ADDENDUM TO: PROPER HOLOMORPHIC MAPS FROM DOMAINS IN $\mathbb{C}^2$ WITH TRANSVERSE CIRCLE ACTION

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6. Updates

The article [CP] has been reviewed as noted in the list of references, below.

Our contact information has changed, and our current web pages can be found at:

`http://users.pfw.edu/CoffmanA/`
`http://users.pfw.edu/Pan/`

7. Citations

The article is cited in these papers: [CL], [J].

The following Sections of this Addendum include some details omitted from the published version of [CP]. In Section 8 we review the definition of "finite type," and check an elementary equality in a special case, based on conversations with Martino Fassina ([FP]). The remaining Sections give some details on the "general properties of actions of Lie groups" mentioned in [CP] §2, and also present some detailed notes on elementary results of point-set topology and smooth manifolds related to constructions in [CP].

8. Finite type conditions

Definition 2.1 of [CP] states, and claims the equivalence of, two definitions of the notion of "finite type" for points on the three-dimensional boundary $b\Omega = \{r(\vec{z}) = 0\}$ of the smoothly bounded domain $\Omega = \{r < 0\} \subseteq \mathbb{C}^2$. Both definitions are local near any given point $p \in b\Omega$, and do not depend on the local holomorphic coordinate system. To clarify the relation between the definitions, we consider a simple special case where a real hypersurface in $\mathbb{C}^2$ is in a <u>rigid</u> local normal form, meaning that there is a coordinate system in which the hypersurface goes through the origin and is defined by $\{r = 0\}$, where the smooth function r is of the form:

$$r(z_1, z_2) = \rho(z_1, \bar{z}_1) - \mathrm{Im}(z_2), \tag{8.1}$$

for a real valued function ρ depending on z_1 only and not on z_2, with $\rho(\vec{0}) = \frac{\partial}{\partial z}\rho(\vec{0}) = \frac{\partial}{\partial \bar{z}}\rho(\vec{0}) = 0$. Smooth hypersurfaces with a rigid normal form are considered in [CP] §4. However, not every smooth hypersurface considered by [CP] has a rigid local normal form under local biholomorphic equivalence.

Definition 8.1. Given a smooth function $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$, define the <u>order of vanishing</u> of f to be:

$$\nu(f) = \min\left\{ N : \exists b_{\alpha\beta} : \left. \sum_{\alpha+\beta\le N} b_{\alpha\beta} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta} f(z)\right]_{z=0} \neq 0 \right\}.$$

$\nu(f)$ is a positive integer unless there is no such number N — then define $\nu(f) = \infty$.

We remark that it would be equivalent to find just one lowest degree, non-zero derivative term instead of a complex linear combination, or to use x, y derivatives instead of z, $\bar{z}$, but this formulation will be convenient later and does not depend too much on the coordinate system.

Theorem 8.2. *For a smooth real hypersurface in $\mathbb{C}^2$ given by an equation in the rigid normal form* (8.1)*, the following are equivalent.*

(1) $\nu(\Lambda_r) < \infty$.
(2) $\nu(\rho_{z\bar{z}}) < \infty$.
(3) $\Delta^1_{reg} = \sup_{\gamma^2} \nu(r(z, \gamma^2(z))) < \infty$.
(4) $\Delta^1 = \sup_{(\gamma^1,\gamma^2)} \dfrac{\nu(r(\gamma^1(z), \gamma^2(z)))}{\min\{\nu(\gamma^1), \nu(\gamma^2)\}} < \infty$.

The supremum in items (3) *and* (4) *is over holomorphic functions in a neighborhood of* 0 *with $\gamma^1(0) = \gamma^2(0) = 0$ and $\gamma^1 \not\equiv 0$. If any of the four quantities is finite then this equality holds:*

$$\nu(\Lambda_r) + 2 = \nu(\rho_{z\bar{z}}) + 2 = \Delta^1_{reg} = \Delta^1.$$

Proof. Condition (3) is called "regular finite type," measuring the highest order of contact of the hypersurface with an embedded holomorphic curve, which must be tangent to the z_1 axis, and parametrized in the form $(z_1, \gamma^2(z_1))$. Condition (4) is "singular finite type," with any, possibly singular, holomorphic curve $(\gamma^1(z), \gamma^2(z))$, so $\gamma^1(z) = z$ is a special case and (4) $\implies$ (3) trivially with $\Delta^1_{reg} \le \Delta^1$.

In general (as in [CP] §2), the Levi determinant $\Lambda_r : \mathbb{C}^2 \to \mathbb{R}$ depends on two variables:

$$\Lambda_r = -\det \begin{bmatrix} 0 & r_{\bar{z}_1} & r_{\bar{z}_2} \\ r_{z_1} & r_{z_1\bar{z}_1} & r_{z_1\bar{z}_2} \\ r_{z_2} & r_{z_2\bar{z}_1} & r_{z_2\bar{z}_2} \end{bmatrix},$$

but when r is in the rigid normal form (8.1), Λ_r is proportional to $\rho_{z,\bar{z}}$. This is enough for the equivalence (1) $\iff$ (2).

We recall some calculus formulas for (real) differentiable functions

$$\begin{aligned}\vec{\gamma}:\mathbb{C}^2\to\mathbb{C}^2:(\zeta,\alpha) &\mapsto (\gamma^1(\zeta,\alpha),\gamma^2(\zeta,\alpha)),\\ \rho:\mathbb{C}^2\to\mathbb{C}:(z,w) &\mapsto \rho(z,w),\end{aligned}$$

starting with the chain rule:

$$\begin{aligned}\frac{\partial}{\partial\zeta}(\rho\circ\vec{\gamma}) &= \frac{\partial\rho}{\partial z}(\vec{\gamma}(\zeta,\alpha))\frac{\partial\gamma^1}{\partial\zeta}+\frac{\partial\rho}{\partial w}(\vec{\gamma}(\zeta,\alpha))\frac{\partial\gamma^2}{\partial\zeta}\\ \frac{\partial}{\partial\alpha}(\rho\circ\vec{\gamma}) &= \frac{\partial\rho}{\partial z}(\vec{\gamma}(\zeta,\alpha))\frac{\partial\gamma^1}{\partial\alpha}+\frac{\partial\rho}{\partial w}(\vec{\gamma}(\zeta,\alpha))\frac{\partial\gamma^2}{\partial\alpha}.\end{aligned}$$

Next, consider the special case where γ^1 is holomorphic in ζ and does not depend on α, and

$$\vec{\gamma}(\zeta,\alpha)=\left(\sum_{j=1}^{\infty}\gamma_j\zeta^j,\sum_{j=1}^{\infty}\overline{\gamma_j}\alpha^j\right),$$

so

$$\begin{aligned}\frac{\partial}{\partial\zeta}(\rho\circ\vec{\gamma}) &= \frac{\partial\rho}{\partial z}(\vec{\gamma}(\zeta,\alpha))\frac{\partial\gamma^1}{\partial\zeta}+\frac{\partial\rho}{\partial w}(\vec{\gamma}(\zeta,\alpha))\cdot 0\\ \frac{\partial}{\partial\alpha}(\rho\circ\vec{\gamma}) &= \frac{\partial\rho}{\partial z}(\vec{\gamma}(\zeta,\alpha))\cdot 0+\frac{\partial\rho}{\partial w}(\vec{\gamma}(\zeta,\alpha))\frac{\partial\gamma^2}{\partial\alpha}.\end{aligned}$$

Restricting to $(\zeta,\alpha)=(z,\bar{z})=(z,w)$ and abbreviating $\dfrac{\partial\rho}{\partial z}=\rho_z$, $\dfrac{\partial\rho}{\partial\bar{z}}=\rho_{\bar{z}}$, and $\dfrac{d}{dz}\gamma^1(z)=(\gamma^1)'(z)$,

$$\begin{aligned}\frac{\partial}{\partial z}(\rho(\gamma^1(z),\overline{\gamma^1(z)})) &= \rho_z(\gamma^1(z),\overline{\gamma^1(z)})\cdot(\gamma^1)'(z)\\ \frac{\partial}{\partial\bar{z}}(\rho(\gamma^1(z),\overline{\gamma^1(z)})) &= \rho_{\bar{z}}(\gamma^1(z),\overline{\gamma^1(z)})\cdot\overline{(\gamma^1)'(z)}.\end{aligned}\tag{8.2}$$

Using the relation $\frac{\partial}{\partial\bar{z}}=C\circ\frac{\partial}{\partial z}\circ C$, where C is conjugation on $\mathbb{C}$, if ρ happens to also be real valued, then $\rho_{\bar{z}}(z,\bar{z})=C(\frac{\partial}{\partial z}\rho(z,\bar{z}))=\overline{\rho_z(z,\bar{z})}$, and applying this to the real valued expression $\rho(\gamma^1(z),\overline{\gamma^1(z)})$ is consistent with (8.2).

For second derivatives when ρ is twice differentiable (and continuing to assume γ^1 is holomorphic), the product rule is needed:

$$\begin{aligned}\frac{\partial^2}{\partial z^2}(\rho(\gamma^1(z),\overline{\gamma^1(z)})) &= \rho_{zz}(\gamma^1,\overline{\gamma^1})\cdot((\gamma^1)')^2+\rho_z(\gamma^1,\overline{\gamma^1})\cdot(\gamma^1)''\\ \frac{\partial^2}{\partial z\partial\bar{z}}(\rho(\gamma^1(z),\overline{\gamma^1(z)})) &= \rho_{z\bar{z}}(\gamma^1,\overline{\gamma^1})\cdot(\gamma^1)'\overline{(\gamma^1)'}\\ \frac{\partial^2}{\partial\bar{z}^2}(\rho(\gamma^1(z),\overline{\gamma^1(z)})) &= \rho_{\bar{z}\bar{z}}(\gamma^1,\overline{\gamma^1})\cdot(\overline{(\gamma^1)'})^2+\rho_z(\gamma^1,\overline{\gamma^1})\cdot\overline{(\gamma^1)''}.\end{aligned}$$

For higher derivatives when ρ is smooth, a special case of the Faà di Bruno formula is needed. Abbreviating $\frac{\partial^\alpha}{\partial z^\alpha}=\partial^\alpha$, $\frac{\partial^\alpha}{\partial\bar{z}^\alpha}=\overline{\partial}^\alpha$, and $\frac{d^\alpha}{dz^\alpha}\gamma^1(z)=\gamma^{(\alpha)}$, the claim is that for nonnegative integers α, β, there exist positive constants $c_{\vec{m},\vec{n}}$ so that

$$\begin{aligned}&\partial^\alpha\overline{\partial}^\beta(\rho(\gamma^1(z),\overline{\gamma^1(z)}))\\ &= \sum_{\vec{m},\vec{n}} c_{\vec{m},\vec{n}}(\partial^{m_1+\cdots+m_\alpha}\overline{\partial}^{n_1+\cdots+n_\beta}\rho)(\gamma^1,\overline{\gamma^1})\left(\prod_{j=1}^{\alpha}(\gamma^{(j)}(z))^{m_j}\right)\left(\prod_{j=1}^{\beta}(\overline{\gamma^{(j)}(z)})^{n_j}\right),\end{aligned}$$

where the sum is over all ordered lists $\vec{m}$ and $\vec{n}$ of non-negative integers satisfying $m_1+2m_2+\cdots+\alpha m_\alpha=\alpha$ and $n_1+2n_2+\cdots+\beta n_\beta=\beta$. When $\alpha=0$, the convention is to set the empty product $\prod_{j=1}^{\alpha}=1$ and to have the only index $\vec{m}$ satisfy $m_1+\cdots+m_0=0$; similarly for $\beta=0$ and $\vec{n}$. The claimed formula is consistent with the previously stated cases $(\alpha,\beta)=(1,0)$, $(0,1)$, $(2,0)$, $(1,1)$, and $(0,2)$, so that starts an induction on either α or β. Assume $\alpha\geq\beta\geq0$ and $\alpha\geq1$ (the remaining cases $\beta>\alpha\geq0$ being similar). The next derivative, increasing α by 1, has terms with non-negative coefficients (so there are no cancellations):

$$\begin{aligned}&\partial\left(\partial^\alpha\overline{\partial}^\beta(\rho(\gamma^1(z),\overline{\gamma^1(z)}))\right)\\ =&\left(\sum_{\vec{m},\vec{n}} c_{\vec{m},\vec{n}}(\partial^{1+m_1+\cdots+m_\alpha}\overline{\partial}^{n_1+\cdots+n_\beta}\rho)(\gamma^1,\overline{\gamma^1})\cdot\right.\\ &\left.\left(\gamma^{(1)}(z)\right)^{m_1+1}\left(\prod_{j=2}^{\alpha}(\gamma^{(j)}(z))^{m_j}\right)\left(\prod_{j=1}^{\beta}(\overline{\gamma^{(j)}(z)})^{n_j}\right)\right)\end{aligned}\tag{8.3}$$

$$\begin{aligned}&+\left(\sum_{\vec{m},\vec{n}} c_{\vec{m},\vec{n}}(\partial^{m_1+\cdots+m_\alpha}\overline{\partial}^{n_1+\cdots+n_\beta}\rho)(\gamma^1,\overline{\gamma^1})\cdot\right.\\ &\left(\sum_{k=1}^{\alpha}\left(\prod_{j\neq k}(\gamma^{(j)}(z))^{m_j}\right)m_k\left(\gamma^{(k)}(z)\right)^{m_k-1}\gamma^{(k+1)}(z)\right)\cdot\\ &\left.\left(\prod_{j=1}^{\beta}(\overline{\gamma^{(j)}(z)})^{n_j}\right)\right).\end{aligned}\tag{8.4}$$

To prove the claim, we need to show that for any list $\vec{m}'$ satisfying $1m_1'+2m_2'+\cdots+\alpha m_\alpha'+(\alpha+1)m_{\alpha+1}'=\alpha+1$, there is some corresponding term in the above expression with a positive coefficient. The indices m_k' can't all be zero, so there are two cases. If $m_1'>0$ then $m_{\alpha+1}'=0$; let $m_1=m_1'-1$ and $m_k=m_k'$ for $1<k\leq\alpha$. Then $1m_1+2m_2+\cdots+\alpha m_\alpha=1(m_1'-1)+2m_2'+\cdots+\alpha m_\alpha'+(\alpha+1)\cdot0=(\alpha+1)-1=\alpha$. The coefficient $c_{\vec{m}',\vec{n}}$ corresponds to a term in (8.3), so it is positive. If $m_{k+1}'>0$ for some $1\leq k\leq\alpha$, let $m_k=m_k'+1$, $m_{k+1}=m_{k+1}'-1$ (which $=0$ for $k=\alpha$), and $m_j = m'_j$ for all other j. Then $m_1 + \cdots + m_\alpha = m'_1 + \cdots + m'_{\alpha+1}$ and $1m_1 + 2m_2 + \cdots + \alpha m_\alpha = 1m'_1 + \cdots + k(m'_k + 1) + (k+1)(m'_{k+1} - 1) + \cdots = (\alpha + 1) + k - (k+1) = \alpha$. The coefficient $c_{\vec{m}',\vec{n}}$ corresponds to a term in (8.4) with $m_k > 0$, so it is positive.

Returning to the smooth, real valued functions r and ρ as in (8.1), consider a holomorphic function $\gamma^2(z) = \sum_{k=1}^{\infty} \gamma_k^2 z^k$, the composite $r(z, \gamma^2(z))$ as in (3), and its degree N_1 Taylor polynomial:

$$
\begin{aligned}
(8.5)\qquad & \sum_{\alpha+\beta\le N_1} \frac{1}{\alpha!\beta!} \frac{\partial^\alpha}{\partial z^\alpha}\frac{\partial^\beta}{\partial \bar{z}^\beta} \left(\rho(z,\bar{z}) - \mathrm{Im}(\gamma^2(z))\right)\Big]_{z=0} z^\alpha \bar{z}^\beta \\
= & \left(\sum_{\alpha=2}^{N_1} \frac{1}{\alpha!}\partial^\alpha \rho(0) z^\alpha\right) + \frac{1}{2i}\left(\sum_{\alpha=1}^{N_1} \gamma_\alpha^2 z^\alpha\right) \\
& + \left(\sum_{\alpha+\beta\le N_1, \alpha\neq 0, \beta\neq 0} \frac{1}{\alpha!\beta!}\partial^{\alpha-1}\overline{\partial}^{\beta-1}\rho_{z\bar{z}}(0) z^\alpha \bar{z}^\beta\right) \\
& + \left(\sum_{\alpha=2}^{N_1} \frac{1}{\alpha!}\overline{\partial}^\alpha \rho(0) \bar{z}^\alpha\right) - \frac{1}{2i}\left(\sum_{\alpha=1}^{N_1} \overline{\gamma_\alpha^2} \bar{z}^\alpha\right).
\end{aligned}
$$

Assuming (2), let $N_2 = \nu(\rho_{z\bar{z}}) \ge 0$, so there exist coefficients $b_{\alpha\beta}$ with

$$\sum_{\alpha+\beta\le N_2} b_{\alpha\beta}\partial^\alpha \overline{\partial}^\beta \rho_{z\bar{z}}(0) \neq 0.$$

Define coefficients $b'_{\alpha\beta} = 0$ for $\alpha = 0$ or $\beta = 0$ and $b'_{\alpha\beta} = b_{\alpha-1,\beta-1}$ for $\alpha > 0$ and $\beta > 0$. Then, for any γ^2, referring to (8.5) shows:

$$
\begin{aligned}
& \sum_{\alpha'+\beta'\le N_2+2} b'_{\alpha'\beta'}\partial^{\alpha'}\overline{\partial}^{\beta'} \left(r(z,\gamma^2(z))\right)\big]_{z=0} \\
= & \sum_{\alpha'+\beta'\le N_2+2, \alpha'\neq 0, \beta'\neq 0} b'_{\alpha'\beta'}\partial^{\alpha'-1}\overline{\partial}^{\beta'-1}\rho_{z\bar{z}}(0) \\
= & \sum_{\alpha+\beta\le N_2} b'_{\alpha+1,\beta+1}\partial^\alpha\overline{\partial}^\beta \rho_{z\bar{z}}(0) \neq 0.
\end{aligned}
$$

This shows (2) $\implies$ (3), with $\Delta^1_{reg} \le \nu(\rho_{z\bar{z}}) + 2$.

Now assuming (3), ν being integer valued means that the supremum N_3 is attained by some particular function γ^2. Considering (8.5) and the assumption that $\rho(0) = \rho_z(0) = \rho_{\bar{z}}(0) = 0$, if $N_3 = \nu(r(z,\gamma^2(z))) = 1$ then γ^2 would have a non-zero linear coefficient γ_1^2, but then it could be replaced by $\gamma^2 - \gamma_1^2 z$ to get $\nu \ge 2$, contradicting N_3 being the supremum over choices of γ^2. So, $N_3 \ge 2$ and we can assume $\gamma_1^2 = 0$. Considering (8.5) again, by definition of $N_3 = \nu(r(z, \gamma^2(z)))$, there are coefficients $b_{\alpha\beta}$ so that

$$
\begin{aligned}
0 \neq & \left(\sum_{\alpha=2}^{N_3} b_{\alpha 0}\left(\partial^\alpha \rho(0) + \frac{\alpha!}{2i}\gamma^2_\alpha\right)\right) \\
& + \left(\sum_{\alpha+\beta\le N_3, \alpha\neq 0, \beta\neq 0} b_{\alpha\beta}\partial^{\alpha-1}\overline{\partial}^{\beta-1}\rho_{z\bar{z}}(0)\right) \\
& + \left(\sum_{\alpha=2}^{N_3} b_{0\alpha}\left(\overline{\partial}^\alpha \rho(0) - \frac{\alpha!}{2i}\overline{\gamma^2_\alpha}\right)\right).
\end{aligned} \tag{8.6}
$$

If we suppose, toward a contradiction, that for any choice of coefficients $b'_{\alpha'\beta'}$,

$$\sum_{\alpha'+\beta'\le N_3-2} b'_{\alpha'\beta'}\partial^{\alpha'}\overline{\partial}^{\beta'}\rho_{z\bar{z}}(0) = 0,$$

then the middle term in (8.6) is 0. Consider the polynomial

$$\gamma^3(z) = \gamma^3_{N_3+1}z^{N_3+1} + \sum_{\alpha=2}^{N_3}\left(-\frac{2i}{\alpha!}\partial^\alpha\rho(0)\right)z^\alpha.$$

Replacing γ^2 by γ^3 in (8.6) changes the total sum to 0 for any choice of $b_{\alpha\beta}$ (recalling that for real ρ, $\overline{\partial}^\alpha\rho(0) = \overline{\partial^\alpha\rho(0)}$). Then the highest degree coefficient can be chosen, $\gamma^3_{N_3+1} \neq -\frac{2i}{(N_3+1)!}\partial^{N_3+1}\rho(0)$, so that $\nu(r(z,\gamma^3)) = N_3+1$, contradicting N_3 being the supremum over choices of γ^2. The conclusion is that $\nu(\rho_{z\bar{z}}) \le N_3 - 2$. This shows (3) $\implies$ (2), and that if either number is finite, then $\Delta^1_{reg} = \nu(\rho_{z\bar{z}}) + 2$.

More generally, consider holomorphic components:

$$
\begin{aligned}
\gamma^1(z) &= \sum_{k=1}^{\infty}\gamma^1_k z^k \not\equiv 0, \\
\gamma^2(z) &= \sum_{k=1}^{\infty}\gamma^2_k z^k,
\end{aligned}
$$

the composite $r(\gamma^1(z), \gamma^2(z))$ as in (4), and its degree N_1 Taylor polynomial using the previously proved FdB formula:

$$\sum_{\alpha+\beta\le N_1} \frac{1}{\alpha!\beta!} \frac{\partial^\alpha}{\partial z^\alpha} \frac{\partial^\beta}{\partial \bar{z}^\beta} \left(\rho(\gamma^1(z), \overline{\gamma^1(z)}) - \mathrm{Im}(\gamma^2(z))\right)\Big]_{z=0} z^\alpha \bar{z}^\beta \tag{8.7}$$

$$= \left(\sum_{\alpha=2}^{N_1} \frac{1}{\alpha!} \left(\sum_{\vec{m}} c_{\vec{m},\vec{0}} (\partial^{m_1+\cdots+m_\alpha}\rho)(0) \left(\prod_{j=1}^{\alpha} (\gamma^1_j j!)^{m_j}\right)\right) z^\alpha\right)$$

$$+\frac{1}{2i}\left(\sum_{\alpha=1}^{N_1} \gamma^2_\alpha z^\alpha\right)$$

$$+\left(\sum_{\alpha+\beta\le N_1, \alpha\neq 0, \beta\neq 0} \frac{1}{\alpha!\beta!} \left(\sum_{\vec{m},\vec{n}} c_{\vec{m},\vec{n}} \left(\partial^{m_1+\cdots+m_\alpha}\overline{\partial}^{n_1+\cdots+n_\beta}\rho\right)(0)\cdot\right.\right.$$

$$\left.\left.\left(\prod_{j=1}^{\alpha}(\gamma^1_j j!)^{m_j}\right)\left(\prod_{j=1}^{\beta}(\overline{\gamma^1_j j!})^{n_j}\right)\right) z^\alpha \bar{z}^\beta\right) \tag{8.8}$$

$$+\left(\sum_{\beta=2}^{N_1} \frac{1}{\beta!} \left(\sum_{\vec{n}} c_{\vec{0},\vec{n}} \left(\overline{\partial}^{n_1+\cdots+n_\beta}\rho\right)(0) \left(\prod_{j=1}^{\beta}(\overline{\gamma^1_j j!})^{n_j}\right)\right) \bar{z}^\beta\right)$$

$$-\frac{1}{2i}\left(\sum_{\beta=1}^{N_1} \overline{\gamma^2_\beta} \bar{z}^\beta\right).$$

There is some $q > 0$ so that $\gamma^1_k = 0$ for $0 \le k < q$ and $\gamma^1_q \neq 0$, so $\nu(\gamma^1) = q$. Assuming (2), let $N_2 = \nu(\rho_{z\bar{z}}) \ge 0$, so there exist coefficients $b_{\alpha\beta}$ with

$$\sum_{\alpha+\beta\le N_2} b_{\alpha\beta}\partial^\alpha\overline{\partial}^\beta\rho_{z\bar{z}}(0) \neq 0. \tag{8.9}$$

By the minimality of N_2,

$$\alpha+\beta < N_2 \implies \partial^\alpha\overline{\partial}^\beta\rho_{z\bar{z}}(0) = 0. \tag{8.10}$$

For indices $\alpha' \ge 0$, $\beta' \ge 0$ with $\alpha'+\beta' \le q(N_2+2)$, define $b'_{\alpha'\beta'} = 0$ for $\alpha' = 0$ or $\beta' = 0$. To show (2) $\implies$ (4) by estimating the order of vanishing of $r(\gamma^1(z), \gamma^2(z))$, use these $b'_{\alpha'\beta'}$ coefficients, the rest of which will be specified later, and the mixed derivative terms from (8.8):

$$\sum_{\alpha'+\beta'\le q(N_2+2)} b'_{\alpha'\beta'}\partial^{\alpha'}\overline{\partial}^{\beta'}\left(r(\gamma^1(z),\gamma^2(z))\right)\Big]_{z=0}$$

$$= \sum_{\alpha'+\beta'\le q(N_2+2)} b'_{\alpha'\beta'}\left(\sum_{\vec{m},\vec{n}} c^{\alpha'\beta'}_{\vec{m},\vec{n}}(\partial^{m_1+\dots+m_{\alpha'}}\overline{\partial}^{n_1+\dots+n_{\beta'}}\rho)(0)\cdot\right.$$

$$\left.\left(\prod_{j=1}^{\alpha'}(\gamma^1_j j!)^{m_j}\right)\left(\prod_{j=1}^{\beta'}(\overline{\gamma^1_j j!})^{n_j}\right)\right). \tag{8.11}$$

The coefficients in the inside sum are labeled to emphasize the dependence on $\alpha'>0$ and $\beta'>0$. $\vec{m}=(m_1,\dots,m_{\alpha'})$ satisfies $m_1+2m_2+\dots+\alpha' m_{\alpha'}=\alpha'$ and similarly $n_1+2n_2+\dots+\beta' n_{\beta'}=\beta'$. Any term in (8.11) with $m_j>0$ or $n_j>0$ for any $j=1,\dots,q-1$ will be zero because $\gamma^1_j=0$. So the only candidates for non-zero terms are of the form $\vec{m}=(0,\dots,0,m_q,\dots,m_{\alpha'})$ for $\alpha'\ge q$, and similarly for $\vec{n}$. If there is some $m_j>0$ or $n_j>0$ for $j\ge q+1$, then there is a strict inequality:

$$\begin{aligned} & m_1+m_2+\dots+m_{\alpha'}+n_1+n_2+\dots+n_{\beta'} \\ < \ & \frac{1}{q}(qm_q+(q+1)m_{q+1}+\dots+\alpha' m_{\alpha'}+qn_q+(q+1)n_{q+1}+\dots+\beta' n_{\beta'}) \\ = \ & \frac{1}{q}(\alpha'+\beta')\le\frac{1}{q}\cdot q(N_2+2)=N_2+2. \end{aligned}$$

As noted in (8.10), terms in (8.11) with such indices will also be zero because there are not enough derivatives of ρ. So, for each α', β', there is at most one possible non-zero term remaining in (8.11), with $\vec{m}=(0,\dots,0,m_q,0,\dots,0)$ and $\vec{n}=(0,\dots,0,n_q,0,\dots,0)$, satisfying $qm_q=\alpha'$, and $qn_q=\beta'$. The sum (8.11) can be simplified to:

$$\sum_{\substack{\alpha'+\beta'\le q(N_2+2),\\ \alpha'=qm_q,\\ \beta'=qn_q}} b'_{\alpha'\beta'}c^{\alpha'\beta'}_{(0,\dots,m_q,\dots 0),(0,\dots,n_q,\dots,0)}(\partial^{m_q}\overline{\partial}^{n_q}\rho)(0)(\gamma^1_q q!)^{m_q}(\overline{\gamma^1_q q!})^{n_q}. \tag{8.12}$$

Now we can choose, for indices $\alpha'+\beta'\le q(N_2+2)$,

$$b'_{\alpha'\beta'}=\frac{b_{\frac{\alpha'}{q}-1,\frac{\beta'}{q}-1}}{c^{\alpha'\beta'}_{(0,\dots,\frac{\alpha'}{q},\dots 0),(0,\dots,\frac{\beta'}{q},\dots,0)}(\gamma^1_q q!)^{\frac{\alpha'}{q}}(\overline{\gamma^1_q q!})^{\frac{\beta'}{q}}},$$

when α' and β' are positive multiples of q, and $b_{\alpha'\beta'}=0$ otherwise. This is the key step, using the previously checked property that the FdB coefficients c are non-zero. Then, re-indexing the sum (8.12) with $\alpha'=q(\alpha+1)=qm_q$ and $\beta'=q(\beta+1)=qn_q$ gives exactly the sum (8.9), which is non-zero by hypothesis. The conclusion so far is that for any γ^2, $\nu(r(\gamma^1(z),\gamma^2(z))) \leq q(N_2+2)$.

Finally, for a fixed γ^1, if γ^2 satisfies $\nu(\gamma^2) < \nu(\gamma^1) = q$, then inspecting the Taylor polynomial (8.8) shows that $\nu(r(\gamma^1(z),\gamma^2(z))) = \nu(\gamma^2)$ — the only non-zero terms are from γ^2 because all the derivatives of γ^1 in degree $< q$ are zero. There are two cases: if $\nu(\gamma^2) < \nu(\gamma^1)$, then $\frac{\nu(r(\gamma^1(z),\gamma^2(z)))}{\min\{\nu(\gamma^1),\nu(\gamma^2)\}} = 1$, and otherwise $\frac{\nu(r(\gamma^1(z),\gamma^2(z)))}{\min\{\nu(\gamma^1),\nu(\gamma^2)\}} \leq \frac{q(N_2+2)}{q} = N_2+2$. This upper bound does not depend on γ^1 so we can conclude that

$$\Delta^1 = \sup_{(\gamma^1,\gamma^2)} \frac{\nu(r(\gamma^1(z),\gamma^2(z)))}{\min\{\nu(\gamma^1),\nu(\gamma^2)\}} \leq N_2+2 = \Delta^1_{reg}.$$

∎

9. Definition of T-action

Let Ω be a bounded open subset of $\mathbb{C}^2$. Denote its boundary by $b\Omega$, so that the closure of Ω in $\mathbb{C}^2$ is $\overline{\Omega} = \Omega \cup b\Omega$. Assume $b\Omega$ is a smoothly embedded real submanifold of $\mathbb{C}^2$. Let $Aut(\Omega)$ denote the set of holomorphic automorphisms of Ω, with identity element e.

Proposition 9.1. *$Aut(\Omega)$ has a Lie group structure so that the evaluation map*

$$A : Aut(\Omega) \times \Omega \to \Omega : (g,z) \mapsto g(z)$$

is real analytic.

Remark. This result of H. Cartan is proved in [N]. ∎

Let S^1 denote the unit circle with its standard Lie group structure, and let

$$\phi : S^1 \to Aut(\Omega) : t \mapsto \phi_t$$

be a homomorphism: $\phi_{st} = \phi_s \circ \phi_t$.

Theorem 9.2. *If ϕ is continuous, then ϕ is real analytic.*

Proof. This follows from [P] Theorem 2.3.1. ∎

Denote the image of ϕ by $\phi(S^1) = \mathbf{T} \subseteq Aut(\Omega)$.

Theorem 9.3. *If ϕ is continuous, then $\mathbf{T}$, with the induced subspace topology, is a Lie group.*

Proof. It follows from the continuity of ϕ, the compactness of S^1, and the Hausdorff property of $Aut(\Omega)$ that $\mathbf{T} = \phi(S^1)$ is a closed subset of $Aut(\Omega)$ ([M] Theorems 26.5, 26.3). By elementary group theory, the image of any homomorphism is a subgroup. Since $\mathbf{T}$ is a closed subgroup of the Lie group $Aut(\Omega)$, it follows from [P] Theorem 3.3.1 (or [W] Theorem 3.42) that $\mathbf{T}$ has a unique real analytic (or smooth) manifold structure so that its inclusion, as a map from $\mathbf{T}$ with this manifold structure to $\mathbf{T}$ with its induced subspace topology from $Aut(\Omega)$, is a homeomorphism — $\mathbf{T}$ is a "regular submanifold" in the sense of [20] §III.5, III.6. ∎

Theorem 9.4. *If ϕ is continuous then either ϕ is constant, so $\mathbf{T}$ is the trivial group $\{e\}$, or ϕ is an immersion, and $\mathbf{T}$ is isomorphic as a Lie group to S^1.*

Proof. If ϕ is not a constant map, then, since S^1 is connected, and ϕ is smooth by Theorem 9.2, there must be some point $t \in S^1$ so that $d\phi : T_tS^1 \to T_{\phi_t}Aut(\Omega)$ is not the zero map ([W] Theorem 1.24), so it has rank 1, the maximum possible. By [20] Theorem III.6.14, the rank of $d\phi$ is constant on S^1. This means ϕ is an immersion.

$\mathbf{T}$ is an example of a "one-parameter subgroup" of $Aut(\Omega)$ ([20] Def. IV.5.9, [P] Def. 2.2.2, [W] Def. 3.29), in the sense that it is the image of a smooth homomorphism $\phi \circ c : \mathbb{R}^1 \to Aut(\Omega)$, where $c : \mathbb{R}^1 \to S^1$ is the immersion $x \to e^{2\pi ix}$. The statement that $\mathbf{T}$ is isomorphic as a Lie group (that is, both diffeomorphic and algebraically isomorphic) to S^1 follows from [20] Exercise IV.5.6. ∎

Theorem 9.5. *If ϕ is continuous and one-to-one then $\phi : S^1 \to \mathbf{T}$ is an isomorphism of Lie groups.*

Proof. If ϕ is one-to-one, then it is not constant, so by the previous Theorem, ϕ is a one-to-one immersion. Using the compactness of S^1, it follows either from [W] Theorem 3.21 or [20] Theorem III.5.7 that ϕ is an embedding into $Aut(\Omega)$, meaning $\phi : S^1 \to \mathbf{T}$ is a homeomorphism, and an embedding is a diffeomorphism onto its image. ∎

Given a continuous homomorphism ϕ, the evaluation map A from Proposition 9.1 restricts to a real analytic map which is onto and continuous with respect to the subspace topology on $\mathbf{T} \times \Omega \subseteq Aut(\Omega) \times \Omega \subseteq Aut(\Omega) \times \mathbb{C}^2$.

$$A_\phi : \mathbf{T} \times \Omega \to \Omega : (\phi_t, z) \mapsto \phi_t(z), \tag{9.1}$$

called a <u>$\mathbf{T}$-action on Ω</u>. The $\mathbf{T}$-action is <u>trivial</u> means: ϕ is the constant map to the identity in $Aut(\Omega)$ (the first case from Theorem 9.4).

10. Continuous extension

Theorem 10.1. *For any subspace D of any topological space X, with closure $\overline{D}$ in X, and any topological space Y, if $g : D \to Y$ is continuous and $h : \overline{D} \to Y$ is continuous with $g(x) = h(x)$ for all $x \in D$, then the image $h(\overline{D})$ is contained in $\overline{g(D)}$, the closure of the image $g(D)$ in Y. If, further, Y is Hausdorff and $h' : \overline{D} \to Y$ is also continuous with $g(x) = h'(x)$ for all $x \in D$, then $h = h'$.*

Proof. The closure of D in $\overline{D}$ is $\overline{D}$ ([M] Theorem 17.4), and the image of the closure of D in $\overline{D}$ is contained in the closure of the image $g(D) = h(D)$ in Y, by the continuity of h ([M] Theorem 18.1). The equality $h = h'$ is ([M] Ex. 18.13). ∎

Theorem 10.2. *Let X and Y be topological spaces with X Hausdorff, and let Ω be an open set in X. Suppose $g : \overline{\Omega} \to Y$ is continuous and the restriction $g|_\Omega : \Omega \to g(\Omega)$ is a homeomorphism, with $g(\Omega)$ open in Y. If $p \in b\Omega$ then $g(p) \in b(g(\Omega))$.*

Proof. By [M] Ex. 17.19.c., $b\Omega = \overline{\Omega} \setminus \Omega$, and $b(g(\Omega)) = \overline{g(\Omega)} \setminus g(\Omega)$. By the previous Theorem, $g(\overline{\Omega}) \subseteq \overline{g(\Omega)}$, so it remains only to show that if $p \in \overline{\Omega} \setminus \Omega$, then $g(p) \notin g(\Omega)$.

Suppose, toward a contradiction, that $g(p) \in g(\Omega)$, so $g(p) = g(x)$ for some $x \in \Omega$. Since X is Hausdorff, there exist open sets U, W in X so that $x \in U$, $p \in W$, and $U \cap W = \emptyset$. Let $U_1 = U \cap \Omega$, so $x \in U_1$, U_1 is open in Ω, and $g(U_1) = (g|_\Omega)(U_1)$ is open in $g(\Omega)$ (because $g|_\Omega$ is a homeomorphism). Since $g(\Omega)$ is open in Y, $g(U_1)$ is open in Y ([M] Lemma 16.2), so $g^{-1}(g(U_1))$ is open in $\overline{\Omega}$. Since $g(p) = g(x) \in g(U_1)$, $p \in g^{-1}(g(U_1))$, and $W \cap \overline{\Omega}$ is open in $\overline{\Omega}$, so $p \in (W \cap \overline{\Omega}) \cap g^{-1}(g(U_1))$, which is an open set in $\overline{\Omega}$, equal to $V \cap \overline{\Omega}$ for some open set V in Y. Since $p \in V \cap \overline{\Omega}$, there is some point v in $V \cap \Omega$ ([M] Theorem 17.5.a.).

$$\begin{aligned} (g|_\Omega)(v) &= g(v) \in g(W \cap \overline{\Omega} \cap g^{-1}(g(U_1))) \\ &\subseteq g(g^{-1}(g(U_1))) \subseteq g(U_1) = (g|_\Omega)(U_1), \end{aligned}$$

so $v \in U_1 \subseteq U$ and $v \in V \cap \Omega \subseteq V \cap \overline{\Omega} = (W \cap \overline{\Omega}) \cap g^{-1}(g(U_1)) \subseteq W$, contradicting $U \cap W = \emptyset$. ∎

Definition 10.3. For $\phi, \Omega \subseteq \mathbb{C}^2$ as in Section 9, the $\mathbf{T}$-action A_ϕ from (9.1) <u>extends continuously</u> to $\overline{\Omega}$ means: there is a continuous map $A_\phi^E : \mathbf{T} \times \overline{\Omega} \to \mathbb{C}^2$ so that $A_\phi^E(\phi_t, z) = A_\phi(\phi_t, z)$ for all $(\phi_t, z) \in \mathbf{T} \times \Omega$.

Corollary 10.4. *If A_ϕ extends continuously to $\overline{\Omega}$, then $A_\phi^E(\mathbf{T} \times \overline{\Omega}) \subseteq \overline{\Omega}$ and A_ϕ^E is unique and takes boundary points to boundary points: for any $\phi_t \in \mathbf{T}$, if $p \in b\Omega$, then $A_\phi^E(\phi_t, p) \in b\Omega$. The restriction to the boundary is continuous.*

Proof. Theorem 10.1 applies with $D = \mathbf{T} \times \Omega$, $X = \mathbf{T} \times \mathbb{C}^2$, $Y = \mathbb{C}^2$. The closure of $\mathbf{T} \times \Omega$ in $\mathbf{T} \times \mathbb{C}^2$ is $\overline{D} = \mathbf{T} \times \overline{\Omega}$ ([M] Ex. 17.9), so the conclusion from Theorem 10.1 that the image of the extension A_ϕ^E is contained in $\overline{\Omega}$ and A_ϕ^E is uniquely defined by ϕ. Given $\phi_t \in \mathbf{T}$, the composite map

$$\overline{\Omega} \to \mathbf{T} \times \overline{\Omega} \to \mathbb{C}^2 : z \mapsto (\phi_t, z) \mapsto A_\phi^E(\phi_t, z)$$

satisfies the hypothesis for g in Theorem 10.2, since it is a composite of continuous maps and its restriction to Ω is the automorphism $z \mapsto A_\phi^E(\phi_t, z) = A_\phi(\phi_t, z) = \phi_t(z) \in \Omega$. The conclusion from Theorem 10.2 is that if $z \in b\Omega$, then $A_\phi^E(\phi_t, z) \in b\Omega$. The restriction of A_ϕ^E to the boundary can be denoted

$$A_\phi^b : \mathbf{T} \times (b\Omega) \to b\Omega \tag{10.1}$$

and it is continuous by ([M] Theorem 18.2). ∎

Corollary 10.5. *If A_ϕ extends continuously to $\overline{\Omega}$, then A_ϕ^E defines a group action on $\overline{\Omega}$.*

Proof. First, the identity automorphism e extends to the identity map on $\overline{\Omega}$ by the uniqueness from Theorem 10.1: for any $z \in \overline{\Omega}$, $A_\phi^E(e, z) = z$. The main claim is that for ϕ_t, $\phi_s \in \mathbf{T}$,

$$A_\phi^E(\phi_t, A_\phi^E(\phi_s, z)) = A_\phi^E(\phi_t \circ \phi_s, z).$$

Equivalently, the following diagram is commutative, where m is the continuous group operation $\mathbf{T} \times \mathbf{T} \to \mathbf{T}$.

$$\begin{array}{ccc} \mathbf{T} \times \mathbf{T} \times \overline{\Omega} & \xrightarrow{m \times Id_{\overline{\Omega}}} & \mathbf{T} \times \overline{\Omega} \\ \downarrow{\scriptstyle Id_{\mathbf{T}} \times A_\phi^E} & & \downarrow{\scriptstyle A_\phi^E} \\ \mathbf{T} \times \overline{\Omega} & \xrightarrow{A_\phi^E} & \overline{\Omega} \end{array}$$

The uniqueness from Theorem 10.1 applies in the same way as in the Proof of Corollary 10.4 — the closure of $\mathbf{T} \times \mathbf{T} \times \Omega$ in $\mathbf{T} \times \mathbf{T} \times \mathbb{C}^2$ is $\mathbf{T} \times \mathbf{T} \times \overline{\Omega}$ and the two paths in the diagram are both continuous extensions of the equal maps on the interior, so they are equal on the closure. It follows that each automorphism ϕ_t of Ω extends to a homeomorphism of $\overline{\Omega}$, where $z \mapsto A_\phi^E(\phi_t, z)$ has continuous inverse $z \mapsto A_\phi^E(\phi_t^{-1}, z)$. ∎

Similarly, the restriction to the boundary, A_ϕ^b from (10.1), defines a group action on $b\Omega$. For a particular $\phi_t \in \mathbf{T}$, the composite

$$b\Omega \to \mathbf{T} \times (b\Omega) \to b\Omega : z \mapsto (\phi_t, z) \mapsto A_\phi^b(\phi_t, z) = A_\phi^E(\phi_t, z) \tag{10.2}$$

is continuous and can be denoted $\phi_t^b : b\Omega \to b\Omega$. It is a homeomorphism because it has a continuous inverse:

$$(\phi_t^b \circ \phi_{t^{-1}}^b)(z_0) = A_\phi^b(\phi_t, A_\phi^e(\phi_{t^{-1}}, z_0)) = A_\phi^b(\phi_t \circ \phi_{t^{-1}}, z_0) = z_0. \tag{10.3}$$

Similarly, for each $z_0 \in b\Omega$, the composite

$$\mathbf{T} \to \mathbf{T} \times (b\Omega) \to b\Omega : \phi_t \mapsto (\phi_t, z_0) \mapsto A_\phi^b(\phi_t, z_0) = A_\phi^E(\phi_t, z_0) \tag{10.4}$$

is continuous and can be denoted $\psi_{z_0} : \mathbf{T} \to b\Omega$. The image of ψ_{z_0} is the orbit of z_0 in $b\Omega$.

11. Smooth extension

Remark 11.1. The notion of "smooth up to the boundary" is well-known in analysis for subsets of $\mathbb{R}^n$. In this Section, we go into some detail about the global existence and construction of smooth extensions of functions on open subsets of smooth manifolds. This generalization is needed in [CP] to describe smooth extensions of group actions.

Recall that a smooth manifold M is covered by an atlas of charts: open sets $U_j \subseteq M$ and continuous, one-to-one maps $\varphi_j : U_j \to \mathbb{R}^n$ so that $\varphi_j(U_j)$ is open in $\mathbb{R}^n$, $\varphi_j : U_j \to \varphi_j(U_j)$ is a homeomorphism, and $\varphi_k \circ \varphi_j^{-1} : \varphi_j(U_j) \to \varphi_k(U_k)$ is smooth on the open subset $\varphi_j(U_j \cap U_k) \subseteq \varphi_j(U_j)$ where it is defined — all derivatives are continuous with respect to coordinates on $\varphi_j(U_j) \subseteq \mathbb{R}^n$.

Definition 11.2. For an open subset $D \subseteq M$, a function $f : D \to \mathbb{R}^m$ is smooth on D means that for each point $p \in D$, there is some chart φ_j around p so that $f \circ \varphi_j^{-1} : \varphi_j(U_j) \to \mathbb{R}^m$ is smooth on some neighborhood of $\varphi_j(p)$ in $\varphi_j(U_j) \subseteq \mathbb{R}^n$.

Note that if φ_k is some other chart around p, then $f \circ \varphi_k^{-1} : \varphi_k(U_k) \to \mathbb{R}^m$ is equal to a composite of smooth functions, $f \circ \varphi_j^{-1} \circ \varphi_j \circ \varphi_k^{-1}$, on some neighborhood of $\varphi_k(p)$. So (by construction) smoothness does not depend on the coordinate chart. Although smoothness of f is a local property, the following Lemma checks that on any particular chart, the derivatives can be calculated everywhere in the intersection of that chart with the domain of f, with respect to that chart's one coordinate system.

Lemma 11.3. *For a smooth manifold M and an open subset $D \subseteq M$, suppose p is a point in $\overline{D}$, the closure of D in M, and φ_j is a chart around p. If $f : D \to \mathbb{R}^m$ is smooth on D then any derivative $\partial/\partial x_1^{a_1} \dots \partial x_n^{a_n}$ of $f \circ \varphi_j^{-1}$ exists at every point of the non-empty open set $\varphi_j(D \cap U_j) \subseteq \varphi(U_j) \subseteq \mathbb{R}^n$.*

Proof. The open set U_j has a non-empty intersection with D ([M] Theorem 17.5), so $\varphi_j(D \cap U_j)$ is a non-empty open set in $\varphi_j(U_j)$. Let $\vec{x} \in \varphi_j(D \cap U_j)$, so that $\varphi_j^{-1}(\vec{x}) \in D$. Then by Definition 11.2, there is some chart φ_k around $\varphi_j^{-1}(\vec{x})$ so that $f \circ \varphi_k^{-1}$ is smooth on some neighborhood of $\varphi_k(\varphi_j^{-1}(\vec{x}))$. $f \circ \varphi_j^{-1} = f \circ \varphi_k^{-1} \circ \varphi_k \circ \varphi_j^{-1}$ is then the composite of smooth functions on some neighborhood of $\vec{x}$, so it is smooth and the derivative in the $x_1, \dots, x_n$ coordinates is well-defined, not depending on k. ∎

The next Lemma 11.4, used as a technical step in Lemma 11.5 and Theorem 11.7, does not refer to smoothness, it only checks that the topological closure and boundary of a domain D in M are well-defined when viewed in any local coordinate neighborhood. The subsequent Lemma 11.5 shows that if a smooth function f on D and all its derivatives continuously extend locally to the boundary in one coordinate chart, then in any overlapping chart, f and the derivatives in the new coordinates also extend to the boundary.

Lemma 11.4. *For a topological manifold M, an open subset $D \subseteq M$, and any chart $\varphi_j : U_j \to \mathbb{R}^n$, the closure of $\varphi_j(D \cap U_j)$ in $\varphi_j(U_j)$ is equal to $\varphi_j(\overline{D} \cap U_j)$.*

Proof. Temporarily denote D_c the closure of $\varphi_j(D \cap U_j)$ in $\varphi_j(U_j)$. The intersection $\overline{D} \cap U_j$ is closed in U_j (in the subspace topology), and $\varphi_j : U_j \to \varphi_j(U_j)$ is a homeomorphism, so $\varphi_j(\overline{D} \cap U_j)$ is closed in $\varphi_j(U_j)$ and contains $\varphi_j(D \cap U_j)$ — this shows $D_c \subseteq \varphi_j(\overline{D} \cap U_j)$. By ([M] Theorem 17.4), the closure of $D \cap U_j$ in U_j is $\overline{D} \cap U_j$, and by ([M] Theorem 18.1) and the continuity of φ_j, $\varphi_j(\overline{D} \cap U_j) \subseteq D_c$. ∎

Lemma 11.5. *For a smooth manifold M, an open subset $D \subseteq M$, a smooth function $f : D \to \mathbb{R}^m$, and a point $p \in \overline{D}$, the following are equivalent.*

1. *There is some chart around p, $\varphi_j : U_j \to \mathbb{R}^n$, so that every derivative $\partial/\partial x_1^{a_1} \ldots \partial x_n^{a_n}$ of $f \circ \varphi_j^{-1}$ extends continuously to $\varphi_j(\overline{D} \cap U_j)$.*
2. *For any chart around p, $\varphi_k : U_k \to \mathbb{R}^n$, there is some neighborhood $p \in V \subseteq U_k$ so that every derivative $\partial/\partial \tilde{x}_1^{a_1} \ldots \partial \tilde{x}_n^{a_n}$ of $f \circ (\varphi_k|_V)^{-1}$ extends continuously from $(\varphi_k|_V)(D \cap V) \to \mathbb{R}^m$ to $(\varphi_k|_V)(\overline{D} \cap V) \to \mathbb{R}^m$.*

Proof. The easy direction is (2) $\implies$ (1); choose any chart $\varphi_k : U_k \to M$, then for the corresponding neighborhood V, let $U_j = V$, $\varphi_j = \varphi_k|_V$, and the x coordinates are the restriction of the $\tilde{x}$ coordinates to V, so that $f \circ (\varphi_k|_V)^{-1} = f \circ \varphi_j^{-1}$ extends continuously from $(\varphi_k|_V)(D \cap V) = \varphi_j(D \cap U_j)$ to $(\varphi_k|_V)(\overline{D} \cap V) = \varphi_j(\overline{D} \cap U_j)$.

The set $\varphi_j(D \cap U_j)$ is the domain of $f \circ \varphi_j^{-1}$ as in Lemma 11.3, where all the derivatives of $f \circ \varphi_j^{-1}$ are defined in the x coordinate system on $\varphi_j(U_j) \subseteq \mathbb{R}^n$. Assuming (1), for any φ_k define $V = U_j \cap U_k$, so that $\varphi_k|_V : V \to \mathbb{R}^n$ is a chart around p with image $\varphi_k(U_j \cap U_k) \subseteq \varphi_k(U_k)$, and V has $\tilde{x}$ coordinates from U_k. The composite $f \circ (\varphi_k|_V)^{-1}$ is defined, and all its derivatives with respect to $\tilde{x}$ exist, on the set $(\varphi_k|_V)(D \cap V) = \varphi_k(D \cap U_j \cap U_k)$, as in Lemma 11.3. We want to show that $f \circ (\varphi_k|_V)^{-1}$ and all its $\tilde{x}$ derivatives extend to $(\varphi_k|_V)(\overline{D} \cap V)$, the closure of $(\varphi_k|_V)(D \cap V)$ in $(\varphi_k|_V)(V)$ by Lemma 11.4, so to show (2), it will be enough to find some continuous extension to any closed set in $(\varphi_k|_V)(V)$ containing $(\varphi_k|_V)(D \cap V)$.

Because $(\varphi_j^{-1} \circ \varphi_j)(q) = q$ for all q in V, we can expand and then rearrange:

$$\begin{aligned} f \circ (\varphi_k|_V)^{-1} &= f \circ ((\varphi_j^{-1} \circ \varphi_j) \circ (\varphi_k|_V)^{-1}) \\ &= (f \circ \varphi_j^{-1}) \circ (\varphi_j \circ (\varphi_k|_V)^{-1}), \end{aligned} \tag{11.1}$$

where the composite in (11.1) has the same domain

$$(\varphi_j \circ (\varphi_k|_V)^{-1})^{-1}(\varphi_j(D \cap U_j)) = (\varphi_k|_V)(D \cap V). \tag{11.2}$$

However, the factor $(\varphi_j \circ (\varphi_k|_V)^{-1})$ is smooth on all of $(\varphi_k|_V)(V)$ (from the definition of smooth manifold, it is a transition function for the two charts φ_j and $\varphi_k|_V$, with $V \subseteq U_j$), and the factor $(f \circ \varphi_j^{-1})$ and all of its derivatives with respect to x extend continuously to $\varphi_j(\overline{D} \cap U_j)$ by the hypothesis (1).

To calculate a derivative of $f \circ (\varphi_k|_V)^{-1}$ with respect to $\tilde{x}$ on the set $(\varphi_k|_V)(D \cap V)$, and then show it coincides with a function that is continuous a larger closed set, the Faà di Bruno formula applies to the composite (11.1). The result is a multi-indexed linear combinations of derivatives of $f \circ \varphi_j^{-1}$ multiplied by products of derivatives of $\varphi_j \circ (\varphi_k|_V)^{-1}$, of the following abbreviated form with constant coefficients $c_{\mathbf{IJ}}$:

$$\begin{aligned} &\frac{\partial}{\partial \tilde{x}^{\mathbf{J}}} \left(f \circ (\varphi_k|_V)^{-1} \right) \\ (11.3)= \ &\sum_{\mathbf{I}} \left(c_{\mathbf{IJ}} \left[\frac{\partial}{\partial x} (f \circ \varphi_j^{-1}) \right]_{x = (\varphi_j \circ (\varphi_k|_V)^{-1})(\tilde{x})} \cdot \prod \frac{\partial}{\partial \tilde{x}} (\varphi_j \circ (\varphi_k|_V)^{-1}) \right). \end{aligned}$$

In (11.3), the derivatives of $(\varphi_j \circ (\varphi_k|_V)^{-1})$ in the second factor are continuous on all of $(\varphi_k|_V)(V)$. The expressions in the first factor, as functions of $\tilde{x}$ extend continuously as follows: the derivatives of $f \circ \varphi_j^{-1}$ with respect to x extend continuously to $\varphi_j(\overline{D} \cap U_j)$ and are then evaluated at $x = (\varphi_j \circ (\varphi_k|_V)^{-1})(\tilde{x})$, so the expressions are continuous in $\tilde{x}$ on the inverse image $(\varphi_j \circ (\varphi_k|_V)^{-1})^{-1}(\varphi_j(\overline{D} \cap U_j))$, which is a closed set in $(\varphi_k|_V)(V)$, containing $(\varphi_k|_V)(D \cap V)$ as in (11.2). ∎

In the following Lemma, let $a > 0$, let $B = (-a, a)^n \subseteq \mathbb{R}^n$ be an open box neighborhood of the origin, and denote the upper half-box $B^+ = (-a, a)^{n-1} \times (0, a)$ and lower half-box $B^- = (-a, a)^{n-1} \times (-a, 0)$.

Lemma 11.6. *For the open box B, if $f : B \to \mathbb{R}^m$ is continuous on B, and every first partial derivative $\frac{\partial f}{\partial x_j}$ exists on $B^+ \cup B^-$ and extends continuously to B, and there is a function $F : B \to \mathbb{R}^m$ such that F has continuous first partial derivatives on B and $F(\vec{x}) = f(\vec{x})$ for all $x \in B^+$, then every first partial derivative $\frac{\partial f}{\partial x_j}$ is continuous on B.*

Proof. For $k = 1, \ldots n$, denote the continuous extension of $\frac{\partial f}{\partial x_k}$ to B by $g_k : B \to \mathbb{R}^m$.

At a point $\vec{x}_0 = (x_1, \ldots, x_{n-1}, 0) \in B$, the last derivative $\frac{\partial f}{\partial x_n}(\vec{x}_0)$ exists and is continuous, without assuming anything about F. By an elementary property of a derivative with respect to one variable ([C]) that follows from the Mean Value Theorem, if f is continuous at $\vec{x}_0$ and

$$\lim_{t \to 0^+} \left(\left. \frac{\partial f}{\partial x_n} \right|_{(x_1, \ldots, x_{n-1}, t)} \right) = g_n(\vec{x}_0),$$

then

$$\lim_{h\to 0^+} \frac{f(x_1,\ldots,x_{n-1},h) - f(x_1,\ldots,x_{n-1},0)}{h} = g_n(\vec{x}_0),$$

and similarly the $h \to 0^-$ limit is also $g_n(\vec{x}_0)$, so the two-sided derivative exists at $\vec{x}_0$:

$$\left.\frac{\partial f}{\partial x_n}\right|_{(x_1,\ldots,x_{n-1},0)} = g_n(\vec{x}_0).$$

The conclusion is that for any point in B, $\frac{\partial f}{\partial x_n}$ coincides with g_n, so it is continuous on B.

At the same point $\vec{x}_0 = (x_1,\ldots,x_{n-1},0) \in B$, for $j = 1,\ldots,n-1$ we can calculate the $\frac{\partial}{\partial x_j}$ derivatives of f by noting that F is continuous on B (in fact, differentiable, as a consequence of its partial derivatives being continuous), so $F(\vec{x}) = f(\vec{x})$ on the closure $\overline{B^+} = \{x_n \geq 0\}$, by Theorem 10.1.

$$\begin{aligned}
\left.\frac{\partial f}{\partial x_j}\right|_{\vec{x}_0} &= \lim_{h\to 0} \frac{f(x_1,\ldots,x_j+h,\ldots,x_{n-1},0) - f(x_1,\ldots,x_j,\ldots,x_{n-1},0)}{h} \\
&= \lim_{h\to 0} \frac{F(x_1,\ldots,x_j+h,\ldots,x_{n-1},0) - F(x_1,\ldots,x_j,\ldots,x_{n-1},0)}{h} \\
&= \left.\frac{\partial F}{\partial x_j}\right|_{\vec{x}_0},
\end{aligned}$$

so this derivative exists. So, for any point in $\overline{B^+}$, $\frac{\partial f}{\partial x_j}$ coincides with the continuous function $\frac{\partial F}{\partial x_j}$, and by the uniqueness from Theorem 10.1 again, must also coincide with g_j on $\overline{B^+}$ and therefore on all of B. ∎

Theorem 11.7. *Given an open set D of a smooth manifold M such that the boundary of D in M is a smooth embedded submanifold bD in M, and a smooth function $f : D \to \mathbb{R}^m$, the following are equivalent:*

(1) *For every point $p \in bD$, there is some chart around p, $\varphi_j : U_j \to \mathbb{R}^n$, so that every derivative $\partial/\partial x_1^{a_1} \ldots \partial x_n^{a_n}$ of $f \circ \varphi_j^{-1}$ extends continuously from $\varphi_j(D \cap U_j)$ to $\varphi_j(\overline{D} \cap U_j)$.*
(2) *There exists an open set $U \subseteq M$ containing $\overline{D}$ and an extension of f to a smooth function $U \to \mathbb{R}^m$.*

Proof. Assuming (2), let $F : U \to \mathbb{R}^m$ be an extension of f. For any $p \in bD \subseteq U$, by Definition 11.2 there is some chart $\varphi_k : U_k \to \mathbb{R}^n$ so that $F \circ \varphi_k^{-1} : \varphi_k(U_k) \to \mathbb{R}^m$ is smooth on some neighborhood U_j' of $\varphi_k(p)$ in $\varphi_k(U_k) \subseteq \mathbb{R}^n$, with $U_j' \subseteq \varphi_k(U \cap U_k)$. Let $U_j = \varphi_k^{-1}(U_j')$ and $\varphi_j = \varphi_k|_{U_j}$, so φ_j is a chart around p, and the x coordinates in U_k restrict to x coordinates on U_j. For $\vec{x} \in \varphi_j(D \cap U_j)$, $(F \circ \varphi_j^{-1})(\vec{x}) = (f \circ \varphi_j^{-1})(\vec{x})$. By Lemma 11.3, every derivative with respect to $\vec{x}$ of $F \circ \varphi_j^{-1}$ is continuous on $\varphi_j(U_j)$, and every derivative with respect to $\vec{x}$ of $f \circ \varphi_j^{-1}$ exists on $\varphi_j(D \cap U_j)$ and is equal to the derivative of $F \circ \varphi_j^{-1}$, so the derivative of $F \circ \varphi_j^{-1}$ is a continuous extension (not necessarily unique) and the conclusion is that (1) holds. Each derivative extends uniquely to $\varphi_j(\overline{D}\cap U_j)$ by Theorem 10.1 and Lemma 11.4.

Now assume (1), where for any $p\in bD$ there is a corresponding chart φ_j. For the same point p, from the hypothesis about the boundary bD being a smoothly embedded submanifold of M, it follows from ([20] Chapter 3) that there is another chart $\varphi_k : U_k \to \mathbb{R}^n$ with $\varphi(U_k) = (-1,1)^n \subseteq \mathbb{R}^n$, $\varphi_k(p) = \vec{0}$, and $\varphi_k(bD\cap U_k) = \{x_n = 0\}$. $\varphi_k(D\cap U_k)$ is a non-empty open subset of $\varphi_k(U_k)$ with closure $\varphi_k(\overline{D}\cap U_k)$ in $\varphi_k(U_k)$ as in Lemma 11.4, and by elementary properties of the topological boundary in $\varphi_k(U_k)$ ([M] §2.17) (and using the one-to-one property of φ_k),

$$\begin{aligned} b(\varphi_k(D\cap U_k)) &= \overline{\varphi_k(D\cap U_k)}\setminus \varphi_k(D\cap U_k) = \varphi_k(\overline{D}\cap U_k)\setminus\varphi_k(D\cap U_k)\\ &= \varphi_k((\overline{D}\cap U_k)\setminus(D\cap U_k)) = \varphi_k((\overline{D}\setminus D)\cap U_k)\\ &= \varphi_k((bD)\cap U_k) = \{x_n = 0\}.\end{aligned}$$

These are disjoint unions:

$$\begin{aligned} U_k &= (D\cap U_k)\cup(bD\cap U_k)\cup(U_k\setminus\overline{D}),\\ \varphi_k(U_k) &= \varphi_k(D\cap U_k)\cup\{x_n=0\}\cup\varphi_k(U_k\setminus\overline{D})\\ &= \varphi_k(D\cap U_k)\cup\{x_n=0\}\cup(\varphi_k(U_k)\setminus\varphi_k(\overline{D}\cap U_k)).\end{aligned}$$

The upper half-cube $B^+ = \{\vec{x}\in\varphi_k(U_k) = (-1,1)^n : x_n > 0\}$ is equal to the union of these disjoint open sets:

$$B^+ = (B^+\cap\varphi_k(D\cap U_k))\cup(B^+\setminus\varphi_k(\overline{D}\cap U_k)),$$

but because B^+ is a connected set, it must equal exactly one of those open sets. In particular, if there is any point $\vec{x}\in B^+$ with $\varphi_k^{-1}(\vec{x})\in D$, then $B^+\subseteq\varphi_k(D\cap U_k)$. Similarly, if there is any such point in the lower half-cube $B^- = \{\vec{x}\in U_k : x_n < 0\}$, then $B^-\subseteq\varphi_k(D\cap U_k)$. The conclusion is that the non-empty open subset $\varphi_k(D\cap U_k)\subseteq B^+\cup B^-$ must be either B^+, B^-, or $B^+\cup B^-$. At this point we assume $B^+\subseteq\varphi_k(D\cap U_k)$, the B^- case being analogous.

By the (1) $\implies$ (2) direction of Lemma 11.5 applied to U_j and U_k, there is some neighborhood V of p in U_k so that every derivative $\partial/\partial\tilde{x}_1^{a_1}\ldots\partial\tilde{x}_n^{a_n}$ of $f\circ(\varphi_k|_V)^{-1}$ extends continuously to $(\varphi_k|_V)(\overline{D}\cap V)\subseteq\varphi_k(V)\subseteq\varphi_k(U_k)$. Just to avoid any problems along the boundary of V, consider two small open cubes around the origin, $\vec{0}\in B_p\subsetneq B'_p\subsetneq\varphi_k(V)$ and let $\chi(\vec{x})$ be a smooth bump function $\mathbb{R}^n\to\mathbb{R}$ with $\chi(x)\equiv 1$ on B_p, $\chi(x)\equiv 0$ on $\mathbb{R}^n\setminus B'_p$. The product $\chi(\vec{x})\cdot(f\circ(\varphi_k|_V)^{-1})(\vec{x})$ coincides with $f\circ(\varphi_k|_V)^{-1}$ on the upper half-cube B_p^+, it extends in the obvious way to a smooth function on the open upper half-space $\{\vec{x}\in\mathbb{R}^n : x_n > 0\}$, and every derivative extends continuously to $\{x_n\geq 0\}$. By a version of Whitney's Extension Theorem (see [S]), this smooth function extends to a smooth function on $\mathbb{R}^n$. In particular, there is a smooth function $F_p:\mathbb{R}^n\to\mathbb{R}^m$ coinciding with $f\circ(\varphi_k|_V)^{-1}$ on B_p^+. The composite $F_p\circ(\varphi_k|_V) : V\to\mathbb{R}^m$ is smooth (as in Definition 11.2) and coincides with the smooth function f on the set $(\varphi_k|_V)^{-1}(B_p^+) \subseteq D$.

In the special case where $\varphi_k(D \cap U_k) = B^+ \cup B^-$ and $f \circ (\varphi_k|_V)^{-1}$ is smooth on both B_p^+ and B_p^-, Lemma 11.6 applies to every derivative of $f \circ (\varphi_k|_V)^{-1}$ and F_p; the conclusion is that the continuous extension of $f \circ (\varphi_k|_V)^{-1}$ from $B_p \setminus \{x_n = 0\}$ to B_p is itself smooth on B_p. (These are the boundary points of D contained in the interior of $\overline{D}$.)

Having found a local smooth extension of f in a neighborhood of each boundary point p, let $\{D\} \cup \{(\varphi_k|_V)^{-1}(B_p) : p \in bD\}$ be an open cover of $\overline{D}$ in M, and let U be the union of all these open sets. There exists a smooth partition of unity $\{\alpha_*\} \cup \{\alpha_p : p \in bD\}$ with respect to this open cover ([W]), so that each α_p is compactly supported in $(\varphi_k|_V)^{-1}(B_p)$. The sum

$$\alpha_* \cdot f + \sum_{p \in bD} \alpha_p \cdot (F_p \circ (\varphi_k|_V))$$

has a finite number of non-zero terms in some neighborhood of each point in $\overline{D}$, so it is smooth on U. For $y \in D$,

$$\alpha_*(y) \cdot f(y) + \sum_{p \in bD} \alpha_p(y) \cdot (F_p((\varphi_k|_V)(y))) = \left(\alpha(y) + \sum_{b \in bD} \alpha_p(y)\right) \cdot f(y) = f(y).$$

Such a smooth extension to U is uniquely defined on $\overline{D}$ but obviously not necessarily unique away from $\overline{D}$. ∎

Remark 11.8. In general, defining smooth functions on closed sets as extensions or restrictions of smooth functions on open sets is the subject of Whitney's Extension Theorem, considered by Whitney for closed sets in $\mathbb{R}^n$. The one-dimensional case of an interval in $\mathbb{R}$ is sketched in [D], and the two-dimensional case of smoothly bounded open sets in $\mathbb{C}$ is considered in [BG] §4.8. For a more general approach to smooth (and $\mathcal{C}^r$) extension problems in $\mathbb{R}^n$, see [BB] Chapter 2. For a description of a global "collar" for bD in M, see [H] Chapter 4 or [KM] Chapter 5.

Given M, D and f, f has a smooth extension to $\overline{D}$ means that either of the properties from Theorem 11.7 holds. The following Definition applies this to the constructions from Section 9.

Definition 11.9. Given $\Omega \subseteq \mathbb{C}^2$ such that its boundary $b\Omega$ is a smooth embedded submanifold, and a continuous homomorphism ϕ, a $\mathbf{T}$-action extends smoothly to the boundary means that the map A_ϕ from (9.1) extends to a smooth map

$$A_\phi^e : \mathbf{T} \times \overline{\Omega} \to \mathbb{C}^2.$$

More precisely, let U be some open set as in Theorem 11.7, $\mathbf{T} \times \overline{\Omega} \subseteq U \subseteq \mathbf{T} \times \mathbb{C}^2$, so that A_ϕ^e is smooth on U.

Any such extension A_ϕ^e restricts to the unique continuous extension A_ϕ^E as in Definition 10.3. The boundary of $\mathbf{T} \times \Omega$ in $\mathbf{T} \times \mathbb{C}^2$ is $\mathbf{T} \times (b\Omega)$, which is smoothly embedded in U, so A^e_ϕ restricts to the unique continuous map $A^b_\phi : \mathbf{T} \times (b\Omega) \to b\Omega$ from (10.1). The restriction A^b_ϕ is a smooth map ([W] Theorem 1.32), and is a group action. For each $\phi_t \in \mathbf{T}$, the restriction of $A^e_\phi : U \to \mathbb{C}^2$ to $(\{\phi_t\} \times \mathbb{C}^2) \cap U \to \mathbb{C}^2$ is smooth, so the automorphism $\phi_t : \Omega \to \Omega$ has a smooth extension to an open neighborhood (depending on t) of the closure:

$$\overline{\Omega} \hookrightarrow (\{\phi_t\} \times \mathbb{C}^2) \cap U \to \mathbb{C}^2.$$

Using the compactness of $\mathbf{T}$ and $\overline{\Omega}$, by ([M] Exercise 3.26.9) there is a neighborhood U_0 of $\overline{\Omega}$ in $\mathbb{C}^2$ so that

$$\mathbf{T} \times \overline{\Omega} \subseteq \mathbf{T} \times U_0 \subseteq U \subseteq \mathbf{T} \times \mathbb{C}^2,$$

which means that all the automorphisms $\phi_t : \Omega \to \Omega$ extend smoothly to the same domain $U_0 \to \mathbb{C}^2$, not depending on t.

The composite $\phi^b_t : b\Omega \to b\Omega$ from (10.2),

$$b\Omega \to \mathbf{T} \times (b\Omega) \to b\Omega : z \mapsto (\phi_t, z) \mapsto A^b_\phi(\phi_t, z) = A^e_\phi(\phi_t, z)$$

is a smooth map. It is a diffeomorphism because it has a smooth inverse as in (10.3).

Similarly, for each $z_0 \in b\Omega$, the composite $\psi_{z_0} : \mathbf{T} \to b\Omega$ from (10.4),

$$\mathbf{T} \to \mathbf{T} \times (b\Omega) \to b\Omega : \phi_t \mapsto (\phi_t, z_0) \mapsto A^b_\phi(\phi_t, z_0) = A^e_\phi(\phi_t, z_0)$$

is a smooth map.

Theorem 11.10. *The orbit of* z_0 *(the image of* ψ_{z_0}*) is a connected embedded submanifold of* $b\Omega$*, of dimension* 0 *or* 1.

Proof. Given z_0, let H_{z_0} denote the subgroup of $\mathbf{T}$ that fixes z_0. It is proved in ([GG] Appendix A) that H_{z_0} is a closed Lie subgroup, that $\mathbf{T}/H_{z_0}$ is a smooth manifold with dimension $\leq \dim(\mathbf{T}) \leq 1$ (by Theorem 9.4), and that ψ_{z_0} induces a one-to-one immersion $\mathbf{T}/H_{z_0} \to b\Omega$ whose image is the orbit of z_0. Since S^1 is compact and connected, so are $\mathbf{T}$ and $\mathbf{T}/H_{z_0}$, and the immersion is an embedding ([20] Theorem III.5.7). ∎

The orbit could be the singleton $\{z_0\}$, if $\mathbf{T}$ fixes z_0.

12. Transverse action

The boundary $b\Omega$ is a smooth 3-manifold embedded in $\mathbb{C}^2$, so at each point $z_0 \in b\Omega$, the tangent space $T_{z_0}b\Omega$ contains exactly one complex line through z_0, $T^h_{z_0}b\Omega = T_{z_0}b\Omega \cap J_{z_0}T_{z_0}b\Omega$, where J_{z_0} is the complex structure operator on $T_{z_0}\mathbb{C}^2$.

Definition 12.1. Given a $\mathbf{T}$-action on Ω which extends smoothly to the boundary as in Definition 11.9, and a point $z_0 \in b\Omega$, the $\mathbf{T}$-action is transverse at z_0 means that the differential map $d(\psi_{z_0}) : T_e\mathbf{T} \to T_{z_0}b\Omega$ has image not contained in $T^h_{z_0}b\Omega$.

For example, a trivial $\mathbf{T}$-action is not transverse at any point.

Theorem 12.2. *If a* **T***-action is transverse at* z_0*, then the orbit of* z_0 *is a smoothly embedded curve in* $b\Omega$*.*

Proof. It follows from Theorem 11.10 that the orbit must be either an embedded curve in $b\Omega$, or a point, in which case ψ_{z_0} is the constant map and $d(\psi_{z_0})$ is the 0 map, so the **T**-action is not transverse. ∎

Definition 12.3. The domain Ω admits a transverse **T**-action means: there exists a continuous homomorphism $\phi : S^1 \to Aut(\Omega)$ so that the corresponding **T**-action is transverse at each point of $b\Omega$.

Proposition 12.4. *If* Ω *is a bounded open subset of* $\mathbb{C}^2$ *with smooth boundary, and* Ω *admits a transverse* **T***-action, then* Ω *satisfies "Condition R."*

Remark. This was proved more generally (for higher-dimensional domains and groups) by [1]. ∎

Proposition 12.5. *If* Ω_1 *and* Ω_2 *are bounded open subsets of* $\mathbb{C}^2$ *with smooth boundaries, and both satisfy Condition R, then every biholomorphic map* $F : \Omega_1 \to \Omega_2$ *extends smoothly to* $\overline{\Omega_1}$*, so that the restriction* $b\Omega_1 \to b\Omega_2$ *is a diffeomorphism.*

Remark. This is proved in [10], Theorem VII.8.10, Ex. VII.8.6, and stated as [5] Corollary 7.1. ∎

In particular, if Ω admits a transverse **T**-action, then every automorphism extends to a diffeomorphism of the closure — this was already shown for automorphisms in **T**. A stronger result is:

Proposition 12.6. *If* Ω *is a bounded open subset of* $\mathbb{C}^2$ *with smooth boundary that satisfies Condition R, then the action* $A : Aut(\Omega) \times \Omega \to \Omega$ *extends smoothly to* $Aut(\Omega) \times \overline{\Omega} \to \overline{\Omega}$*.*

Remark. This is proved in [6]. ∎

13. Orbits of curves

In this Section we forget about $\mathbb{C}^2$ — we want to consider the local geometry of a one-parameter group action on a smooth arc in a real three-dimensional manifold, so it will be enough to work with neighborhoods in Euclidean spaces. Consider the Lie group $\mathbb{R}$ with operation $+$ and coordinate function θ. Let M be an open subset of $\mathbb{R}^3$, with coordinate functions (x, y, z), and let $G : \mathbb{R} \times M \to M$ be a smooth map which is a group action. For $m \in M$, the map has components $G(\theta, m) = (g_1(\theta, m), g_2(\theta, m), g_3(\theta, m)) \in M$, and G has the properties: $G(0, m) = m$ (the action of the identity element is the identity transformation), and $G(\theta + \phi, m) = G(\theta, G(\phi, m))$ (composition of transformations respects the group operation). It follows that for each fixed $\theta_0 \in \mathbb{R}$, the map

$$\Delta_{\theta_0} : M \to M : \Delta_{\theta_0}(m) = G(\theta_0, m)$$

is a diffeomorphism, since it is the restriction of a smooth map and has a smooth inverse, $\Delta_{-\theta_0}$. Also for each fixed θ_0, define the inclusion

$$\iota_{\theta_0} : M \to \mathbb{R} \times M : \iota_{\theta_0}(m) = (\theta_0, m),$$

so that $\Delta_{\theta_0} = G \circ \iota_{\theta_0}$. The Jacobian matrix at point $m \in M$ of the diffeomorphism Δ_{θ_0} is denoted: $d\Delta_{\theta_0}|_m$, and it has rank 3. By the Chain rule,

$$d\Delta_{\theta_0}|_m = d(G \circ \iota_{\theta_0})|_m = dG|_{(\theta_0,m)} \cdot d\iota_{\theta_0}|_m,$$

so we can conclude that the Jacobian of G at (θ_0, m) has rank 3. Specifically, in coordinates (θ, x, y, z) of $\mathbb{R} \times M$,

$$\begin{aligned}
d\iota_{\theta_0}|_m &= \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
dG|_{(\theta_0,m)} &= \begin{bmatrix} \frac{dg_1}{d\theta} & \frac{dg_1}{dx} & \frac{dg_1}{dy} & \frac{dg_1}{dz} \\ \frac{dg_2}{d\theta} & \frac{dg_2}{dx} & \frac{dg_2}{dy} & \frac{dg_2}{dz} \\ \frac{dg_3}{d\theta} & \frac{dg_3}{dx} & \frac{dg_3}{dy} & \frac{dg_3}{dz} \end{bmatrix} \\
d\Delta_{\theta_0}|_m &= \begin{bmatrix} \frac{dg_1}{dx} & \frac{dg_1}{dy} & \frac{dg_1}{dz} \\ \frac{dg_2}{dx} & \frac{dg_2}{dy} & \frac{dg_2}{dz} \\ \frac{dg_3}{dx} & \frac{dg_3}{dy} & \frac{dg_3}{dz} \end{bmatrix}.
\end{aligned}$$

For each point $m_0 \in M$, define a map

$$\psi_{m_0} : \mathbb{R} \to M : \psi_{m_0}(\theta) = G(\theta, m_0).$$

The "orbit" of m_0 is defined to be the image of the map ψ_{m_0}. Since ψ_{m_0} is a path, its Jacobian at θ is the velocity column vector $d\psi_{m_0}|_\theta$. Denote another inclusion

$$\iota_{m_0} : \mathbb{R} \to \mathbb{R} \times M : \iota_{m_0}(\theta) = (\theta, m_0),$$

so $\psi_{m_0} = G \circ \iota_{m_0}$, and by the chain rule,

$$d\psi_{m_0}|_{\theta=\theta_0} = dG|_{(\theta_0,m_0)} \cdot d\iota_{m_0}|_{\theta=\theta_0} = \begin{bmatrix} \frac{dg_1}{d\theta} & \frac{dg_1}{dx} & \frac{dg_1}{dy} & \frac{dg_1}{dz} \\ \frac{dg_2}{d\theta} & \frac{dg_2}{dx} & \frac{dg_2}{dy} & \frac{dg_2}{dz} \\ \frac{dg_3}{d\theta} & \frac{dg_3}{dx} & \frac{dg_3}{dy} & \frac{dg_3}{dz} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{dg_1}{d\theta} \\ \frac{dg_2}{d\theta} \\ \frac{dg_3}{d\theta} \end{bmatrix}.$$

This is the tangent vector to the orbit ψ_{m_0} at the point $\psi_{m_0}(\theta_0) = G(\theta_0, m_0)$. It depends only on G and the point $G(\theta_0, m_0)$, in the following sense: let

$$\psi_{G(\theta_0,m_0)}(\theta) = G(\theta, G(\theta_0, m_0)) = G(\theta + \theta_0, m_0),$$

so $\psi_{G(\theta_0,m_0)}(\theta-\theta_0) = \psi_{m_0}(\theta)$. By the chain rule, $d\psi_{G(\theta_0,m_0)}|_{\theta=0} = d\psi_{m_0}|_{\theta=\theta_0}$, so the velocity vector at $G(\theta_0, m_0)$ is the same if we start at any point on the orbit.

Now, fix $m_0 \in M$, and let $I = (-\epsilon, \epsilon)$ be an open interval in $\mathbb{R}$, and let $\gamma : I \to M$ be any smooth map, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$, with $\gamma(0) = m_0$. Define a map

$$P_\gamma : \mathbb{R} \times I \to M : P_\gamma(\theta, t) = G(\theta, \gamma(t)).$$

The image of P_γ is a union of orbits of the points $\gamma(t)$, so we could call it the orbit of the image of γ, and we are interested in the behavior of P_γ near points on the orbit $\psi_{m_0}(\mathbb{R})$. P_γ is a composite of the form $G \circ (Id \times \gamma)$, where $(Id \times \gamma)(\theta, t) = (\theta, \gamma(t))$, and by the chain rule, at the point $(\theta_0, 0)$,

$$\begin{aligned} dP_\gamma|_{(\theta_0,0)} &= dG|_{(\theta_0,m_0)} \cdot d(Id \times \gamma)|_{(\theta_0,0)} \\ &= \begin{bmatrix} \frac{dg_1}{d\theta} & \frac{dg_1}{dx} & \frac{dg_1}{dy} & \frac{dg_1}{dz} \\ \frac{dg_2}{d\theta} & \frac{dg_2}{dx} & \frac{dg_2}{dy} & \frac{dg_2}{dz} \\ \frac{dg_3}{d\theta} & \frac{dg_3}{dx} & \frac{dg_3}{dy} & \frac{dg_3}{dz} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \frac{d\gamma_1}{dt} \\ 0 & \frac{d\gamma_2}{dt} \\ 0 & \frac{d\gamma_3}{dt} \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} \frac{dg_1}{d\theta} \\ \frac{dg_2}{d\theta} \\ \frac{dg_3}{d\theta} \end{bmatrix} & \left[\begin{bmatrix} \frac{dg_1}{dx} & \frac{dg_1}{dy} & \frac{dg_1}{dz} \\ \frac{dg_2}{dx} & \frac{dg_2}{dy} & \frac{dg_2}{dz} \\ \frac{dg_3}{dx} & \frac{dg_3}{dy} & \frac{dg_3}{dz} \end{bmatrix} \cdot \begin{bmatrix} \frac{d\gamma_1}{dt} \\ \frac{d\gamma_2}{dt} \\ \frac{d\gamma_3}{dt} \end{bmatrix} \right] \end{bmatrix}_{3\times 2}. \end{aligned}$$

The left column is the tangent vector to the orbit ψ_{m_0} at the point $G(\theta_0, m_0)$. The right column is the product $d\Delta_{\theta_0}|_{m_0} \cdot d\gamma|_0 = d(\Delta_{\theta_0} \circ \gamma)|_0$, the tangent vector of the path $\Delta_{\theta_0} \circ \gamma : I \to M$ at the point $G(\theta_0, m_0)$.

We can conclude that $dP_\gamma|_{(\theta_0,0)}$ has rank 2 if the vectors

$$\frac{d}{d\theta}\psi_{m_0}|_{\theta=\theta_0} = \frac{d}{d\theta}(G(\theta, m_0))|_{\theta=\theta_0}$$

and

$$\frac{d}{dt}(\Delta_{\theta_0} \circ \gamma)|_{t=0} = \frac{d}{dt}(G(\theta_0, \gamma(t)))|_{t=0}$$

are linearly independent. The first vector depends only on G and the point $G(\theta_0, m_0) \in M$, and not on the path γ. In the special case $\theta_0 = 0$, the second vector is just the velocity vector of γ at m_0. This rank condition at the one point implies, by continuity of the derivatives, the rank of the Jacobian of P_γ is 2 in some neighborhood of that point, so P_γ is an immersion of that neighborhood, and there is some (possibly smaller) neighborhood of $(\theta_0, 0)$ where P_γ is a homeomorphism onto its image in M, so the image is a two-dimensional ([20] Ch. III) surface containing $G(\theta_0, m_0)$. The two-dimensionality can fail at singular points — if the image of the path and the orbit have velocity vectors in the same direction, or if one of the two velocities is zero.

14. Miscellaneous topological facts

Lemma 14.1. *Given a topological space X and $W \subseteq \Omega \subseteq X$, denote the closure of Ω in X by $\bar{\Omega}$. Then the closure of W in $\bar{\Omega}$ equals the closure of W in X.*

Proof. By [M] Theorem 17.4, the closure of W in $\bar{\Omega}$ equals the intersection of $\bar{\Omega}$ with the closure of W in X, which is contained in the closure of W in X.

Since the closure of W in $\bar{\Omega}$ is closed in $\bar{\Omega}$, and $\bar{\Omega}$ is closed in X, we can conclude the closure of W in $\bar{\Omega}$ is closed in X ([M] Theorem 17.3), and contains W, so the closure of W in X is contained in the closure of W in $\bar{\Omega}$. ∎

As an application of the above Lemma, the notation $\overline{W}$ is unambiguous for the closure of W in either $\bar{\Omega}$ or X. As another application, if B is any set in X, then the set $W \cap B$ has the same closure in $\overline{W}$, $\bar{\Omega}$, and X, and can be denoted $\overline{W \cap B}$.

Theorem 14.2. *Given $W \subseteq \Omega \subseteq X$ and $B \subseteq X$, assume W is closed in Ω and let $E = \overline{W} \cap b\Omega$. Then,*

$$\overline{W \cap B} \subseteq (W \cap B) \cup (\bar{B} \cap E) \cup (bB \cap W).$$

Proof. W equals its own closure in Ω, which equals $\overline{W} \cap \Omega$ by [M] Theorem 17.4 again.

$$\begin{aligned} \overline{W \cap B} &\subseteq \overline{W} \cap \bar{B} \\ &= (\overline{W} \cap (\Omega \cup b\Omega)) \cap \bar{B} \\ &= ((\overline{W} \cap \Omega) \cup (\overline{W} \cap b\Omega)) \cap \bar{B} \\ &= (W \cup E) \cap \bar{B} \\ &= (W \cap \bar{B}) \cup (E \cap \bar{B}) \\ &= (W \cap B) \cup (W \cap bB) \cup (E \cap \bar{B}). \end{aligned}$$

∎

The above Theorem 14.2 applies to the set W of [CP] §3, an analytic set which is closed in $\Omega \subseteq \mathbb{C}^2$. The following Theorem 14.3 is used in the Proof of [CP] Lemma 3.1. The notation $B(q, \epsilon)$ refers to the ball with center $q \in \mathbb{R}^n$ and radius $\epsilon > 0$ as in [CP] §2.

Theorem 14.3. *Let D be an open ball in $\mathbb{R}^n$. Given a subset C closed in D, let bC denote the boundary of C in D, and denote the set:*

$$K = \{p \in C : \exists \epsilon > 0, q \in D \setminus C : \overline{B(q, \epsilon)} \cap C = \{p\}\}.$$

Then K is a dense subset of bC.

Proof. The Theorem is vacuous if $C = D$. If p is an interior point of C, then $p \in B(p, \eta) \subseteq C$ for some $\eta > 0$, so $\overline{B(q, \epsilon)} \cap C$ contains $\overline{B(q, \epsilon)} \cap B(p, \eta)$, which has infinitely many elements if it contains p. This shows $K \subseteq bC$.

Consider any $p \in bC$. We want to show that for any open neighborhood U of p in D, $U \cap (K \cap bC) \neq \emptyset$. For some $\delta > 0$, $B(p, \delta) \subseteq U$, and since p is a boundary point, there is some $q \in B(p, \delta/2) \cap (U \setminus C)$, an open set, so the set

$$\{\epsilon > 0 : \overline{B(q,\epsilon)} \cap C = \emptyset\}$$

is non-empty, and bounded above by $\delta/2$. Let ϵ_0 be the least upper bound of this set, so $\epsilon \leq \delta/2$ and $\overline{B(q,\epsilon_0)} \subseteq B(p,\delta) \subseteq U$. The claim is that the boundary of $B(q,\epsilon_0)$ meets C somewhere in $U \cap bC$. Suppose first that $\overline{B(q,\epsilon_0)} \cap C = \emptyset$. However, contradicting the upper bound property, we can pick $\epsilon > \epsilon_0$, so a larger ball $\overline{B(q,\epsilon)}$ also has empty intersection with C, as follows: since D is normal, $\overline{B(q,\epsilon_0)}$ and C can be separated by disjoint open sets in D; then since the boundary sphere of $\overline{B(q,\epsilon_0)}$ is compact, it can be covered by finitely many balls centered on the boundary which are contained in the separating neighborhood, and the larger sphere centered at q will fit inside these finitely many balls.

Alternatively, suppose the open ball $B(q,\epsilon_0)$ meets C at some point x at distance $\epsilon < \epsilon_0$ from q. Then $B(q,\epsilon)$ is a smaller upper bound than the least upper bound, another contradiction. The only remaining possibility is that $\overline{B(q,\epsilon_0)}$ meets C, and only at points on the boundary sphere.

Such points may not be unique, but by the geometry of the ball, if it meets C at any point, then the ball $B(q,\epsilon_0)$ contains a smaller ball with a different center, whose boundary meets C at only that point. The conclusion is that such a point is in the set K and also in the set $U \cap C$. ∎

Theorem 14.4. *Given topological spaces X, B, with B compact, and a continuous function $F : X \times B \to \mathbb{R}$, let $G : X \to \mathbb{R}$ be defined by*

$$G(a) = \max\{F(a,b) : b \in B\}.$$

Then G is continuous.

Proof. Given any $a \in X$, $\epsilon > 0$, we need to find a neighborhood $U_{a,\epsilon}$ of a so that $x \in U_{a,\epsilon} \implies |G(x) - G(a)| < \epsilon$.

For any $(a,b) \in \{a\} \times B$, there is some product neighborhood $U_b \times V_b$ of (a,b) in $X \times B$ so that $(x,y) \in U_b \times V_b \implies |F(x,y) - F(a,b)| < \frac{\epsilon}{2}$. The compact set $\{a\} \times B$ can be covered by finitely many such neighborhoods, $U_{b_1} \times V_{b_1}, \ldots, U_{b_n} \times V_{b_n}$. The claim is that defining $U_{a,\epsilon} = \bigcap_{k=1}^{n} U_{b_k}$ is the required neighborhood of a.

$G(a) = F(a,c_1)$ for some $c_1 \in B$, so $(a,c_1) \in U_{b_k} \times V_{b_k}$ for some k. For any $x \in U_{a,\epsilon}$, $x \in U_{b,k} \implies (x,c_1) \in U_{b_k} \times V_{b_k}$, so

$$\begin{aligned}
|F(x,c_1) - F(a,c_1)| &= |F(x,c_1) - F(a,b_k) + F(a,b_k) - F(a,c_1)| \\
&\leq |F(x,c_1) - F(a,b_k)| + |F(a,b_k) - F(a,c_1)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\
\implies G(a) - \epsilon &= F(a,c_1) - \epsilon < F(x,c_1) \leq G(x).
\end{aligned}$$

For the same x, $G(x) = F(x, c_2)$ for some $c_2 \in B$, so $(a, c_2) \in U_{b_j} \times V_{b_j}$ for some j, and $x \in U_{a,\epsilon} \subseteq U_{b_j} \implies (x, c_2) \in U_{b_j} \times V_{b_j}$, so

$$\begin{aligned} |F(x,c_2) - F(a,c_2)| &= |F(x,c_2) - F(a,b_j) + F(a,b_j) - F(a,c_2)| \\ &\leq |F(x,c_2) - F(a,b_j)| + |F(a,b_j) - F(a,c_2)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \\ \implies G(x) &= F(x,c_2) < F(a,c_2) + \epsilon \leq G(a) + \epsilon. \end{aligned}$$

■

Theorem 14.5. *For any continuous action of a compact group Γ on a compact metric space X without a fixed point, there is an orbit with a positive minimum diameter.*

Proof. Denote the distance function on X by d, and the action of Γ on X by $A : \Gamma \times X \to X$. At any point $x \in X$, the diameter of the orbit containing x is

$$D(x) = \max\{d(A(g_1, x), A(g_2, x)) : (g_1, g_2) \in \Gamma \times \Gamma\}.$$

Since the function

$$F : X \times (\Gamma \times \Gamma) \to \mathbb{R} : F(x, (g_1, g_2)) = d(A(g_1, x), A(g_2, x))$$

is continuous, and $\Gamma \times \Gamma$ is compact, Theorem 14.4 applies, and $D : X \to \mathbb{R}$ is continuous. Since D achieves its minimum value in $[0, \infty)$ on the compact space X, and there is no x with $D(x) = 0$ (since there is no fixed point), D must achieve a positive minimum at some point x, so the orbit through x has the minimum diameter. ■

For example, a transverse **T**-action on Ω as in Definition 12.3 has no fixed point in $b\Omega$, and $b\Omega$ is compact with metric induced from the standard metric on $\mathbb{R}^4$, so the circles that are orbits of the action have a positive minimum diameter.

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