p_j\|_\rho \leq \|p_j\|_R$ and the bounds from Theorem 7.6: using $\frac{1}{2} < R < r \leq 1 \leq \frac{1}{r} < \frac{1}{R} < 2$, the quantities C_m^1 and C_m^2 are bounded above by some positive constant (depending on m), and $(r-R)^{-2}$ satisfies: $16 < \frac{1}{(r-R)^2} = \frac{4}{(r-\rho)^2}$. The bounds for the derivatives of p_k follow from this consequence ([**A**]) of Cauchy's estimate:

If $0 < R_2 < R_1$ and $f(\zeta)$ is holomorphic and bounded by K for $|\zeta| < R_1$, then $\frac{df}{d\zeta}$ is bounded by $\frac{K}{R_1 - R_2}$ for $|\zeta| < R_2$.

This fact can be applied with $K = \|p_\ell\|_R$ or $\|P_1\|_R$, and $R_1 - R_2 = R - \rho = \frac{1}{2}(r-\rho)$ for the $z_1, \ldots, z_{m-1}$ and $w_1, \ldots, w_k$ derivatives, $R_1 - R_2 = R^2 - \rho^2 > R - \rho = \frac{1}{2}(r-\rho)$ for the z_{n-1} derivatives, $R_1 - R_2 = 2R^2 - 2\rho^2 > r - \rho$ for the $z_m, \ldots, z_{n-2}$ derivatives, and $R_1 - R_2 = (3R^2 + R^3) - (3\rho^2 + \rho^3) > \frac{15}{8}(r-\rho)$ for the z_n derivatives.

Of course, some derivatives can be given sharper estimates, for example, $dP_1/dz_j = 0$ by construction, but it will be simpler later to treat them all the same. ∎

The lower bound $r > \frac{1}{2}$ was important for the previous Corollary, but it is not a significant *a priori* restriction on the manifold $\widehat{M}$. By a real rescaling:

$$(113)\quad (\vec{z}, \vec{w}) \mapsto (a_1 z_1, a_1 z_2, \sqrt{a_1} z_3, a_1 z_4, \ldots, a_1 z_{m-1}, (a_1)^2 z_m, \ldots, (a_1)^2 z_n, a_1 \cdot \vec{w})^T,$$

$a_1 > 0$ (a special case of (12)), the equations (75) can be assumed to define $\widehat{M}$ for $|z_1| < 1$, $|x_\sigma| < 1$, $|t_\alpha| < 1$, and, further, for any $\eta_0 > 0$, there is a rescaling making $|\vec{e}|_1 < \eta_0$.

7.3. The new defining equations and some estimates.

To get a solution of the nonlinear system of equations (80) by iterating the solution of the linear system, the rapid convergence technique will apply, closely following the methods used by [**Moser**] on a different CR singularity problem. Each step along the way to a proof of Main Theorem 6.5 is stated as a Theorem.

Substituting the linear equation's normalized solution $\vec{p}$ from Theorem 7.6 into $E_2, \ldots, e_n$ in the RHS of the nonlinear equations (80–82) gives a quantity $\vec{q}$ depending on z_1, $\bar{z}_1$, x, t. Equation (82) is satisfied exactly by the P_1 constructed in Theorem 7.6, so for now, only (80–81), where the solution of the linear equations is merely an approximation to a solution of the nonlinear equations, will be considered. Let

$$(114)\quad \vec{z} = (z_1, x, \ldots, (x_{2(\tau-m+2)} + i x_{2(\tau-m+2)+1})\bar{z}_1, \ldots, \bar{z}_1^2, (z_1 + x_2 + i t_1 + i x_3^2)\bar{z}_1)^T,$$

as in (88), let

$$\begin{aligned}(115)\qquad \vec{z}+\vec{e} \;=\; & (z_1, x_2+iE_2, \dots, x_{m-1}+iE_{m-1},\\ & \dots, (x_{2(\tau-m+2)}+ix_{2(\tau-m+2)+1})\bar{z}_1+e_\tau, \dots,\\ & \bar{z}_1^2+e_{n-1}, (z_1+x_2+it_1+ix_3^2)\bar{z}_1+e_n)^T,\end{aligned}$$

as in (79), and then define $\vec{q}(z_1, \bar{z}_1, x, t) = (Q_2, \dots, Q_{m-1}, q_m, \dots, q_n)$ by:

$$\begin{aligned}(116)\quad Q_\sigma \;=\; & \mathrm{Im}(p_\sigma(\vec{z}+\vec{e},t)-p_\sigma(\vec{z},t))\\ (117)\quad q_\tau \;=\; & p_\tau(\vec{z}+\vec{e},t)-p_\tau(\vec{z},t)\\ & -(x_{2(\tau-m+2)}+ix_{2(\tau-m+2)+1})\cdot\overline{(p_1(\vec{z}+\vec{e},t)-p_1(\vec{z},t))}\\ & -\bar{z}_1\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z}+\vec{e},t)-p_{2(\tau-m+2)}(\vec{z},t))\\ & -i\bar{z}_1\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z}+\vec{e},t)-p_{2(\tau-m+2)+1}(\vec{z},t))\\ & -\overline{p_1(\vec{z}+\vec{e},t)}\cdot\mathrm{Re}(p_{2(\tau-m+2)}(\vec{z}+\vec{e},t))\\ & -i\overline{p_1(\vec{z}+\vec{e},t)}\cdot\mathrm{Re}(p_{2(\tau-m+2)+1}(\vec{z}+\vec{e},t))\\ (118)\quad q_{n-1} \;=\; & p_{n-1}(\vec{z}+\vec{e},t)-p_{n-1}(\vec{z},t)\\ & -2\bar{z}_1\overline{(p_1(\vec{z}+\vec{e},t)-p_1(\vec{z},t))}-\overline{(p_1(\vec{z}+\vec{e},t))}^2\\ (119)\quad q_n \;=\; & p_n(\vec{z}+\vec{e},t)-p_n(\vec{z},t)-\bar{z}_1\cdot(p_1(\vec{z}+\vec{e},t)-p_1(\vec{z},t))\\ & -\bar{z}_1\mathrm{Re}(p_2(\vec{z}+\vec{e},t)-p_2(\vec{z},t))\\ & -2i\bar{z}_1x_3\mathrm{Re}(p_3(\vec{z}+\vec{e},t)-p_3(\vec{z},t))\\ & -i\bar{z}_1\cdot(\mathrm{Re}(p_3(\vec{z}+\vec{e},t)))^2\\ & -(z_1+x_2+it_1+ix_3^2)\cdot\overline{(p_1(\vec{z}+\vec{e},t)-p_1(\vec{z},t))}\\ & -\overline{(p_1(\vec{z}+\vec{e},t))}p_1(\vec{z}+\vec{e},t)\\ & -\overline{(p_1(\vec{z}+\vec{e},t))}\cdot(\mathrm{Re}(p_2(\vec{z}+\vec{e},t))+iP_1(t))\\ & -\overline{(p_1(\vec{z}+\vec{e},t))}\cdot(2ix_3\mathrm{Re}(p_3(\vec{z}+\vec{e},t)))\\ & -\overline{(p_1(\vec{z}+\vec{e},t))}\cdot(i(\mathrm{Re}(p_3(\vec{z}+\vec{e},t)))^2).\end{aligned}$$

To outline the role of $\vec{q}$ in the argument, the next step (Theorem 7.10) will suppose $\vec{p}(z_1, \dots, z_n, w_1, \dots, w_k)$ is complex analytic on Δ_ρ, and $|\vec{e}|_\kappa$ is small enough so that $(\vec{z}, t) \in \Delta_\kappa \implies (\vec{z}+\vec{e}, t) \in \Delta_\rho$, and so $\vec{q}$ is a real analytic function for $(z_1, \bar{z}_1, x, t) \in D_\kappa$. The quantity $\vec{q}$ can be thought of as an analogue of the RHS of Equation (23); it is the higher degree part of the new defining equations, but expressed in terms of the old coordinates. If $\vec{q}(z_1, \bar{z}_1, x, t)$ happens to be identically zero, then the manifold $\widehat{M}$ has been brought to normal form by the functions $\vec{p}$. Otherwise, the degree of $\vec{q}$ is at least $2d-3$ by the construction of the solution $\vec{p}$, and defining $\vec{q}(z_1, \zeta, x, t)$ by (116–119), with ζ formally substituted for $\bar{z}_1$ and allowing complex x and t, the norm $|\vec{q}|_\kappa$ can be bounded in terms of the norm of $\vec{e}$. Then later, in the Proof of Theorem 7.14, converting $\vec{q}(z_1, \bar{z}_1, x, t)$ into an expression in $\tilde{z}_1$, $\bar{\tilde{z}}_1$, $\tilde{x}$, $\tilde{t}$, and equating it to the $\tilde{z}$ polynomial expression in (78) gives the defining equations of $\widehat{M}$ in the $(\tilde{z}, \tilde{w})$ coordinate system, in analogy with the substitution in the next step after Equation (23).

The $\tilde{N} = N$ case of the following Lemma is proved in [$\mathbf{C}_4$] (Lemma 4.1.).

LEMMA 7.9. *Let $f = (f_1, \dots, f_{\tilde{N}}) : \mathbb{D}_{\mathbf{r}} \to \mathbb{C}^{\tilde{N}}$ be a holomorphic map with*

$$\max_{j=1,\dots,N} \left\{ \sum_{\ell=1}^{\tilde{N}} \sup_{\vec{Z} \in \mathbb{D}_{\mathbf{r}}} \left| \frac{df_\ell}{dZ_j}(\vec{Z}) \right| \right\} \le K.$$

Then, for $\vec{Z} = (Z_1, \dots, Z_N)$, $\vec{Z}' = (Z'_1, \dots, Z'_N) \in \mathbb{D}_{\mathbf{r}}$,

$$\sum_{\ell=1}^{\tilde{N}} |f_\ell(\vec{Z}') - f_\ell(\vec{Z})| \le K \sum_{j=1}^{N} |Z'_j - Z_j|.$$

∎

THEOREM 7.10. *There are some constants $c_2 > 0$ and $\delta_1 > 0$ (depending on m) such that if $\frac{1}{2} < \kappa < r \le 1$, and $\vec{e}$ is as in Theorem 7.6, with $|\vec{e}|_r \le \delta_1(r - \kappa)$, then*

$$|\vec{q}|_\kappa \le \frac{c_2 |\vec{e}|_r^2}{(r - \kappa)^5}.$$

PROOF. Let $\rho = \frac{1}{2}(r + \kappa)$. Note that if $\delta_1 \le \frac{1}{2}$, the formal series for $\vec{q}$ is convergent on D_κ, since for $(z_1, \zeta, x, t) \in D_\kappa \subseteq D_r$, $|x_\sigma + iE_\sigma| < \kappa + \delta_1(r - \kappa) \le \kappa + (\rho - \kappa) = \rho$, $|(x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\zeta + e_\tau| < 2\kappa^2 + (\rho - \kappa) < 2\kappa^2 + (\rho - \kappa)(2(\rho + \kappa)) = 2\rho^2$, and similarly $|\zeta^2 + e_{n-1}| < \rho^2$ and $|(z_1 + x_2 + it_1 + ix_3^2)\zeta + e_n| < 3\rho^2 + \rho^3$, so $(\vec{z} + \vec{e}, t) \in \Delta_\rho$, which is contained in the domain of $\vec{p}$ by Theorem 7.6. The $N = n + k$, $\tilde{N} = 1$, $\mathbb{D}_{\mathbf{r}} = \Delta_\rho$ case of Lemma 7.9 applies to $p_\ell : \Delta_\rho \to \mathbb{C}$, with

$$\max_{j=1,\dots,n,\ \alpha=1,\dots,k} \left\{ \left\| \frac{dp_\ell}{dz_j} \right\|_\rho, \left\| \frac{dp_\ell}{dw_\alpha} \right\|_\rho \right\} \le K = \frac{c_1 |\vec{e}|_r}{(r - \rho)^3}$$

by Corollary 7.8, and $\vec{Z} = (\vec{z}, t)$, $\vec{Z}' = (\vec{z} + \vec{e}, t) \in \Delta_\rho$, so the conclusion is:

$$\begin{aligned} |p_\ell(\vec{z} + \vec{e}, t) - p_\ell(\vec{z}, t)| &\le K(|E_2|_r + \dots + |e_n|_r) \\ &= \frac{c_1 |\vec{e}|_r}{(r - \rho)^3} |\vec{e}|_r = \frac{8 c_1 |\vec{e}|_r^2}{(r - \kappa)^3} < \frac{2 c_1 |\vec{e}|_r^2}{(r - \kappa)^5}. \end{aligned}$$

This provides bounds for the differences that appear in (116–119), and the remaining terms are the products, where we can also use $\frac{1}{2} < \kappa < \rho < r \leq 1$ and the bound of Corollary 7.8 on the p_ℓ and P_1 factors. For example, in a case of (117) where $m > 4$, part of the expression is the product:

$$\begin{aligned}
&\sup_{D_\kappa} \left| (p_1(\vec{z}+\vec{e},t))' \frac{p_{2(\tau-m+2)}(\vec{z}+\vec{e},t) + (p_{2(\tau-m+2)}(\vec{z}+\vec{e},t))'}{2} \right| \\
\leq\ & \|p_1\|_\rho \cdot \|p_{2(\tau-m+2)}\|_\rho \leq \left(\frac{c_1|\vec{e}|_r}{(r-\rho)^2} \right)^2 = \frac{16(c_1)^2|\vec{e}|_r^2}{(r-\kappa)^4} < \frac{8(c_1)^2|\vec{e}|_r^2}{(r-\kappa)^5}.
\end{aligned}$$

Bounds for all of the other terms that are products of p_ℓ or P_1 can be found similarly, except for the last line in (119), which is a product of three quantities:

$$\begin{aligned}
&\sup_{D_\kappa} \left| (p_1(\vec{z}+\vec{e},t))' \cdot i \left(\frac{p_3(\vec{z}+\vec{e},t) + (p_3(\vec{z}+\vec{e},t))'}{2} \right)^2 \right| \\
\leq\ & \|p_1\|_\rho \cdot \|p_3\|_\rho^2 \leq \left(\frac{c_1|\vec{e}|_r}{(r-\rho)^2} \right)^3 = \frac{64(c_1)^3|\vec{e}|_r^3}{(r-\kappa)^6} \\
\leq\ & \frac{64(c_1)^3|\vec{e}|_r^2\delta_1(r-\kappa)}{(r-\kappa)^6} \leq \frac{32(c_1)^3|\vec{e}|_r^2}{(r-\kappa)^5}.
\end{aligned}$$

∎

The cubic nonlinear term (from the $i\bar{z}_1x_3^2$ term in the normal form) in the last step of the above Proof is the only reason for the exponent 5 on $\frac{1}{r-\kappa}$ in the statement of Theorem 7.10. There were no such cubic terms in the problems considered by [$\mathbf{C}_4$] or [$\mathbf{C}_6$], but the analysis will proceed in a similar way.

The following Lemma on inverse functions will be used twice, in the construction of the new coordinate system and the new defining equations; a proof by a standard iteration procedure is sketched in ([$\mathbf{C}_4$]).

LEMMA 7.11. *Suppose* $0 < R_{2,\ell} < R_{1,\ell}$ *for* $\ell = 1, \ldots, N$, *so that*

$$\mathbb{D}^2 = \mathbb{D}_{(R_{2,1},\ldots,R_{2,N})} \subseteq \mathbb{D}^1 = \mathbb{D}_{(R_{1,1},\ldots,R_{1,N})}.$$

Let $f(\vec{Z}) = (f_1(Z_1,\ldots,Z_N),\ldots,f_N(Z_1,\ldots,Z_N))$ *be holomorphic on* $\mathbb{D}^1$, *with*

$$\max_{j=1,\ldots,N} \left\{ \sum_{\ell=1}^N \sup_{\vec{Z}\in\mathbb{D}^1} \left| \frac{df_\ell}{dZ_j}(\vec{Z}) \right| \right\} \leq K < 1,$$

and

$$\sum_{\ell=1}^{N} \sup_{\vec{Z}\in\mathbb{D}^2} |f_\ell(\vec{Z})| \leq (1-K) \min_{\ell=1,\ldots,N}\{R_{1,\ell}-R_{2,\ell}\}.$$

Then, given $\vec{W} \in \mathbb{D}^2$*, there exists a unique solution* $\vec{Z} \in \mathbb{D}^1$ *of the equation*

$$\vec{W} = \vec{Z} + f(\vec{Z}),$$

and this solution satisfies

$$\sum_{\ell=1}^{N} |Z_\ell - W_\ell| \leq \frac{1}{1-K}\sum_{\ell=1}^{N} |f_\ell(\vec{W})|.$$

∎

THEOREM 7.12. *There is some constant* $\delta_2 > 0$ *(depending on* m*) so that for any radius lengths* $\frac{1}{2} < \kappa < r \leq 1$*, and* $\vec{e}$*,* $\vec{p}$ *as in Theorem 7.6, with* $|\vec{e}|_r \leq \delta_2(r-\kappa)^3$ *and* $\rho = \frac{1}{2}(r+\kappa)$*, the transformation*

$$\begin{aligned} \Psi \quad &: \quad (\vec{z},\vec{w}) = (z_1,\ldots,z_n,w_1,\ldots,w_k)^T \mapsto \\ (\tilde{z},\tilde{w}) \quad &= \quad (z_1+p_1(\vec{z},\vec{w}),\ldots,z_n+p_n(\vec{z},\vec{w}),w_1+P_1(\vec{w}),w_2,\ldots,w_k)^T \end{aligned}$$

has a holomorphic inverse $\psi((\tilde{z},\tilde{w})) = (\vec{z},\vec{w})$ *such that if* $(\tilde{z},\tilde{w}) \in \Delta_\kappa$*, then* $\psi((\tilde{z},\tilde{w})) \in \Delta_\rho$*.*

PROOF. By Corollary 7.8,

$$\begin{aligned} \max_{j=1,\ldots,n}\left\{\left\|\frac{dP_1}{dz_j}\right\|_\rho + \sum_{k=1}^{n}\left\|\frac{dp_k}{dz_j}\right\|_\rho\right\} &\leq \frac{c_1|\vec{e}|_r}{(r-\rho)^3} \\ &< \frac{c_1\delta_2(r-\kappa)^3}{(r-\rho)^3} = 8\delta_2 c_1 \leq \frac{1}{2} = K, \end{aligned}$$

if $\delta_2 \leq \frac{1}{16c_1}$, and similarly,

$$\max_{\alpha=1,\ldots,k}\left\{\left\|\frac{dP_1}{dw_\alpha}\right\|_\rho + \sum_{\ell=1}^{n}\left\|\frac{dp_\ell}{dw_\alpha}\right\|_\rho\right\} \leq K.$$

Also by Corollary 7.8,

$$\|P_1\|_\kappa + \sum_{\ell=1}^{n}\|p_\ell\|_\kappa \leq \frac{(n+1)c_1|\vec{e}|_r}{(r-\kappa)^2} \leq (n+1)c_1\delta_2(r-\kappa) \leq (1-K)(\rho-\kappa),$$

if $\delta_2 \leq \frac{1}{4(n+1)c_1}$. The hypotheses of Lemma 7.11 are satisfied with $\Delta_\kappa \subseteq \Delta_\rho$, and $R_{1,\ell}-R_{2,\ell} \geq \rho-\kappa$, so given $(\tilde{z},\tilde{w}) \in \Delta_\kappa$, there exists a unique $(\vec{z},\vec{w}) \in \Delta_\rho$ such that $(\tilde{z},\tilde{w}) = (z_1+p_1(\vec{z},\vec{w}),\ldots,z_n+p_n(\vec{z},\vec{w}),w_1+P_1(\vec{w}),w_2,\ldots,w_k)^T$. This defines ψ so that $\Psi\circ\psi$ is the identity map on Δ_κ. ∎

For $(z_1, \zeta, x, t) \in D_{R_1} \subseteq \mathbb{C}^{m+k}$, define $z^c \in \mathbb{C}^n$ by:

$$\begin{aligned} z^c &= (z_1,\ x_2 + iE_2(z_1,\zeta,x,t), \dots, x_{m-1} + iE_{m-1}(z_1,\zeta,x,t), \\ &\quad \dots, (x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\zeta + e_\tau(z_1,\zeta,x,t), \dots, \\ &\quad \zeta^2 + e_{n-1}(z_1,\zeta,x,t), (z_1 + x_2 + it_1 + ix_3^2)\zeta + e_n(z_1,\zeta,x,t))^T, \end{aligned}$$

and define a map $\Phi : D_{R_1} \to \mathbb{C}^{m+k}$ by

$$\begin{aligned} \Phi(z_1,\zeta,x,t) &= (\Phi_1(z_1,\zeta,x,t), \dots, \Phi_{m+k}(z_1,\zeta,x,t)) \\ &= (z_1 + p_1(z^c,t), \zeta + (p_1(z^c,t))', \\ &\quad x_2 + \frac{1}{2}(p_2(z^c,t) + (p_2(z^c,t))'), \dots, \\ &\quad x_{m-1} + \frac{1}{2}(p_{m-1}(z^c,t) + (p_{m-1}(z^c,t))'), \\ &\quad t_1 + P_1(t), t_2, \dots, t_k). \end{aligned}$$

THEOREM 7.13. *There is some constant $\delta_3 > 0$ (depending on m) so that for any radius lengths $\frac{1}{2} < r' < r \le 1$, and $\vec{e}$, $\vec{p}$ as in Theorem 7.6, with $|\vec{e}|_r \le \delta_3(r-r')^3$, and $\kappa = r' + \frac{1}{3}(r-r')$, the transformation $\Phi : (z_1,\zeta,x,t) \mapsto (\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t})$ has a holomorphic inverse $\phi(\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t}) = (z_1,\zeta,x,t)$ such that if $(\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t}) \in D_{r'}$, then $\phi(\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t}) \in D_\kappa$.*

PROOF. Let $\rho = r' + \frac{2}{3}(r-r')$, so $\kappa - r' = \rho - \kappa = r - \rho = \frac{1}{3}(r-r') < \frac{1}{6}$, and let $\bar{r} = \frac{1}{2}(r+r')$, so $\frac{1}{2} < r' < \kappa < \bar{r} < \rho < r \le 1$. If $(z_1,\zeta,x,t) \in D_{\bar{r}}$, and $\delta_3 \le \frac{2}{3}$, then $|E_2(z_1,\zeta,x,t)| \le \delta_3(r-r')^3 = 216\delta_3(\rho-\bar{r})^3 < \frac{216}{12^2}\delta_3(\rho-\bar{r}) \le \rho - \bar{r}$, and similarly $|e_{n-1}(z_1,\zeta,x,t)| < \rho^2 - \bar{r}^2$, etc., so $(z^c,t) \in \Delta_\rho$, and $\vec{p}(z^c,t)$ and Φ are well-defined and holomorphic on $D_{\bar{r}}$. Using Cauchy's estimate as in Corollary 7.8, for $(z_1,\zeta,x,t) \in D_\kappa$,

$$\begin{aligned} \left|\frac{d}{dz_1}p_2(z^c,t)\right| &\le \frac{|p_2(z^c,t)|_{\bar{r}}}{\bar{r}-\kappa} \le \frac{\|p_2\|_\rho}{\frac{1}{2}(\rho-\kappa)} \\ &\le \frac{2c_1|\vec{e}|_r}{(\rho-\kappa)(r-\rho)^2} = \frac{54c_1|\vec{e}|_r}{(r-r')^3}. \end{aligned}$$

Similarly, the derivative of each term, $p_1(z^c,t)$, $p_\sigma(z^c,t)$, $(p_1(z^c,t))'$, $(p_\sigma(z^c),t)'$, $P_1(t)$, with respect to each variable z_1, ζ, x_σ, t_α is bounded by a comparable quantity, so there is some constant $c_3 > 0$ (depending on m) so that

$$\max_{j=2,\dots,m-1,\ \alpha=1,\dots,k}$$
$$\left\{\left|\frac{dp_1(z^c,t)}{dz_1}\right|_\kappa+\left|\frac{d((p_1(z^c,t))')}{dz_1}\right|_\kappa+\sum_{\sigma=2}^{m-1}\left|\frac{d(\frac{1}{2}(p_\sigma(z^c,t)+(p_\sigma(z^c,t))'))}{dz_1}\right|_\kappa,\right.$$
$$\left|\frac{dp_1(z^c,t)}{d\zeta}\right|_\kappa+\left|\frac{d((p_1(z^c,t))')}{d\zeta}\right|_\kappa+\sum_{\sigma=2}^{m-1}\left|\frac{d(\frac{1}{2}(p_\sigma(z^c,t)+(p_\sigma(z^c,t))'))}{d\zeta}\right|_\kappa,$$
$$\left|\frac{dp_1(z^c,t)}{dx_j}\right|_\kappa+\left|\frac{d((p_1(z^c,t))')}{dx_j}\right|_\kappa+\sum_{\sigma=2}^{m-1}\left|\frac{d(\frac{1}{2}(p_\sigma(z^c,t)+(p_\sigma(z^c,t))'))}{dx_j}\right|_\kappa,$$
$$\left|\frac{dp_1(z^c,t)}{dt_\alpha}\right|_\kappa+\left|\frac{d((p_1(z^c,t))')}{dt_\alpha}\right|_\kappa+\sum_{\sigma=2}^{m-1}\left|\frac{d(\frac{1}{2}(p_\sigma(z^c,t)+(p_\sigma(z^c,t))'))}{dt_\alpha}\right|_\kappa$$
$$\left.+\left|\frac{dP_1(t)}{dt_\alpha}\right|_\kappa\right\}$$
$$\leq \frac{c_3|\vec{e}|_r}{(r-r')^3}\leq c_3\delta_3\leq\frac{1}{2},$$

if $\delta_3 \leq \frac{1}{2c_3}$. It also follows from Corollary 7.8 that

$$\begin{aligned}
&|p_1(z^c,t)|_{r'}+|(p_1(z^c,t))'|_{r'}\\
&+\left(\sum_{\sigma=2}^{m-1}\left|\frac{1}{2}(p_\sigma(z^c,t)+(p_\sigma(z^c,t))')\right|_{r'}\right)+|P_1(t)|_{r'}\\
\leq\ & 2\|p_1\|_\rho+\left(\sum_{\sigma=2}^{m-1}\|p_\sigma\|_\rho\right)+\|P_1\|_\rho\\
\leq\ & \frac{(m+1)c_1|\vec{e}|_r}{(r-\rho)^2}\leq\frac{(m+1)c_1\delta_3(r-r')^3}{(r-\rho)^2}\\
=\ & 9(m+1)c_1\delta_3(r-r')\leq\frac{1}{2}(\kappa-r'),
\end{aligned}$$

if $\delta_3 \leq \frac{1}{54(m+1)c_1}$. So, by Lemma 7.11, given $(\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t})\in D_{r'}$, there exists a unique $(z_1,\zeta,x,t)\in D_\kappa$ such that $(\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t})=\Phi(z_1,\zeta,x,t)$. ∎

By inspection of the form of Φ, for $(z_1,\zeta,x,t)\in D_\kappa$,

$$\Phi(z_1,\zeta,x,t)=(\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t}) \implies \Phi(\bar{\zeta},\bar{z}_1,\bar{x},\bar{t})=(\bar{\tilde{\zeta}},\bar{\tilde{z}}_1,\bar{\tilde{x}},\bar{\tilde{t}}).$$

If, further, $(\tilde{z}_1,\tilde{\zeta},\tilde{x},\tilde{t})=(\bar{\tilde{\zeta}},\bar{\tilde{z}}_1,\bar{\tilde{x}},\bar{\tilde{t}})\in D_{r'}$, then $(z_1,\zeta,x,t)=(\bar{\zeta},\bar{z}_1,\bar{x},\bar{t})$ by uniqueness of the inverse. In particular, if $|\tilde{z}_1|<r'$, and for $\sigma=2,\dots,m-1$, $\tilde{x}_\sigma$ is real and $|\tilde{x}_\sigma|<r'$, and for $\alpha=1,\dots,k$, $\tilde{t}_\alpha$ is real

and $|\tilde{t}| < r'$, then $\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t})$ is of the form $(z_1, \bar{z}_1, x, t)$ for some z_1 with $|z_1| < \kappa$, x real with $|x_\sigma| < \kappa$, and t real with $|t_\alpha| < \kappa$. Such (z_1, x, t) is unique, given $(\tilde{z}_1, \tilde{x}, \tilde{t})$: suppose there were (z_1^0, x^0, t^0) with $|z_1^0| < \kappa$, $|x_\sigma^0| < \kappa$, x^0 real, $|t_\alpha^0| < \kappa$, t^0 real, such that

$$\begin{aligned} \tilde{z}_1 &= \Phi_1(z_1^0, \overline{z_1^0}, x^0, t^0) \\ \tilde{x}_\sigma &= \Phi_{\sigma+1}(z_1^0, \overline{z_1^0}, x^0, t^0),\ \sigma = 2, \dots, m-1, \\ \tilde{t}_\alpha &= \Phi_{m+\alpha}(z_1^0, \overline{z_1^0}, x^0, t^0),\ \alpha = 1, \dots, k. \end{aligned}$$

Then the second component $\Phi_2(z_1^0, \overline{z_1^0}, x^0, t^0)$ can be calculated to have some value $\tilde{\zeta}$, so $\Phi(z_1^0, \overline{z_1^0}, x^0, t^0) = (\tilde{z}_1, \tilde{\zeta}, \tilde{x}, \tilde{t})$. By the formula for Φ, $\tilde{\zeta} = \bar{\tilde{z}}_1$, so $(\tilde{z}_1, \tilde{\zeta}, \tilde{x}, \tilde{t}) \in D_{r'}$ and $(z_1^0, \overline{z_1^0}, x^0, t^0) = \phi(\tilde{z}_1, \tilde{\zeta}, \tilde{x}, \tilde{t}) = \phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) = (z_1, \bar{z}_1, x, t)$, so we can conclude from the uniqueness of Lemma 7.11 that $z_1^0 = z_1$, $x^0 = x$, and $t^0 = t$.

THEOREM 7.14. *There exist constants* $c_4 > 0$ *and* $\delta_4 > 0$ *(depending on* m*) such that for any* $\frac{1}{2} < r' < r \le 1$ *(with* κ, ρ *as in the previous Theorem), and any* $\vec{e}$ *as in Theorem 7.6 with* $|\vec{e}|_r \le \delta_4(r - r')^3$*, there exist a holomorphic map*

$$\Psi : \Delta_\rho \to \mathbb{C}^{n+k} : (z_1, \dots, z_n, w_1, \dots, w_k)^T \mapsto (\tilde{z}_1, \dots, \tilde{z}_n, \tilde{w}_1, \dots, \tilde{w}_k)^T,$$

with a holomorphic inverse $\psi : \Delta_\kappa \to \Delta_\rho$*, and a holomorphic map* $\tilde{e} = (\tilde{E}_2, \dots, \tilde{e}_n) : D_{r'} \to \mathbb{C}^{n-1}$*, such that the defining equations for* $\widehat{M}$ *are*

$$\begin{aligned} \tilde{y}_\sigma = \mathrm{Im}(\tilde{z}_\sigma) &= \tilde{E}_\sigma(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) \\ \tilde{z}_\tau &= (\tilde{x}_{2(\tau-m+2)} + i\tilde{x}_{2(\tau-m+2)+1})\bar{\tilde{z}}_1 + \tilde{e}_\tau(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) \\ \tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 + \tilde{e}_{n-1}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) \\ \tilde{z}_n &= (\tilde{z}_1 + \tilde{x}_2 + i\tilde{t}_1 + i\tilde{x}_3^2)\bar{\tilde{z}}_1 + \tilde{e}_n(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) \\ s_\alpha = \mathrm{Im}(\tilde{w}_\alpha) &= 0, \end{aligned}$$

for $|\tilde{z}_1| < r'$, $|\tilde{x}_\sigma| < r'$, $|\tilde{t}_\alpha| < r'$*. Further, the degree of* $\tilde{e}$ *is at least* $2d - 3$*, and*

$$|\tilde{e}|_{r'} \le \frac{c_4 |\vec{e}|_r^2}{(r - r')^5}.$$

PROOF. Initially, choose $\delta_4 \le \min\{\frac{8}{3}\delta_1, \frac{8}{27}\delta_2, \delta_3\}$, so that Theorems 7.10, 7.12, 7.13 apply, and define Ψ, ψ, $\vec{q}$, and ϕ in terms of the given $\vec{e}$ and the functions $\vec{p}$ constructed in Theorem 7.6. Define $\tilde{e}$ to be the composite of holomorphic maps $\vec{q} \circ \phi : D_{r'} \to \mathbb{C}^{n-1}$, so that by Theorem 7.10,

$$|\tilde{e}|_{r'} \le |\vec{q}|_\kappa \le \frac{c_2 |\vec{e}|_r^2}{(r - \kappa)^5} = \frac{c_2 |\vec{e}|_r^2}{(\frac{2}{3}(r - r'))^5}.$$

Since $\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t})$ has no constant terms, and $\vec{q}$ has degree $\geq 2d-3$ by construction, $\tilde{e}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t})$ also has degree at least $2d-3$.

Given $\tilde{z}_1$, $\tilde{x}$, $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_k)$ such that $|\tilde{z}_1| < r'$, $\tilde{x}$ is real with $|\tilde{x}_\sigma| < r'$, and $\tilde{t}$ is real with $|\tilde{t}_\alpha| < r'$, define quantities $\tilde{z}_2, \ldots, \tilde{z}_n$ by:

$$\begin{aligned} \tilde{z}_\sigma &= \tilde{x}_\sigma + i\tilde{E}_\sigma(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) \\ \tilde{z}_t &= (\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1})\bar{\tilde{z}}_1 + \tilde{e}_t(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) \\ \tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 + \tilde{e}_{n-1}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}) \\ \tilde{z}_n &= (\tilde{z}_1 + \tilde{x}_2 + i\tilde{t}_1 + i\tilde{x}_3^2)\bar{\tilde{z}}_1 + \tilde{e}_n(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t}), \end{aligned} \tag{120}$$

and denote $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)^T$. The claim of the Theorem is that $\psi((\tilde{z}, \tilde{t})) \in \widehat{M}$.

If $(\delta_4)^2 \leq \frac{32}{729c_2}$, then

$$|\tilde{e}|_{r'} \leq \frac{c_2(\delta_4(r-r')^3)^2}{(r-\kappa)^5} = c_2(\delta_4)^2 \frac{3^6}{2^5}(\kappa - r') \leq \kappa - r',$$

so $(\tilde{z}, \tilde{t}) \in \Delta_\kappa$, the domain of ψ.

By Theorem 7.13, there exists a unique (z_1, x, t) (the first and last components of $(z_1, \bar{z}_1, x, t) = \phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}, \tilde{t})$) such that $|z_1| < \kappa$, x is real with $|x_\sigma| < \kappa$, t is real with $|t_\alpha| < \kappa$, and

$$\begin{aligned} \tilde{z}_1 &= z_1 + p_1(z_1, x_2 + iE_2(z_1, \bar{z}_1, x, t), \ldots, \\ &\qquad (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 + e_n(z_1, \bar{z}_1, x, t), t) \\ \tilde{x}_\sigma &= x_\sigma + \mathrm{Re}(p_\sigma(z_1, x_2 + iE_2(z_1, \bar{z}_1, x, t), \ldots, \\ &\qquad (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 + e_n(z_1, \bar{z}_1, x, t), t)) \\ \tilde{t}_1 &= t_1 + P_1(t) \\ \tilde{t}_j &= t_j,\ j = 2, \ldots, k. \end{aligned}$$

Then, define quantities $z_2, \ldots, z_n$ by:

$$\begin{aligned} z_\sigma &= x_\sigma + iE_\sigma(z_1, \bar{z}_1, x, t) \\ z_\tau &= (x_{2(\tau-m+2)} + ix_{2(\tau-m+2)+1})\bar{z}_1 + e_\tau(z_1, \bar{z}_1, x, t) \\ z_{n-1} &= \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x, t) \\ z_n &= (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1 + e_n(z_1, \bar{z}_1, x, t), \end{aligned}$$

and denote, as in (88) and (114), $\vec{z} = (z_1, x, \ldots, (z_1 + x_2 + it_1 + ix_3^2)\bar{z}_1)^T$ and $\vec{z} + \vec{e} = (z_1, z_2, \ldots, z_n)^T$ as in (79) and (115). Since $|z_1| < \kappa < r$, $|x_\sigma| < \kappa < r$, and $|t_\alpha| < \kappa < r$, $(\vec{z} + \vec{e}, t) \in \widehat{M}$, and if $\delta_4 \leq \frac{4}{3}$, then

$$|\vec{e}|_\kappa \leq |\vec{e}|_r \leq \delta_4(r-r')^3 = \delta_4 \cdot 27(\rho - \kappa)^3 < \delta_4 \frac{27}{6^2}(\rho - \kappa) \leq (\rho - \kappa),$$

so $(\vec{z} + \vec{e}, t) \in \Delta_\rho$, which is contained in the domain of $\vec{p}$.

$$
\begin{aligned}
&\Psi(\vec{z}+\vec{e},t)\\
=\ &(z_1+p_1((\vec{z}+\vec{e},t)),\dots,z_n+p_n((\vec{z}+\vec{e},t)),t_1+P_1(t),t_2,\dots,t_k)^T\\
=\ &(\tilde{z}_1,\dots,\tilde{x}_\sigma+iE_\sigma(z_1,\bar{z}_1,x,t)+i\mathrm{Im}(p_\sigma((\vec{z}+\vec{e},t))),\dots,\\
&\overline{(\tilde{z}_1-p_1((\vec{z}+\vec{e},t)))}\cdot(\tilde{x}_{2(\tau-m+2)}-\mathrm{Re}(p_{2(\tau-m+2)}((\vec{z}+\vec{e},t))))\\
&+i\overline{(\tilde{z}_1-p_1((\vec{z}+\vec{e},t)))}\cdot(\tilde{x}_{2(\tau-m+2)+1}-\mathrm{Re}(p_{2(\tau-m+2)+1}((\vec{z}+\vec{e},t))))\\
&+e_\tau(z_1,\bar{z}_1,x,t)+p_\tau((\vec{z}+\vec{e},t)),\dots\\
&\overline{(\tilde{z}_1-p_1((\vec{z}+\vec{e},t)))}^2+e_{n-1}(z_1,\bar{z}_1,x,t)+p_{n-1}((\vec{z}+\vec{e},t)),\\
&\overline{(\tilde{z}_1-p_1((\vec{z}+\vec{e},t)))}\cdot(\tilde{z}_1-p_1((\vec{z}+\vec{e},t))+\tilde{x}_2-\mathrm{Re}(p_2((\vec{z}+\vec{e},t))))\\
&+i\overline{(\tilde{z}_1-p_1((\vec{z}+\vec{e},t)))}\cdot(\tilde{t}_1-P_1(t))\\
&+i\overline{(\tilde{z}_1-p_1((\vec{z}+\vec{e},t)))}\cdot(\tilde{x}_3-\mathrm{Re}(p_3((\vec{z}+\vec{e},t))))^2\\
&+e_n(z_1,\bar{z}_1,x,t)+p_n((\vec{z}+\vec{e},t)),\tilde{t})^T\\
=\ &(\tilde{z}_1,\dots,\tilde{x}_\sigma+iQ_\sigma(z_1,\bar{z}_1,x,t),\dots,\\
&\bar{\tilde{z}}_1\cdot(\tilde{x}_{2(\tau-m+2)}+i\tilde{x}_{2(\tau-m+2)+1})+q_\tau(z_1,\bar{z}_1,x,t),\dots,\\
&\bar{\tilde{z}}_1^2+q_{n-1}(z_1,\bar{z}_1,x,t),\\
&\bar{\tilde{z}}_1\cdot(\tilde{z}_1+\tilde{x}_2+i\tilde{t}_1+i\tilde{x}_3^2)+q_n(z_1,\bar{z}_1,x,t),\tilde{t})^T\\
=\ &(\tilde{z}_1,\dots,\tilde{x}_\sigma+iQ_\sigma(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t})),\dots,\\
&\bar{\tilde{z}}_1\cdot(\tilde{x}_{2(\tau-m+2)}+i\tilde{x}_{2(\tau-m+2)+1})+q_\tau(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t})),\dots,\\
&\bar{\tilde{z}}_1^2+q_{n-1}(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t})),\\
&\bar{\tilde{z}}_1\cdot(\tilde{z}_1+\tilde{x}_2+i\tilde{t}_1+i\tilde{x}_3^2)+q_n(\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t})),\tilde{t})^T\\
=\ &(\tilde{z}_1,\dots,\tilde{x}_\sigma+i\tilde{E}_\sigma(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t}),\dots,\\
&\bar{\tilde{z}}_1\cdot(\tilde{x}_{2(\tau-m+2)}+i\tilde{x}_{2(\tau-m+2)+1})+\tilde{e}_\tau(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t}),\dots,\\
&\bar{\tilde{z}}_1^2+\tilde{e}_{n-1}(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t}),\\
&\bar{\tilde{z}}_1\cdot(\tilde{z}_1+\tilde{x}_2+i\tilde{t}_1+i\tilde{x}_3^2)+\tilde{e}_n(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x},\tilde{t}),\tilde{t})^T=(\tilde{z},\tilde{t})
\end{aligned}
$$

by construction of $\vec{q}$, $\tilde{e}$, and $\tilde{z}$, and using the fact that $\vec{p}$ is a solution of (84–88). By the uniqueness of Theorem 7.12, $\psi((\tilde{z},\tilde{t}))=(\vec{z}+\vec{e},t)\in\widehat{M}$. ∎

7.4. Composition of approximate solutions.

The previous Theorem's quadratic estimate on the size of $\tilde{e}$ in terms of $\vec{e}$ allows for the rapid convergence of a sequence of approximations. A couple technical Lemmas will be needed to measure the behavior of composite mappings. Theorem 7.21, which is the last step in proving

Main Theorem 6.5, uses these Lemmas and the estimates of the previous Subsection to prove convergence of a sequence of transformations, following the ideas of [**Moser**].

NOTATION 7.15. For $R_1 > 0$ and a $(n+k) \times (n+k)$ matrix of complex valued functions $F = (F_{\ell j}((\vec{z}, \vec{w})))$ on Δ_{R_1}, define

$$|||F|||_{R_1} = \max_{j=1,\ldots,n+k} \left\{ \sum_{\ell=1}^{n+k} \sup_{(\vec{z},\vec{w}) \in \Delta_{R_1}} |F_{\ell j}((\vec{z}, \vec{w}))| \right\}.$$

This "maximum column sum" norm appeared already, in Corollary 7.8 and Lemmas 7.9, 7.11, in the case where $F = \mathrm{D}f = \mathrm{D}_{(\vec{z},\vec{w})}f$, the Jacobian matrix of some map $f : \Delta_{R_1} \to \mathbb{C}^{n+k}$ at $(\vec{z}, \vec{w}) \in \Delta_{R_1}$.

The 3×3 case of the following Lemma was proved in [$\mathbf{C}_4$].

LEMMA 7.16. *If $|||A|||_{R_1} < 1$, then $\mathbb{1} + A$ is invertible (where $\mathbb{1}$ is the $(n+k) \times (n+k)$ identity matrix), and*

$$|||(\mathbb{1} + A)^{-1}|||_{R_1} \le \frac{1}{1 - |||A|||_{R_1}}.$$

∎

Also, the following elementary fact from the calculus of one real variable will be used.

LEMMA 7.17. *If μ_ℓ is a sequence such that $0 \le \mu_\ell < 1$ and $\sum_{\ell=0}^{\infty} \mu_\ell$ is a convergent series, then the sequence of partial products*

$$\prod_{\ell=0}^{N} \frac{1}{1 - \mu_\ell}$$

is bounded above by some positive limit. ∎

NOTATION 7.18. Define a sequence $\left\{1, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \ldots\right\}$ indexed by $\nu = 0, 1, 2, 3, \ldots$, by the formula

$$r_\nu = \frac{1}{2}\left(1 + \frac{1}{\nu + 1}\right).$$

Note that $\frac{1}{2} < r_\nu \le 1$, and the sequence is decreasing, with

$$\begin{aligned} r_\nu - r_{\nu+1} &= \frac{1}{2(\nu+1)(\nu+2)} \le \frac{1}{4}, \\ \frac{r_{\nu+1} - r_{\nu+2}}{r_\nu - r_{\nu+1}} &= \frac{\nu+1}{\nu+3} \ge \frac{1}{3}. \end{aligned}$$

NOTATION 7.19. Define $\kappa_\nu = r_{\nu+1} + \frac{1}{3}(r_\nu - r_{\nu+1})$, $\rho_\nu = r_{\nu+1} + \frac{2}{3}(r_\nu - r_{\nu+1})$, as in Theorem 7.13.

Recall that given $\eta_0 > 0$, there is some scaling transformation (113) so that $\widehat{M} \cap \Delta_1$ is defined by (75), with $\vec{e}$ holomorphic on D_1, degree $d \geq 4$ (with the cubic terms of $\widehat{M}$ already in normal form as in Remark 7.7), and $|\vec{e}|_1 \leq \eta_0$.

NOTATION 7.20. Denote $\vec{e}_0 = \vec{e}$ (so $|\vec{e}_0|_{r_0} = |\vec{e}|_1 \leq \eta_0$), and inductively define the formal series $\vec{e}_{\nu+1}(z_1, \zeta, x, t)$ in terms of $\vec{e}_\nu(z_1, \zeta, x, t)$, by the $\vec{e} \mapsto \tilde{e}$ procedure of Theorem 7.14, with $r = r_\nu$, $r' = r_{\nu+1}$. Each $\vec{e}_\nu$ defines, as in the previous Theorems, functions $\vec{p}_\nu$, $\vec{q}_\nu$, Ψ_ν, ψ_ν, ϕ_ν, and the degree of $\vec{e}_\nu$ is denoted d_ν.

Also recall that the degree $d_{\nu+1}$ of $\vec{e}_{\nu+1}$ is at least $2d_\nu - 3$; it can be checked that this, together with $d_0 = d \geq 4$, implies $d_\nu \geq 2^\nu + 3$.

The plan is to show that the bound for $\vec{e}_\nu$ in the hypothesis of Theorem 7.14 holds for all ν, to get a sequence of transformations $\psi_\nu : \Delta_{\kappa_\nu} \to \Delta_{\rho_\nu}$, so that the composition $\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu : \Delta_{\kappa_\nu} \to \Delta_{\rho_0}$ is well-defined, $\vec{e}_\nu$ is holomorphic on D_{r_ν}, and $\lim_{\nu\to\infty} |\vec{e}_\nu|_{r_\nu} = 0$.

THEOREM 7.21. *There exists $\eta_0 > 0$ (depending on m) so that if $\vec{e}_0$ and $\widehat{M}$ are as described above, then there exists a holomorphic transformation $\psi : \Delta_{\frac{1}{2}} \to \mathbb{C}^{n+k}$, with a holomorphic inverse Ψ, and such that if $(\tilde{z}, \tilde{t}) \in \widehat{\widehat{M}} \cap \Delta_{\frac{1}{2}}$, then $\psi((\tilde{z}, \tilde{t})) \in \widehat{M}$.*

PROOF. Let $\delta_5 = \min\{16\delta_4, \frac{1}{243c_4}\}$, and choose

$$0 < \eta_0 < \min\left\{\frac{\delta_5}{1024}, \frac{1}{1728c_1}\right\}.$$

It will be shown that

$$|\vec{e}_\nu|_{r_\nu} \leq \delta_5(r_\nu - r_{\nu+1})^5 \implies |\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \delta_5(r_{\nu+1} - r_{\nu+2})^5.$$

By Theorem 7.14, $|\vec{e}_\nu|_{r_\nu} \leq \delta_5(r_\nu - r_{\nu+1})^5 \leq \delta_4(r_\nu - r_{\nu+1})^3$ and $|\vec{e}_\nu|_{r_\nu} \leq \frac{1}{243c_4}(r_\nu - r_{\nu+1})^5$ imply

$$|\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \frac{c_4|\vec{e}_\nu|^2_{r_\nu}}{(r_\nu - r_{\nu+1})^5} \leq \frac{1}{243}|\vec{e}_\nu|_{r_\nu};$$

this already suggests a geometric decrease in the sequence of norms. Then, using the properties of the sequence r_ν,

$$\frac{1}{243}|\vec{e}_\nu|_{r_\nu} \leq \frac{1}{243}\delta_5(r_\nu - r_{\nu+1})^5 \leq \delta_5(r_{\nu+1} - r_{\nu+2})^5,$$

which proves the claimed implication. Using this as an inductive step, and starting the induction with $|\vec{e}_0|_{r_0} \le \eta_0 < \frac{1}{1024}\delta_5 = \delta_5(r_0 - r_1)^5$, the hypothesis of Theorem 7.14 is satisfied for all ν. The first of three conclusions from Theorem 7.14 is that $\vec{e}_\nu$ is holomorphic on D_{r_ν}, with degree $d_\nu \ge 2^\nu + 3$, and $|\vec{e}_\nu|_{r_\nu} \le 243^{-\nu}\eta_0$. Secondly, $\psi_0 \circ \ldots \circ \psi_\nu$ is a well-defined holomorphic map $\Delta_{\kappa_\nu} \to \Delta_{\rho_0}$, and $\Psi_\nu \circ \ldots \circ \Psi_0$ is well-defined and holomorphic on the image $(\psi_0 \circ \ldots \circ \psi_\nu)(\Delta_{\kappa_\nu})$, so that $\Psi_\nu \circ \ldots \circ \Psi_0 \circ \psi_0 \circ \ldots \circ \psi_\nu$ is the identity on Δ_{κ_ν}. The third conclusion is that if $|\tilde{z}_1| < r_{\nu+1}$, $|\tilde{x}_\sigma| < r_{\nu+1}$, and $|\tilde{t}_\alpha| < r_{\nu+1}$, and $(\tilde{z}, \tilde{t})$ is defined as in (120) with $\tilde{e} = \vec{e}_{\nu+1}$, then $(\psi_0 \circ \ldots \circ \psi_\nu)((\tilde{z}, \tilde{t})) \in \widehat{M}$. For any $(\vec{z}, \vec{w}) = (z_1, \ldots, z_n, w_1, \ldots, w_k) \in \Delta_{\frac{1}{2}}$, the sequence (depending on ν) $(\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu)((\vec{z}, \vec{w}))$ is contained in $\Delta_{\rho_0} = \Delta_{11/12}$. The following argument, beginning with several applications of Lemma 7.9, shows this sequence is a Cauchy sequence, and converges to some value $\psi((\vec{z}, \vec{w}))$.

$$\begin{aligned}
& \sum_{\ell=1}^{n+k} |(\psi_0 \circ \ldots \circ \psi_{\nu+1})_\ell((\vec{z}, \vec{w})) - (\psi_0 \circ \ldots \circ \psi_\nu)_\ell((\vec{z}, \vec{w}))| \\
= & \sum_{\ell=1}^{n+k} |(\psi_0)_\ell((\psi_1 \circ \ldots \circ \psi_{\nu+1})((\vec{z}, \vec{w}))) \\
& \qquad -(\psi_0)_\ell((\psi_1 \circ \ldots \circ \psi_\nu)((\vec{z}, \vec{w})))| \\
\le & \; |||\mathrm{D}\psi_0|||_{\rho_1} \cdot \sum_{j=1}^{n+k} |(\psi_1 \circ \ldots \circ \psi_{\nu+1})_j((\vec{z}, \vec{w})) \\
& \qquad -(\psi_1 \circ \ldots \circ \psi_\nu)_j((\vec{z}, \vec{w}))| \\
\le & \left(\prod_{\ell=0}^{\nu} |||\mathrm{D}\psi_\ell|||_{\rho_{\ell+1}}\right) \cdot \sum_{j=1}^{n+k} |(\psi_{\nu+1})_j((\vec{z}, \vec{w})) - (\vec{z}, \vec{w})_j|, \qquad (121)
\end{aligned}$$

where $(\psi_0)_\ell$ denotes the ℓ^{th} output component of the vector valued function ψ_0, etc., and $(\vec{z}, \vec{w})_j = z_j$ for $j = 1, \ldots, n$, or w_{j-n} for $j = n+1, \ldots, n+k$. By the estimate from Lemma 7.11, with $f = \vec{p}_{\nu+1}$ and $K = \frac{1}{2}$ from the Proof of Theorem 7.12, and then using the bound for

$\vec{p}$ from Corollary 7.8,

$$\begin{aligned}
&\sum_{j=1}^{n+k} |(\psi_{\nu+1})_j((\vec{z},\vec{w})) - (\vec{z},\vec{w})_j| \leq \frac{1}{1-\frac{1}{2}} \sum_{j=1}^{n+k} |(\vec{p}_{\nu+1})_j((\vec{z},\vec{w}))| \\
\leq\quad & 2\sum_{j=1}^{n+k} \|(\vec{p}_{\nu+1})_j\|_{\frac{1}{2}} \leq 2\sum_{j=1}^{n+1} \|(\vec{p}_{\nu+1})_j\|_{\rho_{\nu+1}} \leq 2(n+1)\frac{c_1|\vec{e}_{\nu+1}|_{r_{\nu+1}}}{(r_{\nu+1}-\rho_{\nu+1})^2} \\
=\quad & 18(n+1)\frac{c_1|\vec{e}_{\nu+1}|_{r_{\nu+1}}}{(r_{\nu+1}-r_{\nu+2})^2} = 72(n+1)c_1(\nu+2)^2(\nu+3)^2|\vec{e}_{\nu+1}|_{r_{\nu+1}} \\
\leq\quad & \frac{72(n+1)c_1(\nu+2)^2(\nu+3)^2\eta_0}{243^{\nu+1}}.
\end{aligned}$$

It follows from $\mathrm{D}_{(\vec{z},\vec{w})}\psi_\ell = (\mathbb{1}+\mathrm{D}_{\psi_\ell((\vec{z},\vec{w}))}\vec{p}_\ell)^{-1}$ and Lemma 7.16 that:

$$\begin{aligned}
|||\mathrm{D}\psi_\ell|||_{\rho_{\ell+1}} &= |||(\mathbb{1}+\mathrm{D}_{\psi_\ell((\vec{z},\vec{w}))}\vec{p}_\ell)^{-1}|||_{\rho_{\ell+1}} \\
&\leq |||(\mathbb{1}+\mathrm{D}\vec{p}_\ell)^{-1}|||_{\rho_\ell} \leq \frac{1}{1-|||\mathrm{D}\vec{p}_\ell|||_{\rho_\ell}}.
\end{aligned}$$

Then, by Lemma 7.17, the product from (121) is bounded above by some constant $c_5 > 0$, since by Corollary 7.8,

$$\begin{aligned}
|||\mathrm{D}\vec{p}_\ell|||_{\rho_\ell} &\leq \frac{c_1|\vec{e}_\ell|_{r_\ell}}{(r_\ell-\rho_\ell)^3} = \frac{27c_1|\vec{e}_\ell|_{r_\ell}}{(r_\ell-r_{\ell+1})^3} = 216(\ell+1)^3(\ell+2)^3c_1|\vec{e}_\ell|_{r_\ell} \\
&\leq \frac{216(\ell+1)^3(\ell+2)^3c_1\eta_0}{243^\ell} < 1,
\end{aligned}$$

and by the comparison:

$$\sum_{\ell=0}^{\infty} |||\mathrm{D}\vec{p}_\ell|||_{\rho_\ell} \leq \sum_{\ell=0}^{\infty} \frac{216(\ell+1)^3(\ell+2)^3c_1\eta_0}{243^\ell},$$

the infinite series is convergent, with terms < 1.

The inequality

$$\begin{aligned}
&\sum_{\ell=1}^{n+k} |(\psi_0\circ\ldots\circ\psi_{\nu+1})_\ell((\vec{z},\vec{w})) - (\psi_0\circ\ldots\circ\psi_\nu)_\ell((\vec{z},\vec{w}))| \\
\leq\quad & \frac{72(n+1)c_1c_5(\nu+2)^2(\nu+3)^2\eta_0}{243^{\nu+1}}
\end{aligned}$$

is enough to show that the sequence of composite functions converges pointwise and uniformly to a function ψ on $\Delta_{\frac{1}{2}}$. ∎

8. Some other directions

1. The lists of normal forms from Section 5.1 stopped at $Q \equiv 0$, although some results on surfaces with a higher order contact with the complex tangent line appear in [**H**$_1$], [**H**$_2$], [**Bharali**]. The list of normal forms from Section 6.1 is also incomplete.

2. The calculations of normal forms of unfoldings of surfaces from Section 5.2 stopped at low degree, leaving $O(4)$ or $O(5)$ quantities not normalized. Presumably some higher degree terms in the normal forms could be analyzed by further calculations, but the term-by-term approach of that Section does become complicated.

3. In Example 5.13, the unfolding normal form (36) for generic elliptic points was observed to be similar to the flattened algebraic normal form result of [**MW**] for n-manifolds in $\mathbb{C}^n$ under biholomorphic transformations. This leaves open the question: can the [**MW**] normal form (37) be achieved by a transformation in the subgroup $\mathcal{U}_{2,2,k}$ (27–28), or if not, how is the unfolding classification different?

4. The local geometry of a manifold near a point in N_j (as in (2)), for $m \le n$, $j > 1$, where the tangent space contains a complex plane, has not been studied as much as the $j = 1$ case. Since the codimension of N_2 is generally $4(n - m + 2)$, it could have a nonnegative expected dimension for $m \ge 8$ and some n. It would be interesting to understand the local geometry of M near a point in N_2, and how such a manifold might be deformed so that other types of singularities in N_1 or N_2 appear.

5. The case $\frac{2}{3}(n + 1) < m \le n$, where M has high codimension, but the locus N_1 has a positive expected dimension, is also interesting. Globally, there are characteristic class formulas ([**Webster**$_2$], [**Webster**$_3$], [**C**$_1$]) describing the non-isolated CR singularities, generalizing (3). The formulas of [**Webster**$_2$] involving parabolic singularities in the $m = n$ case have not been generalized to other types of degenerate singularities. Locally, the normal forms in the nondegenerate case for $\frac{2}{3}(n+1) < m < n$ are known ([**C**$_6$]), but the normal forms for degenerate CR singularities are not, and may not be as simple as those in Proposition 6.2. Normal forms for unfolding could also become complicated.

6. Another less-explored case of CR singular m-submanifolds of $\mathbb{C}^n$ is where $m > n$, so the generic point $\mathbf{x}$ of a submanifold in general position satisfies $j = \dim T_{\mathbf{x}} \cap JT_{\mathbf{x}} = m - n$, and the CR singular points are where this dimension jumps. The expected codimension formula codim $N_j = 2j(n - m + j)$ still holds for $j \ge m - n$ and $m > n$, and some $j > m - n$ cases have been considered by [**G**].

7. The use of all real variables to parametrize the deformation seemed to be a more natural, or at least simpler, choice than using complex parameters, since the coefficients appearing in normal forms (for example, the Bishop invariant) can be real quantities, and a real time variable in the $k = 1$ case was useful in the visualization. A more general unfolding construction could use both complex and real deformation parameters. Then, for example, even for M with only $N_1 \setminus N_2$ singularities, the manifold $\widehat{M}$ could be tangent to a complex plane.

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Bibliography

[A] L. Ahlfors, *Complex Analysis*, third ed., McGraw-Hill, New York, 1979. MR 0510197 (80c:30001), Zbl 0395.30001.

[AVGL] V. I. Arnol'd, V. A. Vasil'ev, V. V. Goryunov, and O. V. Lyashko, *Dynamical Systems VI: Singularity Theory I: Local and Global Theory*, transl. by A. Iacob, EMS **6**, Springer-Verlag, Berlin, 1993. MR 1230637 (94b:58018), Zbl 0786.00006.

[BF] T. Banchoff and F. Farris, *Tangential and normal Euler numbers, complex points, and singularities of projections for oriented surfaces in four-space*, Pacific J. Math. (1) **161** (1993), 1–24. MR 1237138 (94j:57026), Zbl 0815.57024.

[BER] M. S. Baouendi, P. Ebenfelt, and L. Preiss Rothschild, *Local geometric properties of real submanifolds in complex space*, Bull. AMS (N.S.) (3) **37** (2000), 309–336. MR 1754643 (2001a:32043), Zbl 0955.32027.

[Beloshapka] V. K. Beloshapka, *The normal form of germs of four-dimensional real submanifolds in* $\mathbb{C}^5$ *at generic* $\mathbb{RC}$*-singular points*, Mat. Zametki (6) **61** (1997), 931–934; transl. in Math. Notes (5–6) **61** (1997), 777–779. MR 1629829 (99f:32012), Zbl 0917.32015.

[Bharali] G. Bharali, *Surfaces with degenerate CR singularities that are locally polynomially convex*, Michigan Math. J. (2) **53** (2005), 429–445. MR 2152709 (2006e:32014), Zbl 1087.32009.

[Bishop] E. Bishop, *Differentiable manifolds in complex Euclidean space*, Duke Math. J. (1) **32** (1965), 1–21. MR 0200476 (34 #369), Zbl 0154.08501.

[Callahan] J. Callahan, *Singularities and plane maps* II*: Sketching catastrophes*, Amer. Math. Monthly (10) **84** (1977), 765–803. MR 0458466 (56 #16669), Zbl 0389.58006.

[C_1] A. Coffman, *Enumeration and Normal Forms of Singularities in Cauchy-Riemann Structures*, dissertation, University of Chicago, 1997.

[C_2] A. Coffman, *CR singular immersions of complex projective spaces*, Beiträge zur Algebra und Geometrie (2) **43** (2002), 451–477. MR 1957752 (2003k:32052), Zbl 1029.32020.

[C_3] A. Coffman, *Real congruence of complex matrix pencils and complex projections of real Veronese varieties*, Linear Algebra and its Applications **370** (2003), 41–83. MR 1994320 (2004f:14026), Zbl 1049.14042.

[C_4] A. Coffman, *Analytic normal form for CR singular surfaces in* $\mathbb{C}^3$, Houston J. Math. (4) **30** (2004), 969–996. MR 2110245 (2006d:32048), Zbl 1074.32013.

[C_5] A. Coffman, *CR singularities of real threefolds in* $\mathbb{C}^4$, Advances in Geometry (1) **6** (2006), 109–137. MR 2242865 (2007d:32033), Zbl 1120.32024.

[C_6] A. Coffman, *Analytic stability of the CR cross-cap*, Pacific J. Math. (2) **226** (2006), 221–258. MR 2247863 (2007j:32038), Zbl 1123.32018.

[DJ] A. Derdzinski and T. Januszkiewicz, *Immersions of surfaces in Spin^c-manifolds with a generic positive spinor*, Ann. Global Anal. Geom. (2) **26** (2004), 175–199; and *Erratum*, Ann. Global Anal. Geom. (3) **26** (2004), 319+25 pp. MR 2071487 (2005g:53084a), Zbl 1080.53057; MR 2097622 (2005g:53084b), Zbl 1067.53037.

[D_1] A. V. Domrin, *The number of $\mathbb{RC}$-singular points on a 4-dimensional submanifold in a 5-dimensional complex manifold*, Mat. Zametki (2) **57** (1995), 240–245; transl. in Math. Notes (1–2) **57** (1995), 167–170. MR 1339443 (96d:32017), Zbl 0853.57028.

[D_2] A. V. Domrin, *A description of characteristic classes of real submanifolds in complex manifolds via RC-singularities*, Izv. Ross. Akad. Nauk Ser. Mat. (English transl.) (5) **59** (1995), 899–918. MR 1360632 (97c:32020), Zbl 0877.57011.

[F] F. Forstnerič, *Complex tangents of real surfaces in complex surfaces*, Duke Math. J. (2) **67** (1992), 353–376. MR 1177310 (93g:32025), Zbl 0761.53032.

[G] T. Garrity, *Global structures on CR manifolds via Nash blow-ups*, W. Fulton Birthday Volume, Michigan Math. J. **48** (2000), 281–294. MR 1786491 (2001h:32058), Zbl 0995.32023.

[GG] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, GTM **14**, Springer, New York, 1973. MR 0341518 (49 #6269), Zbl 0294.58004.

[H_1] G. Harris, *Lowest order invariants for real-analytic surfaces in* $\mathbb{C}^2$, Trans. AMS (1) **288** (1985), 413–422. MR 0773068 (86g:32033), Zbl 0574.32029.

[H_2] G. Harris, *Real k-flat surfaces in* $\mathbb{C}^2$, Houston J. Math. (4) **14** (1988), 501–506. MR 0998452 (90i:32030), Zbl 0694.32011.

[HL] F. R. Harvey and H. B. Lawson, Jr., *Geometric residue theorems*, American Journal of Mathematics (4) **117** (1995), 829–873. MR 1342833 (96c:53112), Zbl 0851.58036.

[HK] X. Huang and S. Krantz, *On a problem of Moser*, Duke Math. J. (1) **78** (1995), 213–228. MR 1328757 (96f:32026), Zbl 0846.32010.

[IS] S. Ivashkovich and V. Shevchishin, *Complex Curves in Almost-Complex Manifolds and Meromorphic Hulls*, `arXiv:math.CV/9912046`.

[Lai] H. F. Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. AMS **172** (1972), 1–33. MR 0314066 (47 #2618), Zbl 0247.32005.

[Lu] Y.-C. Lu, *Singularity Theory and an Introduction to Catastrophe Theory*, 2nd corr. printing, Universitext, Springer-Verlag, New York, 1976. MR 0461562 (57 #1547), Zbl 0354.58008.

[Maple] Maple 9.50, Waterloo Maple Inc., 2004. `www.maplesoft.com`

[Martinet] J. Martinet, *Singularities of Smooth Functions and Maps*, transl. by C. Simon, LMS Lec. Note Ser. **58**, Cambridge, 1982. MR 0671585 (83i:58018), Zbl 0522.58006.

[Moser] J. Moser, *Analytic surfaces in $\mathbb{C}^2$ and their local hull of holomorphy*, Ann. Acad. Scient. Fenn., Ser. A. I. **10** (1985), 397–410. MR 0802502 (87c:32024), Zbl 0585.32007.

[MW] J. Moser and S. Webster, *Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations*, Acta Mathematica (3–4) **150** (1983), 255–296. MR 0709143 (85c:32034), Zbl 0519.32015.

[PS] T. Poston and I. Stewart, *Catastrophe Theory and its Applications*, SRWM **2**, Pitman, London, 1978. MR 0501079 (58 #18535), Zbl 0382.58006.

[POV-Ray] POV-Ray for Windows version 3.6.1c.icl8.win32, Persistence of Vision Raytracer Pty. Ltd., 2006. `www.povray.org`

[S-G] A. Sanabria-García, *Polynomial hulls of smooth discs: a survey*, Irish Math. Soc. Bull. **45** (2000), 135–153. MR 1832326 (2002c:32019), Zbl 1065.32005.

[Slapar] M. Slapar, *Real surfaces in elliptic surfaces*, Internat. J. Math. (4) **16** (2005), 357–363. MR 2133261 (2005k:32050), Zbl 1072.32015.

[Wall] C. T. C. Wall, *Classification and stability of singularities of smooth maps*, in Singularity Theory (Trieste, 1991), 920–952, World Sci. Publishing, River Edge, NJ, 1995. MR 1378436 (97a:58016), Zbl 0991.58507.

[Webster$_1$] S. Webster, *Real submanifolds of $\mathbb{C}^n$ and their complexifications*, in Topics in Several Complex Variables, RNM **112**, Pitman, Boston, 1985, 69–79. MR 0805590 (86k:32019), Zbl 0566.32014.

[Webster$_2$] S. Webster, *The Euler and Pontrjagin numbers of an n-manifold in $\mathbb{C}^n$*, Comment. Math. Helv. **60** (1985), 193–216. MR 0800003 (86m:32034), Zbl 0566.32015.

[Webster$_3$] S. Webster, *On the relation between Chern and Pontrjagin numbers*, Contemp. Math. **49** (1986), 135–143. MR 0833810 (88a:57047), Zbl 0587.57010.

[Wells] R. O. Wells, *Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles*, Math. Ann. **179** (1969), 123–129. MR 0237823 (38 #6104), Zbl 0167.21604.

[Whitney$_1$] H. Whitney, *The general type of singularity of a set of $2n-1$ smooth functions of n variables*, Duke Math. J. **10** (1943), 161–172. MR 0007784 (4,193b), Zbl 0061.37207.

[Whitney$_2$] H. Whitney, *The singularities of a smooth n-manifold in $(2n-1)$-space*, Ann. of Math. (2) **45** (1944), 247–293. MR 0010275 (5,274a), Zbl 0063.08238.