ADDENDUM TO: CR SINGULARITIES OF REAL THREEFOLDS IN $\mathbb{C}^4$

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Last update: May 27, 2018.

6. Updates

The article and its Reference [5] have been reviewed as noted in bibliographic items [C$_3$], [C$_2$], below. Reference [4] has also been indexed in MR: [C$_1$]. Reference [6] has appeared as [C$_4$], and the web address printed in this entry is now obsolete.

My contact information has changed, and my current web page can be found at: `http://users.pfw.edu/CoffmanA/`

7. Citations

The article is cited in these papers: [B$_1$], [B$_2$].

The following Sections of this addendum include some calculations which are lower-degree special cases of some of the comparisons of coefficients that appear in [C$_3$].

8. Addendum to: Section 3, Higher-order terms in the non-degenerate case

The following Subsections could be read in [C$_3$] at the beginning of Section 3, as the construction of a solution $\vec{p}$ in terms of $\vec{e}$ in the case when $\vec{e}$ has cubic terms. This is a special case of the calculation that appears in Section 3.

8.1. **Stabilizer of the non-degenerate normal form.**

We consider M satisfying the first non-degeneracy condition, and in a normal form:

$$\begin{aligned} y_2 &= E_2(z_1, \bar{z}_1, x_2) = O(3) \\ z_3 &= \bar{z}_1^2 + e_3(z_1, \bar{z}_1, x_2) \\ z_4 &= (z_1 + \varepsilon x_2)\bar{z}_1 + e_4(z_1, \bar{z}_1, x_2). \end{aligned} \tag{8.1}$$

with ε either 1 or 0, corresponding to whether M satisfies the second non-degeneracy condition (type (I)) or not (type (II)). To see what transformations will be available to normalize some higher degree terms, we will find the group of transformations which leaves (8.1) invariant.

We start with transformations that have some effect on the quadratic terms, and whose linear part is as in (2.7):

$$\begin{array}{rcl}\tilde{z}_1 & = & c_1z_1 + c_2z_2 \\ \tilde{z}_2 & = & r_2z_2 + p_2(z_1, z_2, z_3, z_4) \\ \tilde{z}_3 & = & c_3z_3 + c_4z_4 + p_3(z_1, z_2, z_3, z_4) \\ \tilde{z}_4 & = & c_5z_3 + c_6z_4 + p_4(z_1, z_2, z_3, z_4),\end{array} \tag{8.2}$$

where p_2 has only weight 2 terms and p_3 and p_4 have only quadratic weight 2 terms. Any terms of higher weight, or any further terms in the $\tilde{z}_1$ expression, would contribute only terms of degree 3 or higher in the new defining equations. The transformation corresponding to Lemma 2.3 is given by $z_1 = \frac{1}{c_1}\tilde{z}_1 - \frac{c_2}{c_1r_2}\tilde{x}_2 + \phi_1(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}_2)$, $x_2 = \frac{1}{r_2}\tilde{x}_2 + \phi_3(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}_2)$.

Let $p_4(z_1, z_2, z_3, z_4) = p_4^{20}z_1^2 + p_4^{11}z_1z_2 + p_4^{02}z_2^2$. A stabilizing transformation should leave no quadratic terms in the following expression:

$$\begin{aligned}
& \tilde{z}_3 - \bar{\tilde{z}}_1^2 \\
= \;& c_3z_3 + c_4z_4 + p_3 - \overline{(c_1z_1 + c_2z_2)}^2 \\
= \;& c_3(\bar{z}_1^2 + e_3) + c_4((z_1 + \varepsilon x_2)\bar{z}_1 + e_4) + p_3(z_1, x_2 + iH_2, h_3, h_4) \\
& -\overline{(c_1z_1 + c_2z_2)}^2 \\
= \;& (c_3 - \overline{c_1}^2)\bar{z}_1^2 + (c_4\varepsilon - 2\overline{c_1c_2})\bar{z}_1x_2 + c_4z_1\bar{z}_1 - \overline{c_2}^2x_2^2 + p_3(z_1, x_2, 0, 0) + O(3) \\
= \;& \frac{c_3 - \overline{c_1}^2}{\overline{c_1}^2}\bar{\tilde{z}}_1^2 + \frac{c_4}{c_1\overline{c_1}}\tilde{z}_1\bar{\tilde{z}}_1 + \frac{p_3^{20}}{c_1^2}\tilde{z}_1^2 + \frac{\varepsilon c_1\overline{c_1}c_4 - 2c_1\overline{c_2}c_3 - \overline{c_1}c_2c_4}{c_1\overline{c_1}^2r_2}\bar{\tilde{z}}_1\tilde{x}_2 \\
& + \frac{p_3^{11}c_1\overline{c_1} - 2p_3^{20}\overline{c_1}c_2 - c_1\overline{c_2}c_4}{c_1^2\overline{c_1}r_2}\tilde{z}_1\tilde{x}_2 \\
& + \frac{p_3^{20}\overline{c_1}^2c_2^2 - p_3^{11}c_1\overline{c_1}^2c_2 + p_3^{02}c_1^2\overline{c_1}^2 + c_1^2\overline{c_2}^2c_3 - \varepsilon c_1^2\overline{c_1c_2}c_4 + c_1\overline{c_1}c_2\overline{c_2}c_4}{c_1^2\overline{c_1}^2r_2^2}\tilde{x}_2^2 \\
& + \tilde{O}(3).
\end{aligned}$$

The only $\tilde{z}_1\bar{\tilde{z}}_1$ term has coefficient $\frac{c_4}{c_1\overline{c_1}}$, so a stabilizing transformation must have $c_4 = 0$, and similarly $p_3^{20} = 0$. The $\bar{\tilde{z}}_1^2$ term has coefficient $\frac{c_3 - \overline{c_1}^2}{\overline{c_1}^2}$, so $c_3 = \overline{c_1}^2$. Making these substitutions into the remaining terms gives the quantity

$$\frac{-2\overline{c_2}}{r_2}\bar{\tilde{z}}_1\tilde{x}_2 + \frac{p_3^{11}}{c_1r_2}\tilde{z}_1\tilde{x}_2 + \frac{p_3^{02}c_1 - p_3^{11}c_2 + c_1\overline{c_2}^2}{c_1r_2^2}\tilde{x}_2^2 + \tilde{O}(3).$$

So, $c_2 = p_3^{11} = 0$ and this implies $p_3^{02} = 0$.

For the next step, we use transformation (8.2), with $c_2 = 0$, so $\tilde{z}_1 = c_1 z_1$.

$$\begin{aligned}
&\tilde{z}_4 - (\tilde{z}_1 + \varepsilon\tilde{x}_2)\bar{\tilde{z}}_1 \\
= \; & c_5 z_3 + c_6 z_4 + p_4 - (c_1 z_1 + \varepsilon \mathrm{Re}(r_2 z_2 + p_2))\overline{(c_1 z_1)} \\
= \; & c_5(\bar{z}_1^2 + e_3) + c_6((z_1 + \varepsilon x_2)\bar{z}_1 + e_4) + p_4(z_1, x_2 + iH_2, h_3, h_4) \\
& -(c_1 z_1 + \varepsilon \mathrm{Re}(r_2(x_2 + iH_2) + p_2))\overline{(c_1 z_1)} \\
= \; & c_5 \bar{z}_1^2 + \varepsilon(c_6 - r_2\overline{c_1})\bar{z}_1 x_2 + (c_6 - c_1\overline{c_1}) z_1 \bar{z}_1 + p_4(z_1, x_2, 0, 0) + O(3) \\
= \; & \frac{c_5}{\overline{c_1}^2}\bar{\tilde{z}}_1^2 + \frac{c_6 - c_1\overline{c_1}}{c_1\overline{c_1}}\tilde{z}_1\bar{\tilde{z}}_1 + \frac{\varepsilon(c_6 - \overline{c_1} r_2)}{\overline{c_1} r_2}\bar{\tilde{z}}_1\tilde{x}_2 \\
& + \frac{p_4^{20}}{c_1^2}\tilde{z}_1^2 + \frac{p_4^{11}}{c_1 r_2}\tilde{z}_1\tilde{x}_2 + \frac{p_4^{02}}{r_2^2}\tilde{x}_2^2 + \tilde{O}(3).
\end{aligned}$$

The only $\bar{\tilde{z}}_1^2$ term has coefficient $\frac{c_5}{\overline{c_1}^2}$, so $c_5 = 0$, and similarly $p_4^{20} = p_4^{11} = p_4^{02} = 0$. The $\tilde{z}_1\bar{\tilde{z}}_1$ term has coefficient $\frac{c_6 - c_1\overline{c_1}}{c_1\overline{c_1}}$, so $c_6 = c_1\overline{c_1}$. The only remaining $\bar{\tilde{z}}_1 x_2$ term has coefficient $\frac{\varepsilon(c_1 - r_2)}{r_2}$, so if $\varepsilon = 1$, then $c_1 = r_2$.

For the final step, consider the transformation (8.2) with $c_2 = 0$ and

$$p_2(\vec{z}) = p_2^1 z_3 + p_2^2 z_4 + p_2^3 z_1^2 + p_2^4 z_1 z_2 + p_2^5 z_2^2.$$

$$\begin{aligned}
\mathrm{Im}(\tilde{z}_2) \; &= \; \mathrm{Im}(r_2 z_2 + p_2) \\
&= \; \mathrm{Im}(r_2 x_2 + i r_2 H_2) + \mathrm{Im}(p_2(z_1, x_2 + iH_2, h_3, h_4)) \\
&= \; r_2 H_2(z_1, \bar{z}_1, x_2) \\
& \quad + \mathrm{Im}(p_2^1(\bar{z}_1^2 + e_3) + p_2^2((z_1 + \varepsilon x_2)\bar{z}_1 + e_4) \\
& \quad + p_2^3 z_1^2 + p_2^4 z_1(x_2 + iH_2) + p_2^5(x_2 + iH_2)^2) \\
&= \; \frac{p_2^1 - \overline{p_2^3}}{2i\overline{c_1}^2}\bar{\tilde{z}}_1^2 + \frac{p_2^3 - \overline{p_2^1}}{2ic_1^2}\tilde{z}_1^2 + \frac{p_2^2 - \overline{p_2^2}}{2ic_1\overline{c_1}}\tilde{z}_1\bar{\tilde{z}}_1 \\
& \quad + \frac{\varepsilon\overline{p_2^2} - p_2^4}{2i\overline{c_1} r_2}\bar{\tilde{z}}_1 x_2 + \frac{\overline{p_2^4} - \varepsilon p_2^2}{2ic_1 r_2}\tilde{z}_1\tilde{x}_2 + \frac{p_2^5 - \overline{p_2^5}}{2ir_2^2}\tilde{x}_2^2 + \tilde{O}(3).
\end{aligned}$$

For all the quadratic terms to be 0, it follows that p_2^2 and p_2^5 are real, $p_2^3 = \overline{p_2^1}$, and $p_2^4 = \varepsilon\overline{p_2^2}$.

We arrive at the following conclusions in the two cases. If $\varepsilon = 1$, the stabilizer of the quadratic normal form is the set of transformations of the form:

$$\begin{aligned}
\tilde{z}_1 &= r z_1 + p_1(\vec{z}) \\
\tilde{z}_2 &= r z_2 + p_2^1 z_3 + p_2^2 z_4 + \overline{p_2^1} z_1^2 + \overline{p_2^2} z_1 z_2 + p_2^5 z_2^2 + p_2(\vec{z}) \\
\tilde{z}_3 &= r^2 z_3 + p_3(\vec{z}) \\
\tilde{z}_4 &= r^2 z_4 + p_4(\vec{z}),
\end{aligned} \tag{8.3}$$

where r is a non-zero real number, p_2^2 and p_2^5 are real, $p_1(\vec{z})$ has weight ≥ 2, and p_2, p_3, p_4 have weight ≥ 3. If $\varepsilon = 0$, the stabilizer is given by:

$$\begin{aligned}
\tilde{z}_1 &= c_1 z_1 + p_1(\vec{z}) \\
\tilde{z}_2 &= r_2 z_2 + p_2^1 z_3 + p_2^2 z_4 + \overline{p_2^1} z_1^2 + p_2^5 z_2^2 + p_2(\vec{z}) \\
\tilde{z}_3 &= \overline{c_1}^2 z_3 + p_3(\vec{z}) \\
\tilde{z}_4 &= c_1\overline{c_1} z_4 + p_4(\vec{z}),
\end{aligned} \tag{8.4}$$

where c_1 is a non-zero complex number, r_2 is a non-zero real number, p_2^2, p_2^5 are real, and the functions p_1, p_2, p_3, p_4 have weights as in the previous case.

8.2. **The cubic normal form in the non-degenerate case.**

We consider M with a type (I) normal form ((8.1), with $\varepsilon = 1$). In analogy with (2.4), any terms of the form $z_1^3, z_1^2x_2, z_1x_2^2, x_2^3$ can be eliminated from the h_3 and h_4 series, without introducing any other quadratic or cubic terms in h_3 or h_4, by a weight 3 transformation of the form $\tilde{z}_3 = z_3 + p_3(z_1, z_2, 0, 0)$, $\tilde{z}_4 = z_4 + p_4(z_1, z_2, 0, 0)$. However, we leave these terms in the expression (8.6) since even if they are eliminated as a first step, they may re-appear when other terms are eliminated. So, to deal simultaneously with all ten cubic terms in each of h_3, h_4, we consider a coordinate transformation of the form $\tilde{z}_1 = z_1 + p_1$, $\tilde{z}_2 = z_2 + p_2$, $\tilde{z}_3 = z_3 + p_3$, $\tilde{z}_4 = z_4 + p_4$, with

$$\begin{aligned}
(8.5)\quad p_1 &= p_1^1 z_3 + p_1^2 z_1^2 + p_1^3 z_1 z_2 + p_1^4 z_2^2 \\
p_2 &= p_2^1 z_1 z_3 + p_2^2 z_2 z_4 + p_2^3 z_1^3 + p_2^4 z_1^2 z_2 + p_2^5 z_1 z_2^2 + p_2^6 z_2^3 \\
p_3 &= p_3^1 z_1 z_3 + p_3^2 z_1 z_4 + p_3^3 z_2 z_3 + p_3^4 z_2 z_4 + p_3^5 z_1^3 + p_3^6 z_1^2 z_2 + p_3^7 z_1 z_2^2 + p_3^8 z_2^3 \\
p_4 &= p_4^1 z_1 z_3 + p_4^2 z_1 z_4 + p_4^3 z_2 z_3 + p_4^4 z_2 z_4 + p_4^5 z_1^3 + p_4^6 z_1^2 z_2 + p_4^7 z_1 z_2^2 + p_4^8 z_2^3,
\end{aligned}$$

so p_1 has weight 2 and p_2, p_3, p_4 have weight 3, and although there is a linear term in p_1, Lemma 2.3 applies. Since it fits the form of (8.3), such a transformation does not affect the quadratic terms of H_2, h_3 or h_4, as considered in Section 8.1. For points on M,

$$\begin{aligned}
(8.6)\qquad & \tilde{z}_3 - \bar{\tilde{z}}_1^2 \\
&= z_3 + p_3(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}^2 \\
&= e_3 + p_3 - 2\bar{z}_1\overline{p_1} - \overline{p_1}^2 \\
&= e_3^{210} z_1^2\bar{z}_1 + e_3^{120} z_1\bar{z}_1^2 + e_3^{111} z_1\bar{z}_1 x_2 + e_3^{030}\bar{z}_1^3 + e_3^{021}\bar{z}_1^2 x_2 + e_3^{012}\bar{z}_1 x_2^2 \\
&\quad + e_3^{300} z_1^3 + e_3^{201} z_1^2 x_2 + e_3^{102} z_1 x_2^2 + e_3^{003} x_2^3 \\
&\quad + p_3(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1) - 2\bar{z}_1\overline{p_1(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1)} + O(4) \\
&= e_3^{210} z_1^2\bar{z}_1 + e_3^{120} z_1\bar{z}_1^2 + e_3^{111} z_1\bar{z}_1 x_2 + e_3^{030}\bar{z}_1^3 + e_3^{021}\bar{z}_1^2 x_2 + e_3^{012}\bar{z}_1 x_2^2 \\
&\quad + e_3^{300} z_1^3 + e_3^{201} z_1^2 x_2 + e_3^{102} z_1 x_2^2 + e_3^{003} x_2^3 \\
&\quad + p_3^1 z_1\bar{z}_1^2 + p_3^2 z_1(z_1 + x_2)\bar{z}_1 + p_3^3 x_2\bar{z}_1^2 + p_3^4 x_2(z_1 + x_2)\bar{z}_1 \\
&\quad + p_3^5 z_1^3 + p_3^6 z_1^2 x_2 + p_3^7 z_1 x_2^2 + p_3^8 x_2^3 \\
&\quad - 2\bar{z}_1\overline{(p_1^1\bar{z}_1^2 + p_1^2 z_1^2 + p_1^3 z_1 x_2 + p_1^4 x_2^2)} + O(4),
\end{aligned}$$

$$
\begin{aligned}
(8.7)\qquad & \tilde{z}_4 - (\tilde{z}_1 + \tilde{x}_2)\bar{\tilde{z}}_1 \\
= \quad & z_4 + p_4(\vec{z}) - (z_1 + p_1(\vec{z}) + x_2 + \mathrm{Re}(p_2(\vec{z})))\overline{(z_1 + p_1(\vec{z}))} \\
= \quad & e_4 + p_4 - (z_1 + x_2)\overline{p_1} - \bar{z}_1(p_1 + \mathrm{Re}(p_2)) - p_1\overline{p_1} - \mathrm{Re}(p_2)\overline{p_1} \\
= \quad & e_4^{210} z_1^2\bar{z}_1 + e_4^{120} z_1\bar{z}_1^2 + e_4^{111} z_1\bar{z}_1 x_2 + e_4^{030}\bar{z}_1^3 + e_4^{021}\bar{z}_1^2 x_2 + e_4^{012}\bar{z}_1 x_2^2 \\
& + e_4^{300} z_1^3 + e_4^{201} z_1^2 x_2 + e_4^{102} z_1 x_2^2 + e_4^{003} x_2^3 \\
& + p_4(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1) \\
& - (z_1 + x_2)\overline{p_1(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1)} - \bar{z}_1 p_1(z_1, x_2, \bar{z}_1^2, (z_1 + x_2)\bar{z}_1) + O(4) \\
= \quad & e_4^{210} z_1^2\bar{z}_1 + e_4^{120} z_1\bar{z}_1^2 + e_4^{111} z_1\bar{z}_1 x_2 + e_4^{030}\bar{z}_1^3 + e_4^{021}\bar{z}_1^2 x_2 + e_4^{012}\bar{z}_1 x_2^2 \\
& + e_4^{300} z_1^3 + e_4^{201} z_1^2 x_2 + e_4^{102} z_1 x_2^2 + e_4^{003} x_2^3 \\
& + p_4^1 z_1\bar{z}_1^2 + p_4^2 z_1(z_1 + x_2)\bar{z}_1 + p_4^3 x_2\bar{z}_1^2 + p_4^4 x_2(z_1 + x_2)\bar{z}_1 \\
& + p_4^5 z_1^3 + p_4^6 z_1^2 x_2 + p_4^7 z_1 x_2^2 + p_4^8 x_2^3 \\
& - (z_1 + x_2)\overline{(p_1^1\bar{z}_1^2 + p_1^2 z_1^2 + p_1^3 z_1 x_2 + p_1^4 x_2^2)} \\
& - \bar{z}_1(p_1^1\bar{z}_1^2 + p_1^2 z_1^2 + p_1^3 z_1 x_2 + p_1^4 x_2^2) + O(4).
\end{aligned}
$$

The last line can be converted to $\tilde{z}_1$, $\tilde{x}_2$ coordinates, but this would only introduce higher order terms, without changing any of the cubic coefficients. Inspecting (8.6) and (8.7) shows that the p_2 quantity does not contribute cubic terms to h_3 or h_4. Each of the defining equations h_3 and h_4 has 10 complex cubic coefficients, and we have a choice of 20 complex coefficients in the three components p_1, p_3, p_4 of the coordinate transformation. By comparing like terms in the above quantities, it turns out that we can find coefficients for p_1, p_3, p_4 that eliminate all the cubic terms of h_3 and h_4, by solving a linear system of 20 unknowns (coefficients of $\vec{p}$) given 20 constants (coefficients of e).

The twenty equations are

$$
\begin{aligned}
0 &= e_3^{210} + p_3^2 - 2\overline{p_1^1} \\
0 &= e_3^{021} + p_3^3 - 2\overline{p_1^3} \\
0 &= e_3^{120} + p_3^1 \\
0 &= e_3^{111} + p_3^2 + p_3^4 \\
0 &= e_3^{030} - 2\overline{p_1^2} \\
0 &= e_3^{012} - 2\overline{p_1^4} + p_3^4 \\
0 &= e_3^{300} + p_3^5 \\
0 &= e_3^{201} + p_3^6 \\
0 &= e_3^{102} + p_3^7 \\
0 &= e_3^{003} + p_3^8 \\
0 &= e_4^{210} + p_4^2 - p_1^2 \\
0 &= e_4^{021} - \overline{p_1^2} + p_4^3 \\
0 &= e_4^{012} - \overline{p_1^3} - p_1^4 + p_4^4 \\
0 &= e_4^{120} + p_4^1 - \overline{p_1^2} \\
0 &= e_4^{030} - p_1^1 \\
0 &= e_4^{111} - \overline{p_1^3} - p_1^3 + p_4^2 + p_4^4 \\
0 &= e_4^{300} - \overline{p_1^1} + p_4^5 \\
0 &= e_4^{201} - \overline{p_1^1} + p_4^6 \\
0 &= e_4^{102} - \overline{p_1^4} + p_4^7 \\
0 &= e_4^{003} - \overline{p_1^4} + p_4^8,
\end{aligned}
$$

and the solutions are

$$
\begin{aligned}
p_1^1 &= e_4^{030}\\
p_1^2 &= \frac{1}{2}\overline{e_3^{030}}\\
p_1^3 &= -e_4^{210} + e_4^{111} - e_4^{012} - e_4^{030} + \frac{1}{2}\overline{e_3^{030}} - \frac{1}{2}\overline{e_3^{111}} + \frac{1}{2}\overline{e_3^{210}} + \frac{1}{2}\overline{e_3^{012}}\\
p_1^4 &= -e_4^{030} - \frac{1}{2}\overline{e_3^{111}} + \frac{1}{2}\overline{e_3^{210}} + \frac{1}{2}\overline{e_3^{012}}\\
p_3^1 &= -e_3^{120}\\
p_3^2 &= -e_3^{210} + 2\overline{e_4^{030}}\\
p_3^3 &= e_3^{030} - e_3^{111} + e_3^{210} + e_3^{012} - 2\overline{e_4^{210}} + 2\overline{e_4^{111}} - 2\overline{e_4^{012}} - 2\overline{e_4^{030}} - e_3^{021}\\
p_3^4 &= -e_3^{111} - 2\overline{e_4^{030}} + e_3^{210}\\
p_3^5 &= -e_3^{300}\\
p_3^6 &= -e_3^{201}\\
p_3^7 &= -e_3^{102}\\
p_3^8 &= -e_3^{003}\\
p_4^1 &= \frac{1}{2}e_3^{030} - e_4^{120}\\
p_4^2 &= \frac{1}{2}\overline{e_3^{030}} - e_4^{210}\\
p_4^3 &= \frac{1}{2}e_3^{030} - e_4^{021}\\
p_4^4 &= \frac{1}{2}e_3^{030} - \frac{1}{2}e_3^{111} + \frac{1}{2}e_3^{210} + \frac{1}{2}e_3^{012} - \overline{e_4^{210}} + \overline{e_4^{111}} - \overline{e_4^{012}}\\
&\quad -\overline{e_4^{030}} - e_4^{012} - e_4^{030} - \frac{1}{2}\overline{e_3^{111}} + \frac{1}{2}\overline{e_3^{210}} + \frac{1}{2}\overline{e_3^{012}}\\
p_4^5 &= \overline{e_4^{030}} - e_4^{300}\\
p_4^6 &= \overline{e_4^{030}} - e_4^{201}\\
p_4^7 &= -\frac{1}{2}e_3^{111} + \frac{1}{2}e_3^{210} + \frac{1}{2}e_3^{012} - \overline{e_4^{030}} - e_4^{102}\\
p_4^8 &= -\frac{1}{2}e_3^{111} + \frac{1}{2}e_3^{210} + \frac{1}{2}e_3^{012} - \overline{e_4^{030}} - e_4^{003}.
\end{aligned}
$$

The cubic terms can similarly be eliminated from H_2, using only the p_2 component of the transformation. For points on M,

$$
\begin{aligned}
(8.8)\quad & \operatorname{Im}(\tilde{z}_2) = \operatorname{Im}(z_2 + p_2(\vec{z}))\\
= \;& \operatorname{Im}(x_2 + iH_2(z_1,\bar{z}_1,x_2) + p_2(z_1, x_2 + iH_2, \bar{z}_1^2 + e_3, (z_1+x_2)\bar{z}_1 + e_4))\\
= \;& H_2(z_1,\bar{z}_1,x_2) + \operatorname{Im}(p_2(z_1,x_2,\bar{z}_1^2,(z_1+x_2)\bar{z}_1)) + O(4)\\
= \;& e_2^{210}z_1^2\bar{z}_1 + \overline{e_2^{210}}z_1\bar{z}_1^2 + e_2^{111}z_1\bar{z}_1x_2 + e_2^{300}z_1^3 + \overline{e_2^{300}}\bar{z}_1^3\\
& +\overline{e_2^{201}}\bar{z}_1^2x_2 + \overline{e_2^{102}}\bar{z}_1x_2^2 + e_2^{201}z_1^2x_2 + e_2^{102}z_1x_2^2 + e_2^{003}x_2^3\\
& +\frac{1}{2i}(p_2^1z_1\bar{z}_1^2 + p_2^2x_2(z_1+x_2)\bar{z}_1 + p_2^3z_1^3 + p_2^4z_1^2x_2 + p_2^5z_1x_2^2 + p_2^6x_2^3)\\
& -\frac{1}{2i}\overline{(p_2^1z_1\bar{z}_1^2 + p_2^2x_2(z_1+x_2)\bar{z}_1 + p_2^3z_1^3 + p_2^4z_1^2x_2 + p_2^5z_1x_2^2 + p_2^6x_2^3)} + O(4).
\end{aligned}
$$

The coefficients e_2^{111} and e_2^{003} are real, but the others can be complex, leading to six equations in six unknowns:

$$\begin{aligned}
0 &= e_2^{300}+\frac{1}{2i}p_2^3\\
0 &= e_2^{201}+\frac{1}{2i}p_2^4\\
0 &= e_2^{111}+\frac{1}{2i}(p_2^2-\overline{p_2^2})\\
0 &= \overline{e_2^{210}}+\frac{1}{2i}p_2^1\\
0 &= e_2^{003}+\frac{1}{2i}(p_2^6-\overline{p_2^6})\\
0 &= e_2^{102}+\frac{1}{2i}p_2^5-\frac{1}{2i}\overline{p_2^2}.
\end{aligned} \tag{8.9}$$

The solutions are

$$\begin{aligned}
p_2^1 &= -2i\overline{e_2^{210}}\\
p_2^2 &= -ie_2^{111}\\
p_2^3 &= -2ie_2^{300}\\
p_2^4 &= -2ie_2^{201}\\
p_2^5 &= -2ie_2^{102}+ie_2^{111}\\
p_2^6 &= -ie_2^{003}.
\end{aligned} \tag{8.10}$$

9. Addendum to: Section 4, Some degenerate cases

The following Subsections could be read in [C$_3$] at the beginning of Section 4.

9.1. **Cubic normal forms in a degenerate case.** We consider M with a type (II) normal form (satisfying the first non-degeneracy condition, but not the second: (8.1), with $\varepsilon=0$). To deal simultaneously with all ten cubic terms in each of h_3, h_4, we consider a coordinate transformation of the form $\tilde z_1=z_1+p_1$, $\tilde z_2=z_2+p_2$, $\tilde z_3=z_3+p_3$, $\tilde z_4=z_4+p_4$, with

$$\begin{aligned}
p_1 &= p_1^1z_3+p_1^2z_1^2+p_1^3z_1z_2+p_1^4z_2^2+p_1^5z_4\\
p_2 &= p_2^1z_1z_3+p_2^2z_2z_4+p_2^3z_1^3+p_2^4z_1^2z_2+p_2^5z_1z_2^2+p_2^6z_2^3\\
p_3 &= p_3^1z_1z_3+p_3^2z_1z_4+p_3^3z_2z_3+p_3^4z_2z_4+p_3^5z_1^3+p_3^6z_1^2z_2+p_3^7z_1z_2^2+p_3^8z_2^3\\
p_4 &= p_4^1z_1z_3+p_4^2z_1z_4+p_4^3z_2z_3+p_4^4z_2z_4+p_4^5z_1^3+p_4^6z_1^2z_2+p_4^7z_1z_2^2+p_4^8z_2^3,
\end{aligned}$$

so p_1 has weight 2 and p_2, p_3, p_4 have weight 3. This is similar to (8.5), but the term $p_1^5z_4$ is included, to give all the available weight 2 terms in p_1, and even with both linear terms, Lemma 2.3 applies. Since it fits the form of (8.4), such a transformation does not affect the quadratic terms of H_2, h_3 or h_4, as considered in Section 8.1. For points on M,

$$
\begin{aligned}
(9.1)\qquad & \tilde z_3 - \bar{\tilde z}_1^2 \\
= \;& z_3 + p_3(\vec z) - \overline{(z_1 + p_1(\vec z))}^2 \\
= \;& e_3 + p_3 - 2\bar z_1\overline{p_1} - \overline{p_1}^2 \\
= \;& e_3^{210} z_1^2\bar z_1 + e_3^{120} z_1\bar z_1^2 + e_3^{111} z_1\bar z_1 x_2 + e_3^{030}\bar z_1^3 + e_3^{021}\bar z_1^2 x_2 + e_3^{012}\bar z_1 x_2^2 \\
& + e_3^{300} z_1^3 + e_3^{201} z_1^2 x_2 + e_3^{102} z_1 x_2^2 + e_3^{003} x_2^3 \\
& + p_3(z_1, x_2, \bar z_1^2, z_1\bar z_1) - 2\bar z_1\overline{p_1(z_1, x_2, \bar z_1^2, z_1\bar z_1)} + O(4) \\
= \;& e_3^{210} z_1^2\bar z_1 + e_3^{120} z_1\bar z_1^2 + e_3^{111} z_1\bar z_1 x_2 + e_3^{030}\bar z_1^3 + e_3^{021}\bar z_1^2 x_2 + e_3^{012}\bar z_1 x_2^2 \\
& + e_3^{300} z_1^3 + e_3^{201} z_1^2 x_2 + e_3^{102} z_1 x_2^2 + e_3^{003} x_2^3 \\
& + p_3^1 z_1\bar z_1^2 + p_3^2 z_1^2\bar z_1 + p_3^3 x_2\bar z_1^2 + p_3^4 x_2 z_1\bar z_1 \\
& + p_3^5 z_1^3 + p_3^6 z_1^2 x_2 + p_3^7 z_1 x_2^2 + p_3^8 x_2^3 \\
& - 2\bar z_1\overline{(p_1^1\bar z_1^2 + p_1^2 z_1^2 + p_1^3 z_1 x_2 + p_1^4 x_2^2 + p_1^5 z_1\bar z_1)} + O(4) \\
= \;& (p_3^1 - 2\overline{p_1^5} + e_3^{120})\bar z_1^2 z_1 + (e_3^{201} + p_3^6) z_1^2 x_2 + (p_3^3 - 2\overline{p_1^3} + e_3^{021})\bar z_1^2 x_2 \\
& + (-2\overline{p_1^1} + p_3^2 + e_3^{210})\bar z_1 z_1^2 + (e_3^{003} + p_3^8) x_2^3 + (p_3^4 + e_3^{111}) z_1\bar z_1 x_2 \\
& + (p_3^7 + e_3^{102}) z_1 x_2^2 + (p_3^5 + e_3^{300}) z_1^3 + (-2\overline{p_1^2} + e_3^{030})\bar z_1^3 \\
& + (e_3^{012} - 2\overline{p_1^4}) x_2^2\bar z_1 + O(4),
\end{aligned}
$$

$$
\begin{aligned}
(9.2)\quad & \tilde z_4 - \tilde z_1\bar{\tilde z}_1 \\
= \;& z_4 + p_4(\vec z) - (z_1 + p_1(\vec z))\overline{(z_1 + p_1(\vec z))} \\
= \;& e_4 + p_4 - z_1\overline{p_1} - \bar z_1 p_1 - p_1\overline{p_1} \\
= \;& e_4^{210} z_1^2\bar z_1 + e_4^{120} z_1\bar z_1^2 + e_4^{111} z_1\bar z_1 x_2 + e_4^{030}\bar z_1^3 + e_4^{021}\bar z_1^2 x_2 + e_4^{012}\bar z_1 x_2^2 \\
& + e_4^{300} z_1^3 + e_4^{201} z_1^2 x_2 + e_4^{102} z_1 x_2^2 + e_4^{003} x_2^3 \\
& + p_4(z_1, x_2, \bar z_1^2, z_1\bar z_1) \\
& - z_1\overline{p_1(z_1, x_2, \bar z_1^2, z_1\bar z_1)} - \bar z_1 p_1(z_1, x_2, \bar z_1^2, z_1\bar z_1) + O(4) \\
= \;& e_4^{210} z_1^2\bar z_1 + e_4^{120} z_1\bar z_1^2 + e_4^{111} z_1\bar z_1 x_2 + e_4^{030}\bar z_1^3 + e_4^{021}\bar z_1^2 x_2 + e_4^{012}\bar z_1 x_2^2 \\
& + e_4^{300} z_1^3 + e_4^{201} z_1^2 x_2 + e_4^{102} z_1 x_2^2 + e_4^{003} x_2^3 \\
& + p_4^1 z_1\bar z_1^2 + p_4^2 z_1^2\bar z_1 + p_4^3 x_2\bar z_1^2 + p_4^4 x_2 z_1\bar z_1 \\
& + p_4^5 z_1^3 + p_4^6 z_1^2 x_2 + p_4^7 z_1 x_2^2 + p_4^8 x_2^3 \\
& - z_1\overline{(p_1^1\bar z_1^2 + p_1^2 z_1^2 + p_1^3 z_1 x_2 + p_1^4 x_2^2 + p_1^5 z_1\bar z_1)} \\
& - \bar z_1(p_1^1\bar z_1^2 + p_1^2 z_1^2 + p_1^3 z_1 x_2 + p_1^4 x_2^2 + p_1^5 z_1\bar z_1) + O(4) \\
= \;& (-p_1^1 + e_4^{030})\bar z_1^3 + (e_4^{120} - \overline{p_1^2} - p_1^5 + p_4^1)\bar z_1^2 z_1 + (e_4^{300} + p_4^5 - \overline{p_1^1}) z_1^3 \\
& + (p_4^3 + e_4^{021})\bar z_1^2 x_2 + (p_4^8 + e_4^{003}) x_2^3 + (p_4^7 + e_4^{102} - \overline{p_1^4}) z_1 x_2^2 \\
& + (p_4^4 - \overline{p_1^3} + e_4^{111} - p_1^3)\bar z_1 z_1 x_2 + (e_4^{210} - \overline{p_1^5} - p_1^2 + p_4^2)\bar z_1 z_1^2 + (p_4^6 + e_4^{201}) z_1^2 x_2 \\
& + (e_4^{012} - p_1^4) x_2^2\bar z_1 + O(4)
\end{aligned}
$$

The last line can be converted to $\tilde z_1$, $\tilde x_2$ coordinates, but this would only introduce higher order terms, without changing any of the cubic coefficients. Inspecting (9.1) and (9.2) shows that the p_2 quantity does not contribute cubic terms to h_3 or h_4. Each of the defining equations h_3 and h_4 has 10 complex cubic coefficients, and we have a choice of 21 complex coefficients in the three components p_1, p_3, p_4 of the coordinate transformation. By inspecting the $\bar{z}_1 x_2^2$ terms, we see that the only coefficient which transforms these coefficients is p_1^4, but it is unlikely to cancel both e_3 and e_4. However, if we choose to eliminate the $\bar{z}_1 x_2^2$ term from e_3 only, it turns out that we can find coefficients for $\vec{p}$ that also eliminate the remaining 18 cubic terms of h_3 and h_4, and in fact this can be done while choosing $p_1^3 = p_1^5 = 0$.

All the cubic terms can be eliminated from E_2 using p_2, in a way similar to (but even simpler than) the calculation from Section 8.2. The six equations are the same except that (8.9) is just $0 = e_2^{102} + \frac{1}{2i} p_2^5$, and the six solutions are the same, except that (8.10) is just $p_2^5 = -2ie_2^{102}$. In fact, a real-valued series in z_1, $\bar{z}_1$, x_2 can always be written, formally, as the real or imaginary part of some series in z_1, x_2, $\bar{z}_1^2$, $z_1\bar{z}_1$, so it is easy to find $p_2(z_1, x_2, \bar{z}_1^2, z_1\bar{z}_1)$ that eliminates the lowest-degree terms in E_2, without the rearrangements that were used in Section 3.

So, after the transformation, the defining equations become

$$\begin{aligned} y_2 &= E_2(z_1, \bar{z}_1, x_2) \\ z_3 &= \bar{z}_1^2 + e_3(z_1, \bar{z}_1, x_2) \\ z_4 &= z_1\bar{z}_1 + e_4^{012}\bar{z}_1 x_2^2 + e_4(z_1, \bar{z}_1, x_2), \end{aligned} \tag{9.3}$$

where E_2, e_3, e_4 have degree 4. A transformation of the form $\tilde{z}_1 = c_1 z_1$, $\tilde{z}_2 = z_2$, $\tilde{z}_3 = \overline{c_1}^2 z_3$, $\tilde{z}_4 = c_1\overline{c_1} z_4$, as in (8.4), changes the coefficient from e_4^{012} to $c_1 e_4^{012}$, so we get two cubic normal forms, corresponding to scaling the $\bar{z}_1 x_2^2$ coefficient to be either 1 or 0.

9.2. **Stabilizer of the cubic degenerate normal form.**

Without going through all the calculations as in Section 8.1, we can state the subgroup of the stabilizer group (8.4), consisting of transformations which preserve the normal form (9.3) with $e_4^{012} = 1$:

$$\begin{aligned} \tilde{z}_1 &= r^2 z_1 + p_{1A} + p_{1B} \\ p_{1A} &= p_1^{11} z_1 z_2 + p_1^0 z_4 \\ \tilde{z}_2 &= r z_2 + p_{2A} + p_{2B} + p_{2C} \\ p_{2A} &= p_2^1 z_3 + p_2^2 z_4 + \overline{p_2^1} z_1^2 + p_2^5 z_2^2 \\ p_{2B} &= p_2^{101} z_1 z_3 + \overline{p_2^{101}} z_1 z_4 + p_2^{011} z_2 z_3 + p_2^{010} z_2 z_4 \\ &\quad + \overline{p_2^{011}} z_1^2 z_2 + p_2^2 z_1 z_2^2 + p_2^{030} z_2^3 \\ \tilde{z}_3 &= r^4 z_3 + p_{3A} + p_{3B} \\ p_{3A} &= 2r^2 \overline{p_1^0} z_1 z_3 + 2r^2 \overline{p_1^{11}} z_2 z_3 \\ \tilde{z}_4 &= r^4 z_4 + p_{4A} + p_{4B} \\ p_{4A} &= r^2 p_1^0 z_1 z_3 + r^2 \overline{p_1^0} z_1 z_4 + r^2 (p_1^{11} + \overline{p_1^{11}}) z_2 z_4, \end{aligned} \tag{9.4}$$

where r is a non-zero real number, p_2^2, p_2^5, p_2^{010}, p_2^{030} are real, $p_{1B}(\vec{z})$ has weight 3, and the functions $p_{2C}(\vec{z})$, $p_{3B}(\vec{z})$, $p_{4B}(\vec{z})$ have weight 4.

9.3. **The quartic normal form for the degenerate case.**

Again considering the normal form (9.3) with $e_4^{012} = 1$, we can in fact eliminate all the degree 4 terms from E_2, e_3, e_4 by a polynomial coordinate change. We begin by considering all transformations of the form (9.4), with $r = 1$.

As in (9.1), for points on M,

$$\begin{aligned} & \tilde{z}_3 - \bar{\tilde{z}}_1^2 \\ = \;& z_3 + p_3(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}^2 \\ = \;& e_3 + p_3 - 2\bar{z}_1\overline{p_1} - \overline{p_1}^2 \\ = \;& e_3 + p_{3A} + p_{3B} - 2\bar{z}_1\overline{(p_{1A} + p_{1B})} - \overline{(p_{1A} + p_{1B})}^2 \\ = \;& e_3^4 + p_{3A} + p_{3B} - 2\bar{z}_1\overline{(p_{1A} + p_{1B})} - \overline{p_{1A}}^2 + O(5), \end{aligned} \tag{9.5}$$

where in line (9.5), e_3^4 represents the degree 4 terms of e_4 and all the cubic terms cancel, by the assumption that the transformation stabilizes the cubic normal form. Similarly, as in (9.2),

$$\begin{aligned} & \tilde{z}_4 - (\tilde{z}_1 + \tilde{x}_2^2)\bar{\tilde{z}}_1 \\ = \;& z_4 + p_4(\vec{z}) - (z_1 + p_1(\vec{z}) + (x_2 + \mathrm{Re}(p_2(\vec{z})))^2)\overline{(z_1 + p_1(\vec{z}))} \\ = \;& e_4 + p_4 - (z_1 + x_2^2)\overline{p_1} - (\bar{z}_1 + \overline{p_1})(p_1 + 2x_2\mathrm{Re}(p_2) + (\mathrm{Re}(p_2))^2) \\ = \;& e_4^4 + p_{4A} + p_{4B} - (p_{1A} + p_{1B} + 2x_2\mathrm{Re}(p_{2A}))\bar{z}_1 - p_{1A}\overline{p_{1A}} \\ & -z_1\overline{(p_{1A} + p_{1B})} - x_2^2\overline{p_{1A}} + O(5). \end{aligned} \tag{9.6}$$

Note that the quantities (9.5) and (9.6) are non-linear in p_{1A}. Rather than give the whole expansion of these expressions, we can consider just the $\bar{z}_1x_2^3$ terms first. If the coefficients of p_{1A} and p_{2A} are as in (9.4), the $\bar{z}_2^3$ term of p_{1B} has coefficient p_1^{030}, and the $\bar{z}_1x_2^3$ terms of e_3 and e_4 are e_3^{013} and e_4^{013}, then the coefficient of the $\bar{z}_1x_2^3$ term in (9.5) is $e_3^{013} - 2\overline{p_1^{030}}$ and in (9.6) is $e_4^{013} - p_1^{030} - 2\mathrm{Re}(p_2^5) - \overline{p_1^{11}}$. We can choose p_1^{030} to eliminate the term from e_3, but to simultaneously eliminate it from e_4, the real number $\mathrm{Re}(p_2^5)$ may not be enough, so we would need to use p_1^{11}.

However, all the other terms in e_3^4 and e_4^4 can be eliminated by p_{1B}, p_{3B}, p_{4B}. Specifically, for $k = 3, 4$,

$$\begin{aligned} e_k^4 \;=\;& e_k^{400}z_1^4 + e_k^{310}z_1^3\bar{z}_1 + e_k^{301}z_1^3x_2 + e_k^{220}z_1^2\bar{z}_1^2 + e_k^{211}z_1^2\bar{z}_1x_2 \\ & +e_k^{202}z_1^2x_2^2 + e_k^{130}z_1\bar{z}_1^3 + e_k^{121}z_1\bar{z}_1^2x_2 + e_k^{112}z_1\bar{z}_1x_2^2 + e_k^{103}z_1x_2^3 \\ & +e_k^{040}\bar{z}_1^4 + e_k^{031}\bar{z}_1^3x_2 + e_k^{022}\bar{z}_1^2x_2^2 + e_k^{013}\bar{z}_1x_2^3 + e_k^{004}x_2^4, \end{aligned}$$

so there are 30 coefficients, which can be eliminated by a transformation of the following form, a special case of (9.4):

$$\begin{aligned} \tilde{z}_1 &= z_1 + p_1^{11}z_1x_2 + p_1 \\ p_1 &= p_1^{210}z_1^2z_2 + p_1^{030}z_2^3 + p_1^{011}z_2z_3 \\ \tilde{z}_2 &= z_2 + p_2(z_1, z_2, z_3, z_4) \\ \tilde{z}_3 &= z_3 + 2\overline{p_1^{11}}z_2z_3 + p_3 \\ p_3 &= p_3^{001}z_3z_4 + p_3^{200}z_1^2z_4 + p_3^{110}z_1z_2z_4 + p_3^{020}z_2^2z_4 + p_3^{400}z_1^4 \\ & \quad +p_3^{310}z_1^3z_2 + p_3^{220}z_1^2z_2^2 + p_3^{201}z_1^2z_3 + p_3^{111}z_1z_2z_3 + p_3^{021}z_2^2z_3 \\ & \quad +p_3^{130}z_1z_2^3 + p_3^{040}z_2^4 + p_3^{002}z_3^2 \\ \tilde{z}_4 &= z_4 + (p_1^{11} + \overline{p_1^{11}})z_2z_4 + p_4 \\ p_4 &= p_4^{001}z_3z_4 + p_4^{200}z_1^2z_4 + p_4^{110}z_1z_2z_4 + p_4^{020}z_2^2z_4 + p_4^{400}z_1^4 \\ & \quad +p_4^{310}z_1^3z_2 + p_4^{220}z_1^2z_2^2 + p_4^{201}z_1^2z_3 + p_4^{111}z_1z_2z_3 + p_4^{021}z_2^2z_3 \\ & \quad +p_4^{130}z_1z_2^3 + p_4^{040}z_2^4 + p_4^{002}z_3^2. \end{aligned} \tag{9.7}$$

The quantity p_1 does not depend on z_4 and has three out of six weight 3 terms, omitting z_1^3, $z_1z_2^2$, and z_1z_3. The quantities p_3 and p_4 have all 13 weight 4 terms. Together with the previously mentioned p_1^{11}, this gives exactly 30 unknowns, enough to find the following solution that eliminates all the quartic terms from (9.5), (9.6):

$$
\begin{aligned}
p_1^{11} &= \frac{1}{2}\overline{e_3^{013}} - e_4^{013} \\
p_1^{011} &= e_4^{031} \\
p_1^{210} &= \frac{1}{2}\overline{e_3^{031}} \\
p_1^{030} &= \frac{1}{2}\overline{e_3^{013}} \\
p_3^{201} &= -e_3^{220} \\
p_3^{111} &= -e_3^{121} \\
p_3^{001} &= -e_3^{130} \\
p_3^{020} &= -e_3^{112} \\
p_3^{040} &= -e_3^{004} \\
p_3^{200} &= -e_3^{310} \\
p_3^{002} &= -e_3^{040} \\
p_3^{400} &= -e_3^{400} \\
p_3^{110} &= -e_3^{211} + 2\overline{e_4^{031}} \\
p_3^{130} &= -e_3^{103} \\
p_3^{310} &= -e_3^{301} \\
p_3^{220} &= -e_3^{202} \\
p_3^{021} &= -e_3^{022} + \frac{1}{4}(e_3^{013})^2 - e_3^{013}\overline{e_4^{013}} + (\overline{e_4^{013}})^2 \\
p_4^{220} &= -e_4^{202} \\
p_4^{200} &= -e_4^{310} \\
p_4^{001} &= -e_4^{130} \\
p_4^{310} &= -e_4^{301} + \overline{e_4^{031}} \\
p_4^{002} &= -e_4^{040} \\
p_4^{400} &= -e_4^{400} \\
p_4^{110} &= \frac{1}{2}\overline{e_3^{031}} - e_4^{211} \\
p_4^{040} &= -e_4^{004} \\
p_4^{111} &= -e_4^{121} + \frac{1}{2}e_3^{031} \\
p_4^{201} &= -e_4^{220} \\
p_4^{130} &= -e_4^{103} + \frac{1}{2}e_3^{013} \\
p_4^{021} &= -e_4^{022} \\
p_4^{020} &= -\frac{1}{2}e_4^{013}e_3^{013} + e_4^{013}\overline{e_4^{013}} + \frac{1}{4}\overline{e_3^{013}}e_3^{013} - \frac{1}{2}\overline{e_3^{013}e_4^{013}} - e_4^{112}.
\end{aligned}
\tag{9.8}
$$

The non-linear solutions for p_3^{021} and p_4^{020} reflect the appearance of non-linear terms in (9.5) and (9.6). The quartic terms can also be eliminated from E_2 using a weight 4 function p_2, as in Section 8.2.

References

[B$_1$] V. Burcea, *A normal form for a real 2-codimensional submanifold in $\mathbb{C}^{N+1}$ near a CR singularity*, Advances in Mathematics **243** (2013), 262–295. MR 3062747.

[B$_2$] V. Burcea, *Normal forms and degenerate CR singularities*, Complex Var. Elliptic Equ. (9) **61** (2016), 1314–1333. MR 3508264.

[C$_1$] A. Coffman, *Enumeration and Normal Forms of Singularities in Cauchy-Riemann Structures*, dissertation, University of Chicago, 1997. MR 2716702.

[C$_2$] A. Coffman, *Analytic normal form for CR singular surfaces in $\mathbb{C}^3$*, Houston J. Math (4) **30** (2004), 969–996. MR 2110245 (2006d:32048), Zbl 1074.32013.

[C$_3$] A. Coffman, *CR singularities of real threefolds in $\mathbb{C}^4$*, Advances in Geometry (1) **6** (2006), 109–137. MR 2242865 (2007d:32033), Zbl 1120.32024.

[C$_4$] A. Coffman, *Real equivalence of complex matrix pencils and complex projections of real Segre varieties*, Electronic Journal of Linear Algebra **17** (2008), 651–698. MR 2470963 (2010c:14003), Zbl 1173:15003.

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