Notes on series in several variables

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These notes are elementary derivations of well-known, but sometimes hard to find, facts on series in several variables. By "elementary" I mean "avoiding the theory of complex differentiation and integration," and the basic ideas of the proofs will be natural generalizations of the first-year calculus treatment of power series in one variable. I will also avoid issues of "uniformity," even though this is the usual approach to some of the theorems. Some books which state some related facts on multi-indexed series are [D] and [GF].

1 Multi-indexed series

Notation 1.1.

- $\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$ is the set of whole numbers (so $\mathbb{N} \subseteq \mathbb{W} \subseteq \mathbb{Z}$).
- $n \in \mathbb{N}$ will be a fixed natural number.
- An element $\boldsymbol{\alpha} \in \mathbb{W}^n$ is a "multi-index." The "order" of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Sometimes to emphasize the number of terms the order will be written $|\boldsymbol{\alpha}|_n$.
- $(\mathbb{K}, |\ |)$ will be either of the fields $\mathbb{R}$ or $\mathbb{C}$, with the usual absolute value and complex conjugation ($z \mapsto \bar{z}$).
- $(\mathbf{B}, \|\ \|)$ will be a Banach space over $\mathbb{K}$.

Definition 1.2. A "multi-indexed sequence in $\mathbf{B}$" is a function

$$c : \mathbb{W}^n \to \mathbf{B} : \boldsymbol{\alpha} \mapsto c_{\boldsymbol{\alpha}}.$$

Definition 1.3. If the set

$$V_c = \{\sum_{\boldsymbol{\alpha} \in \Lambda} \|c_{\boldsymbol{\alpha}}\| : \Lambda \subseteq \mathbb{W}^n,\ \Lambda \text{ finite}\}$$

is a bounded subset of $\mathbb{R}$, we will say "c forms a convergent multi-indexed series."

It looks like an analogue of "absolutely convergent series," but since there is no canonical way to order $\mathbb{W}^n$ for $n > 1$, we won't bother with "conditionally convergent" series, where even when $n = 1$ the sum depends on the ordering.

Theorem 1.4. *If c forms a convergent multi-indexed series, then there exists an element $L \in \mathbf{B}$ with the following property: for any $\epsilon_1 > 0$, there is some $N_1 \in \mathbb{N}$ such that if $N_2 \geq N_1$, then*

$$\left\| \left(\sum_{k=0}^{N_2} \left(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}} \right) \right) - L \right\| < \epsilon_1.$$

Further, L is unique and satisfies $\|L\| \leq \mathrm{lub} V_c$.

Proof. Let β be the least upper bound of the set V_c. Then, given any $\epsilon_2 > 0$, there's some finite set $\Lambda \subseteq \mathbb{W}^n$ such that

$$\beta - \epsilon_2 < \sum_{\boldsymbol{\alpha} \in \Lambda} \|c_{\boldsymbol{\alpha}}\| \leq \beta.$$

Let $N_3 = \max\{|\boldsymbol{\alpha}| : \boldsymbol{\alpha} \in \Lambda\}$. Then,

$$\begin{aligned}
N_4 \geq N_3 \quad &\Longrightarrow \quad \beta - \epsilon_2 < \sum_{\boldsymbol{\alpha} \in \Lambda} \|c_{\boldsymbol{\alpha}}\| \leq \sum_{k=0}^{N_4} \left(\sum_{|\boldsymbol{\alpha}|=k} \|c_{\boldsymbol{\alpha}}\| \right) \leq \beta, \\
N_5 \geq N_4 \geq N_3 \quad &\Longrightarrow \quad \left\| \left(\sum_{k=0}^{N_5} \left(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}} \right) \right) - \left(\sum_{k=0}^{N_4} \left(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}} \right) \right) \right\| \\
&\qquad = \left\| \sum_{k=N_4+1}^{N_5} \left(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}} \right) \right\| \leq \sum_{k=N_4+1}^{N_5} \left(\sum_{|\boldsymbol{\alpha}|=k} \|c_{\boldsymbol{\alpha}}\| \right) \\
&\qquad = \left(\sum_{k=0}^{N_5} \left(\sum_{|\boldsymbol{\alpha}|=k} \|c_{\boldsymbol{\alpha}}\| \right) \right) - \left(\sum_{k=0}^{N_4} \left(\sum_{|\boldsymbol{\alpha}|=k} \|c_{\boldsymbol{\alpha}}\| \right) \right) \\
&\qquad < \beta - (\beta - \epsilon_2) = \epsilon_2.
\end{aligned}$$

This implies that as a sequence depending on N, $\displaystyle\sum_{k=0}^{N} \left(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}} \right)$ is a Cauchy sequence in $\mathbf{B}$, so it converges to some $L \in \mathbf{B}$. The uniqueness of L is the usual uniqueness of a limit, and the bound for $\|L\|$ is given, for $N_2 \geq N_1$, by:

$$\|L\| \leq \left\| \left(\sum_{k=0}^{N_2} \left(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}} \right) \right) - L \right\| + \left(\sum_{k=0}^{N_2} \left(\sum_{|\boldsymbol{\alpha}|=k} \|c_{\boldsymbol{\alpha}}\| \right) \right) < \epsilon_1 + \beta.$$

∎

Notation 1.5. If c forms a convergent multi-indexed series, and $L \in \mathbf{B}$ is the element from the previous Theorem, the following abbreviations make sense:

$$\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} = \sum c_{\boldsymbol{\alpha}} = L.$$

The idea of the Theorem and this Notation is that we can group the multi-indexed series by its "homogeneous" parts, to get a well-defined "sum" of the series. The Theorem also relates the multi-indexed series $\sum_{\boldsymbol{\alpha}}$ to a single-indexed series $\sum_{k=0}^{\infty}$, as defined in first-year calculus. It will usually be convenient to denote the partial sums:

$$\sum_{k=0}^{N}\left(\sum_{|\boldsymbol{\alpha}|=k} c_{\boldsymbol{\alpha}}\right) = \sum_{|\boldsymbol{\alpha}|\le N} c_{\boldsymbol{\alpha}}.$$

To approximate the sum L by a finite partial sum, it is obviously not sufficient to consider arbitrary finite index sets Λ, but the following two Theorems generalize Theorem 1.4 by showing that it is sufficient to consider finite sets that contain "enough" of the lower-order terms.

Theorem 1.6. *If c forms a convergent multi-indexed series, then there exists a unique element $L \in \mathbf{B}$ with the following property: for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that if $\Lambda \subseteq \mathbb{W}^n$ is a finite set and $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \le N\} \subseteq \Lambda$, then*

$$\left\|\left(\sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}}\right) - L\right\| < \epsilon.$$

Proof. Let L be as in Theorem 1.4, and let $\epsilon > 0$. Then, corresponding to $\epsilon_1 = \epsilon/2 > 0$, there's some $N_1 \in \mathbb{N}$ such that if $N_2 \ge N_1$, then

$$\left\|\left(\sum_{|\boldsymbol{\alpha}|\le N_2} c_{\boldsymbol{\alpha}}\right) - L\right\| < \epsilon/2.$$

Also as in Theorem 1.4, corresponding to $\epsilon_2 = \epsilon/2$, there's some N_3 so that

$$N_4 \ge N_3 \implies \beta - \epsilon/2 < \sum_{|\boldsymbol{\alpha}|\le N_4} \|c_{\boldsymbol{\alpha}}\| \le \beta.$$

Let $N = \max\{N_1, N_3\}$, and, for any finite Λ containing $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \le N\}$, let

$N_5 = \max\{|\boldsymbol{\alpha}| : \boldsymbol{\alpha} \in \Lambda\} \geq N \geq N_3$. Then,

$$\begin{aligned}
\left\| \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\boldsymbol{\alpha}} \right) - L \right\| &= \left\| \left(\sum_{|\boldsymbol{\alpha}| \leq N} c_{\boldsymbol{\alpha}} \right) - L + \sum_{\substack{\boldsymbol{\alpha} \in \Lambda \\ |\boldsymbol{\alpha}| > N}} c_{\boldsymbol{\alpha}} \right\| \\
&\leq \left\| \left(\sum_{|\boldsymbol{\alpha}| \leq N} c_{\boldsymbol{\alpha}} \right) - L \right\| + \sum_{\substack{\boldsymbol{\alpha} \in \Lambda \\ |\boldsymbol{\alpha}| > N}} \|c_{\boldsymbol{\alpha}}\| \\
&\leq \left\| \left(\sum_{|\boldsymbol{\alpha}| \leq N} c_{\boldsymbol{\alpha}} \right) - L \right\| + \sum_{N < |\boldsymbol{\alpha}| \leq N_5} \|c_{\boldsymbol{\alpha}}\| \\
&< \epsilon/2 + \epsilon/2 = \epsilon.
\end{aligned}$$

For the uniqueness, suppose L_1 and L_2 have the claimed property. Then, for any $\epsilon > 0$, there's some N so that if Λ is finite and $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N\} \subseteq \Lambda$, then

$$\left\| \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\alpha} \right) - L_1 \right\| < \frac{\epsilon}{2},$$

and there's some N' so that if $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N'\} \subseteq \Lambda$, then

$$\left\| \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\alpha} \right) - L_2 \right\| < \frac{\epsilon}{2}.$$

Let $N'' = \max\{N, N'\}$, so that if $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N''\} \subseteq \Lambda$, then

$$\begin{aligned}
\|L_1 - L_2\| &= \left\| L_1 - \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\alpha} \right) + \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\alpha} \right) - L_2 \right\| \\
&\leq \left\| \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\alpha} \right) - L_1 \right\| + \left\| \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\alpha} \right) - L_2 \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

$\blacksquare$

Theorem 1.7. *If c forms a convergent multi-indexed series with sum L, and $\sigma : \mathbb{W} \to \mathbb{W}^n$ is any bijection, then*

$$\sum_{k=0}^{\infty} c_{\sigma(k)} = L.$$

Proof. Given any $\epsilon > 0$, let N be the corresponding number from the previous Theorem. Then, $\sigma^{-1}(\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N\})$ is a finite subset of $\mathbb{W}$, with largest element M_1. For any $M_2 \geq M_1$, let $\Lambda = \{\sigma(1), \ldots, \sigma(M_2)\}$, a finite subset of $\mathbb{W}^n$ such that $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N\} = \sigma(\sigma^{-1}(\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N\})) \subseteq \sigma(\{1, \ldots, M_1\}) \subseteq \Lambda$. So,

$$\left\| \left(\sum_{k=0}^{M_2} c_{\sigma(k)} \right) - L \right\| = \left\| \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\boldsymbol{\alpha}} \right) - L \right\| < \epsilon.$$

■

Theorem 1.8 (Easy Comparison)**.** *If* $(\mathbf{B}_1, \|\ \|_1)$ *and* $(\mathbf{B}_2, \|\ \|_2)$ *are Banach spaces, and* $c_{\boldsymbol{\alpha}}$ *is a multi-indexed sequence in* $\mathbf{B}_1$ *that forms a convergent multi-indexed series, and* $b_{\boldsymbol{\alpha}}$ *is a multi-indexed sequence in* $\mathbf{B}_2$ *such that* $\|b_{\boldsymbol{\alpha}}\|_2 \leq \|c_{\boldsymbol{\alpha}}\|_1$ *for all but finitely many* $\boldsymbol{\alpha} \in \mathbb{W}^n$*, then* $b_{\boldsymbol{\alpha}}$ *also forms a convergent multi-indexed series.*

Proof. Let U be any upper bound for V_c, and let Φ be a fixed finite set such that $\|b_{\boldsymbol{\alpha}}\|_2 > \|c_{\boldsymbol{\alpha}}\|_1 \implies \boldsymbol{\alpha} \in \Phi$. Then, the set V_b is bounded: for any finite $\Lambda \subseteq \mathbb{W}^n$,

$$\begin{aligned}
\sum_{\boldsymbol{\alpha} \in \Lambda} \|b_{\boldsymbol{\alpha}}\|_2 &= \left(\sum_{\boldsymbol{\alpha} \in \Lambda \setminus \Phi} \|b_{\boldsymbol{\alpha}}\|_2 \right) + \left(\sum_{\boldsymbol{\alpha} \in \Lambda \cap \Phi} \|b_{\boldsymbol{\alpha}}\|_2 \right) \\
&\leq \left(\sum_{\boldsymbol{\alpha} \in \Lambda \setminus \Phi} \|c_{\boldsymbol{\alpha}}\|_1 \right) + \left(\sum_{\boldsymbol{\alpha} \in \Phi} \|b_{\boldsymbol{\alpha}}\|_2 \right) \leq U + \left(\sum_{\boldsymbol{\alpha} \in \Phi} \|b_{\boldsymbol{\alpha}}\|_2 \right).
\end{aligned}$$

■

Corollary 1.9. *Given any set* $\Gamma \subseteq \mathbb{W}^n$*, and a multi-indexed sequence in* $\mathbf{B}$*,* $c_{\boldsymbol{\alpha}}$*, define another multi-indexed sequence in* $\mathbf{B}$*:*

$$d_\alpha = \begin{cases} c_{\boldsymbol{\alpha}} & \text{if } \boldsymbol{\alpha} \in \Gamma \\ 0 & \text{if } \boldsymbol{\alpha} \notin \Gamma \end{cases}.$$

If $c_{\boldsymbol{\alpha}}$ *forms a convergent multi-indexed series, then so does* $d_{\boldsymbol{\alpha}}$*.* ■

Notation 1.10. If $c_{\boldsymbol{\alpha}}$ forms a convergent multi-indexed series, and Γ and $d_{\boldsymbol{\alpha}}$ are as in the previous Corollary, with sum M, denote

$$\sum_{\boldsymbol{\alpha} \in \Gamma} c_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} d_{\boldsymbol{\alpha}} = M.$$

Theorem 1.11 (Comparison with Estimate)**.** *Given* $b_{\boldsymbol{\alpha}}$*, a multi-indexed sequence in* $\mathbf{B}$*, and* $c_{\boldsymbol{\alpha}}$*, a multi-indexed sequence in* $\mathbb{R}$*, if* $\|b_{\boldsymbol{\alpha}}\| \leq c_{\boldsymbol{\alpha}}$ *for all* $\boldsymbol{\alpha} \in \mathbb{W}^n$ *and* $\sum c_{\boldsymbol{\alpha}} = \lambda$*, then* $b_{\boldsymbol{\alpha}}$ *forms a convergent multi-indexed series, with sum* $L \in \mathbf{B}$ *such that* $\|L\| \leq \lambda$*.*

Proof. Note that the hypothesis implies $c_{\boldsymbol{\alpha}} = |c_{\boldsymbol{\alpha}}|$. Let $\beta = \mathrm{lub} V_c$, as in the Proof of Theorem 1.4, so that for any $\epsilon_2 > 0$, there is some N_3 such that if $N_4 \geq N_3$, then

$$\begin{aligned} \beta - \epsilon_2 \quad &< \quad \sum_{|\boldsymbol{\alpha}| \leq N_4} c_{\boldsymbol{\alpha}} \leq \beta \\ &\Longrightarrow \quad \left| \left(\sum_{|\boldsymbol{\alpha}| \leq N_4} c_{\boldsymbol{\alpha}} \right) - \beta \right| < \epsilon_2. \end{aligned}$$

This implies $\beta = \lambda$, by the uniqueness of the sum from Theorem 1.4. For any finite $\Lambda \subseteq \mathbb{W}^n$,

$$\sum_{\boldsymbol{\alpha} \in \Lambda} \|b_{\boldsymbol{\alpha}}\| \leq \sum_{\boldsymbol{\alpha} \in \Lambda} c_{\boldsymbol{\alpha}} \leq \lambda.$$

This shows $b_{\boldsymbol{\alpha}}$ forms a convergent multi-indexed series, with $\mathrm{lub} V_b \leq \lambda$. The inequality $\|L\| \leq \lambda$ follows from the bound from Theorem 1.4. ∎

Theorem 1.12. *If* $\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} = L$, *and* $\sigma : \mathbb{W}^m \to 2^{\mathbb{W}^n}$ *has the property that*

$$\mathbb{W}^n = \bigcup_{\boldsymbol{\gamma} \in \mathbb{W}^m} \sigma(\boldsymbol{\gamma})$$

is a disjoint union, then

$$\sum_{\boldsymbol{\gamma} \in \mathbb{W}^m} \left(\sum_{\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})} c_{\boldsymbol{\alpha}} \right) = L.$$

Proof. (Step 1, establishing convergence.) For each $\boldsymbol{\gamma} \in \mathbb{W}^m$, denote by $d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}}$ the multi-indexed sequence in $\mathbf{B}$ corresponding to Corollary 1.9, applied to $c_{\boldsymbol{\alpha}}$ and $\sigma(\boldsymbol{\gamma})$. Then $d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}}$ forms a convergent multi-indexed series, and as in the above Notation, denote for each $\boldsymbol{\gamma}$,

$$\sum_{\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})} c_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} = L_{\boldsymbol{\gamma}}.$$

Given a finite, non-empty subset $\Lambda \subseteq \mathbb{W}^m$ with $\#\Lambda$ elements, Theorem 1.4 applies to $\epsilon = \frac{1}{\#\Lambda} > 0$, giving $N_1(\boldsymbol{\gamma}, \Lambda) \in \mathbb{N}$ so that if $N_2 \geq N_1(\boldsymbol{\gamma}, \Lambda)$, then

$$\left\| \left(\sum_{|\boldsymbol{\alpha}| \leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) - L_{\boldsymbol{\gamma}} \right\| < \frac{1}{\#\Lambda}.$$

If $N_2 \geq N_1(\Lambda) = \max\{N_1(\boldsymbol{\gamma}, \Lambda) : \boldsymbol{\gamma} \in \Lambda\}$, then

$$\begin{aligned}
\sum_{\boldsymbol{\gamma}\in\Lambda} \left\| \sum_{\boldsymbol{\alpha}\in\sigma(\boldsymbol{\gamma})} c_{\boldsymbol{\alpha}} \right\| &= \sum_{\boldsymbol{\gamma}\in\Lambda} \|L_{\boldsymbol{\gamma}}\| \\
&= \sum_{\boldsymbol{\gamma}\in\Lambda} \left\| L_{\boldsymbol{\gamma}} - \left(\sum_{|\boldsymbol{\alpha}|\leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) + \left(\sum_{|\boldsymbol{\alpha}|\leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) \right\| \\
&< \left(\sum_{\boldsymbol{\gamma}\in\Lambda} \frac{1}{\#\Lambda} \right) + \sum_{\boldsymbol{\gamma}\in\Lambda} \left(\sum_{|\boldsymbol{\alpha}|\leq N_2} \|d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}}\| \right) \\
&= 1 + \sum_{\mathit{finite}} \|c_{\boldsymbol{\alpha}}\| \leq 1 + \beta,
\end{aligned}$$

the last step using the disjointness property of σ, and the lub β as in Theorem 1.4.

(Step 2, establishing the value of the limit.) Let $\epsilon > 0$. Denote

$$\sum_{\boldsymbol{\gamma}\in\mathbb{W}^m} \left(\sum_{\boldsymbol{\alpha}\in\sigma(\boldsymbol{\gamma})} c_{\boldsymbol{\alpha}} \right) = \sum_{\boldsymbol{\gamma}\in\mathbb{W}^m} L_{\boldsymbol{\gamma}} = L_\sigma,$$

with the goal of showing $\|L - L_\sigma\| < \epsilon$. Applying Theorem 1.6 to the hypothesis that $c_{\boldsymbol{\alpha}}$ forms a convergent multi-indexed series with sum L, there's some N corresponding to $\epsilon/3$ so that if Λ is any finite subset of $\mathbb{W}^n$ containing $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N\}$, then

$$\left\| \sum_{\boldsymbol{\alpha}\in\Lambda} c_{\boldsymbol{\alpha}} - L \right\| < \frac{\epsilon}{3}.$$

By the assumed property of σ, for each $\boldsymbol{\alpha} \in \mathbb{W}^n$ there is a unique $\boldsymbol{\gamma} \in \mathbb{W}^m$ so that $\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})$. Let Γ_1 be a finite subset of $\mathbb{W}^m$ so that

$$\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N\} \subseteq \bigcup_{\boldsymbol{\gamma}\in\Gamma_1} \sigma(\boldsymbol{\gamma}).$$

Then, for any $\boldsymbol{\alpha}$ such that $|\boldsymbol{\alpha}| \leq N$, there's some $\boldsymbol{\gamma} \in \Gamma_1$ so that $\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})$, which, by construction, means $c_{\boldsymbol{\alpha}} = d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}}$, and for any $N_2 \geq N$, $c_{\boldsymbol{\alpha}}$ will be exactly one of the terms of

$$\sum_{\boldsymbol{\gamma}\in\Gamma_1} \left(\sum_{|\boldsymbol{\alpha}|\leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right).$$

(The "exactly one" refers to $c_{\boldsymbol{\alpha}}$ as a formal symbol, since of course, some values of the multi-indexed sequence c may repeat, or be equal to 0.) This implies, for any $N_2 \geq N$, and any $\Gamma_2 \subseteq \mathbb{W}^m$ which is finite and contains Γ_1,

$$\left\| \left(\sum_{\boldsymbol{\gamma}\in\Gamma_2} \left(\sum_{|\boldsymbol{\alpha}|\leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) \right) - L \right\| < \frac{\epsilon}{3}. \tag{1}$$

Similarly applying Theorem 1.6 to the multi-indexed sequence $L_{\boldsymbol{\gamma}}$, which was shown to form a convergent multi-indexed series in Step 1, there is some N' so that if $\Gamma_3 \subseteq \mathbb{W}^m$ is a finite set containing $\{\boldsymbol{\gamma} : |\boldsymbol{\gamma}| \leq N'\}$, then

$$\left\| \left(\sum_{\boldsymbol{\gamma} \in \Gamma_3} L_{\boldsymbol{\gamma}} \right) - L_\sigma \right\| < \frac{\epsilon}{3}. \tag{2}$$

In particular, both inequalities (1) and (2) hold for the finite set $\Gamma = \Gamma_1 \cup \{\boldsymbol{\gamma} : |\boldsymbol{\gamma}| \leq N'\}$.

As in Step 1, there is some $N_1(\Gamma) = \max\{N_1(\boldsymbol{\gamma}, \Gamma) : \boldsymbol{\gamma} \in \Gamma\}$ corresponding to the above Γ and $\frac{\epsilon}{3 \cdot \#\Gamma} > 0$, so that if $N_2 \geq N_1(\Gamma)$, then

$$\sum_{\boldsymbol{\gamma} \in \Gamma} \left\| L_{\boldsymbol{\gamma}} - \sum_{|\boldsymbol{\alpha}| \leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right\| < \frac{\epsilon}{3}. \tag{3}$$

Let $N_1 = \max\{N, N_1(\Gamma)\}$, so that for any $N_2 \geq N_1$, inequalities (1), (2), and (3) all hold, and:

$$\begin{aligned}
\|L - L_\sigma\| &= \left\| L - \left(\sum_{\boldsymbol{\gamma} \in \Gamma} L_{\boldsymbol{\gamma}} \right) + \left(\sum_{\boldsymbol{\gamma} \in \Gamma} L_{\boldsymbol{\gamma}} \right) - L_\sigma \right\| \\
&\leq \left\| \sum_{\boldsymbol{\gamma} \in \Gamma} \left(L_{\boldsymbol{\gamma}} - \sum_{|\boldsymbol{\alpha}| \leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) \right\| \\
&\quad + \left\| \left(\sum_{\boldsymbol{\gamma} \in \Gamma} \left(\sum_{|\boldsymbol{\alpha}| \leq N_2} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) \right) - L \right\| \\
&\quad + \left\| \left(\sum_{\boldsymbol{\gamma} \in \Gamma} L_{\boldsymbol{\gamma}} \right) - L_\sigma \right\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}.
\end{aligned}$$

∎

Theorem 1.7 could be considered a special case. The converse statement, that if the double sum converges, then the multi-indexed sum also converges: $\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} = L$, is clearly false. However, under a stronger "absolute convergence" assumption, the following result holds.

Theorem 1.13. *Given a multi-indexed sequence $c_{\boldsymbol{\alpha}}$ in $\mathbf{B}$, and a map σ as in Theorem 1.12, if*

$$\sum_{\boldsymbol{\gamma} \in \mathbb{W}^m} \left(\sum_{\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})} \|c_{\boldsymbol{\alpha}}\| \right)$$

forms a convergent multi-indexed series, with sum $\lambda \in \mathbb{R}$*, then*

$$\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}}$$

and

$$\sum_{\boldsymbol{\gamma} \in \mathbb{W}^m} \left(\sum_{\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})} c_{\boldsymbol{\alpha}} \right)$$

both form convergent multi-indexed series, with the same sum $L \in \mathbf{B}$*, and* $\|L\| \leq \lambda$*.*

Proof. Let $d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}}$ be the multi-indexed sequence in $\mathbf{B}$ as in Notation 1.10, corresponding to the $c_{\boldsymbol{\alpha}}$ terms with indices in the set $\sigma(\boldsymbol{\gamma})$. The hypothesis means that

$$\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} ||d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}}|| = \sum_{\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})} \|c_{\boldsymbol{\alpha}}\|$$

converges, with a sum $\lambda_{\boldsymbol{\gamma}}$, which as in the Proof of Theorem 1.11, is the lub of finite sums of terms $\|c_{\boldsymbol{\alpha}}\|$, $\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})$. Theorem 1.11 then applies to show that

$$\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma})} c_{\boldsymbol{\alpha}}$$

is convergent, with sum $L_{\boldsymbol{\gamma}} \in \mathbf{B}$, and $\|L_{\boldsymbol{\gamma}}\| \leq \lambda_{\boldsymbol{\gamma}}$. The hypothesis also means that $\displaystyle\sum_{\boldsymbol{\gamma} \in \mathbb{W}^m} \lambda_{\boldsymbol{\gamma}} = \lambda$, which by Theorem 1.11 again, implies that $\displaystyle\sum_{\boldsymbol{\gamma} \in \mathbb{W}^m} L_{\boldsymbol{\gamma}}$ is a convergent series, with sum $L \in \mathbf{B}$ such that $\|L\| \leq \lambda$.

To show that $\displaystyle\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}}$ is convergent, let Λ be a finite subset of $\mathbb{W}^n$. Then, there is some finite set Γ so that $\Lambda = \displaystyle\bigcup_{\boldsymbol{\gamma} \in \Gamma} (\Lambda \cap \sigma(\boldsymbol{\gamma}))$, and

$$\sum_{\boldsymbol{\alpha} \in \Lambda} \|c_{\boldsymbol{\alpha}}\| = \sum_{\boldsymbol{\gamma} \in \Gamma} \left(\sum_{\boldsymbol{\alpha} \in \Lambda \cap \sigma(\boldsymbol{\gamma})} \|c_{\boldsymbol{\alpha}}\| \right) \leq \sum_{\boldsymbol{\gamma} \in \Gamma} \lambda_{\boldsymbol{\gamma}} \leq \lambda.$$

By Theorem 1.4, $\displaystyle\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}}$ has sum $L' \in \mathbf{B}$; to show $L' = L$, suppose $\epsilon > 0$. By Theorem 1.6, corresponding to $\epsilon/3 > 0$, there is some $N \in \mathbb{N}$ such that if Λ is a finite subset of $\mathbb{W}^n$ and $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \leq N\} \subseteq \Lambda$, then

$$\left\| \left(\sum_{\boldsymbol{\alpha} \in \Lambda} c_{\boldsymbol{\alpha}} \right) - L' \right\| < \frac{\epsilon}{3}.$$

Also by Theorem 1.4, there is some $N_3 \in \mathbb{N}$ such that if $N_4 \geq N_3$, then

$$\left\| \left(\sum_{|\boldsymbol{\gamma}| \leq N_4} L_{\boldsymbol{\gamma}} \right) - L \right\| < \frac{\epsilon}{3}.$$

We can further pick N_4 large enough so that $\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \le N\} \subseteq \bigcup_{|\boldsymbol{\gamma}| \le N_4} \sigma(\boldsymbol{\gamma})$. Let C be the number of such indices:

$$C = \#\{\boldsymbol{\gamma} \in \mathbb{W}^m : |\boldsymbol{\gamma}| \le N_4\}.$$

For each $\boldsymbol{\gamma}$, there is, corresponding to $\frac{\epsilon}{3C} > 0$, some $N_5(\boldsymbol{\gamma})$ such that if $N_6(\boldsymbol{\gamma}) \ge N_5(\boldsymbol{\gamma})$, then

$$\left\| \left(\sum_{|\boldsymbol{\alpha}| \le N_6(\boldsymbol{\gamma})} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) - L_{\boldsymbol{\gamma}} \right\| < \frac{\epsilon}{3C}.$$

If we choose each $N_6(\boldsymbol{\gamma})$ larger than N, then

$$\{\boldsymbol{\alpha} : |\boldsymbol{\alpha}| \le N\} \subseteq \bigcup_{|\boldsymbol{\gamma}| \le N_4} \{\boldsymbol{\alpha} \in \sigma(\boldsymbol{\gamma}) : |\boldsymbol{\alpha}| \le N_6(\boldsymbol{\gamma})\},$$

and

$$\begin{aligned} \|L - L'\| &\le \left\| L - \sum_{|\boldsymbol{\gamma}| \le N_4} L_{\boldsymbol{\gamma}} \right\| \\ &\quad + \sum_{|\boldsymbol{\gamma}| \le N_4} \left\| \left(\sum_{|\boldsymbol{\alpha}| \le N_6(\boldsymbol{\gamma})} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) - L_{\boldsymbol{\gamma}} \right\| \\ &\quad + \left\| \left(\sum_{|\boldsymbol{\gamma}| \le N_4} \left(\sum_{|\boldsymbol{\alpha}| \le N_6(\boldsymbol{\gamma})} d_{\boldsymbol{\alpha}}^{\boldsymbol{\gamma}} \right) \right) - L' \right\| \\ &< \frac{\epsilon}{3} + C \cdot \frac{\epsilon}{3C} + \frac{\epsilon}{3}. \end{aligned}$$

∎

2 The geometric series

Lemma 2.1. *Given $k \in \mathbb{W}$, the number of multi-indices $\boldsymbol{\alpha} \in \mathbb{W}^n$ such that $|\boldsymbol{\alpha}| = k$ is* $\begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix}$.

Proof. We will first find the number of multi-indices $\boldsymbol{\alpha} \in \mathbb{N}^n$ such that $|\boldsymbol{\alpha}| = K \geq n$. The sum $\alpha_1 + \ldots + \alpha_n = K$ can be visualized as K dots in a row, separated into blocks of size α_i by $n-1$ dividers, for example, $6 = 2+3+1$ is represented:

$$\cdot\cdot|\cdot\cdot\cdot|\cdot$$

Each divider fits between two of the dots, and between any two adjacent dots is at most one divider (since $\alpha_i > 0$). The number of ways to assign $n-1$ dividers to the $K-1$ spaces between the K dots is $\begin{pmatrix} K-1 \\ n-1 \end{pmatrix}$.

The function $(\alpha_1, \ldots, \alpha_n) \mapsto (\alpha_1 + 1, \ldots, \alpha_n + 1)$ is obviously a bijection $\mathbb{W}^n \to \mathbb{N}^n$, which, for any $k \geq 0$, restricts to a bijection from the set of multi-indices of order k in $\mathbb{W}^n$ to the set of multi-indices of order $k+n$ in $\mathbb{N}^n$. Applying the previous paragraph's formula to $K = k+n$ gives the claim of the Lemma. ∎

Theorem 2.2 (Geometric series: convergence)**.** *Given $v \in \mathbf{B}$ and $\mathbf{r} = (r_1, r_2, \ldots, r_n) \in \mathbb{K}^n$ such that $|r_i| < 1$ for $i = 1, \ldots, n$, the multi-indexed sequence in $\mathbf{B}$:*

$$v \cdot \mathbf{r}^{\boldsymbol{\alpha}} = v \cdot r_1^{\alpha_1} \cdot r_2^{\alpha_2} \cdot \ldots \cdot r_n^{\alpha_n}$$

forms a convergent multi-indexed series. Its sum is

$$\sum_{\boldsymbol{\alpha}} v \cdot \mathbf{r}^{\boldsymbol{\alpha}} = v \cdot \prod_{i=1}^{n} \frac{1}{(1-r_i)}.$$

Proof. (Step 1, establishing convergence.) Let $\rho = \max\{|r_1|, \ldots, |r_n|\}$, and given any finite $\Lambda \subseteq \mathbb{W}^n$, let $N = \max\{|\boldsymbol{\alpha}| : \boldsymbol{\alpha} \in \Lambda\}$.

$$\begin{aligned}
\sum_{\boldsymbol{\alpha} \in \Lambda} \|v \cdot \mathbf{r}^{\boldsymbol{\alpha}}\| &= \sum_{\boldsymbol{\alpha} \in \Lambda} \|v\| \cdot |r_1|^{\alpha_1} \cdot |r_2|^{\alpha_2} \cdot \ldots \cdot |r_n|^{\alpha_n} \\
&\leq \|v\| \sum_{k=0}^{N} \left(\sum_{|\boldsymbol{\alpha}|=k} |r_1|^{\alpha_1} \cdot |r_2|^{\alpha_2} \cdot \ldots \cdot |r_n|^{\alpha_n} \right) \\
&\leq \|v\| \sum_{k=0}^{N} \begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix} \rho^k,
\end{aligned}$$

using the previous Lemma. The above finite sum is a partial sum of a single-indexed series, which converges by the Ratio test ([C]):

$$\lim_{k \to \infty} \left| \frac{\begin{pmatrix} k+1+n-1 \\ n-1 \end{pmatrix} \rho^{k+1}}{\begin{pmatrix} k+n-1 \\ n-1 \end{pmatrix} \rho^k} \right| = \lim_{k \to \infty} \frac{k+n}{k+1} \rho = \rho < 1.$$

(Step 2, approximating the geometric series.) The following claim will be proved by induction on n. For any $N \in \mathbb{W}$, there is some multi-indexed sequence in $\mathbb{K}$, $\delta_{\boldsymbol{\alpha}}^{N,n}$, such that $|\delta_{\boldsymbol{\alpha}}^{N,n}| \leq 2^{n-1}$ and

$$\left(\prod_{i=1}^{n}(1-r_i)\right)\sum_{k=0}^{N}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\mathbf{r}^{\boldsymbol{\alpha}}\right) = 1 - \sum_{k=N+1}^{N+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N,n}\mathbf{r}^{\boldsymbol{\alpha}}\right).$$

For $n = 1$, let $\delta_{(\alpha_1)}^{N,1} = 1$ if $\alpha_1 = N + 1$, or 0 otherwise. This works, by the usual calculation:

$$\begin{aligned} LHS &= \left(\prod_{i=1}^{1}(1-r_i)\right)\sum_{k=0}^{N}\left(\sum_{|\boldsymbol{\alpha}|_1=k}\mathbf{r}^{\boldsymbol{\alpha}}\right) = (1-r_1)\sum_{k=0}^{N} r_1^k = 1 - r_1^{N+1}, \\ RHS &= 1 - \sum_{k=N+1}^{N+1}\left(\sum_{|\boldsymbol{\alpha}|_1=k}\delta_{\boldsymbol{\alpha}}^{N,1}\mathbf{r}^{\boldsymbol{\alpha}}\right) = 1 - \delta_{(N+1)}^{N,1} r_1^{N+1}. \end{aligned}$$

Suppose, inductively, that the claim holds for some $n \in \mathbb{N}$. Then, it also holds for $n + 1$, applied to the vector $(r_1, r_2, \ldots, r_n, r_{n+1})$, although we will continue to use the symbol $\mathbf{r}$ for an n-tuple: $(r_1, r_2, \ldots, r_n)$. Starting with the LHS,

$$
\begin{aligned}
&\left(\prod_{i=1}^{n+1}(1-r_i)\right)\sum_{k=0}^{N}\left(\sum_{|\boldsymbol{\alpha}|_{n+1}=k}(r_1,r_2,\ldots,r_n,r_{n+1})^{\boldsymbol{\alpha}}\right)\\
&= (1-r_{n+1})\left(\prod_{i=1}^{n}(1-r_i)\right)\sum_{j=0}^{N}\left(\sum_{k=0}^{N-j}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)r_{n+1}^{j}\\
&= (1-r_{n+1})\sum_{j=0}^{N}\left(1-\sum_{k=N-j+1}^{N-j+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N-j,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)r_{n+1}^{j}\\
&= \left(\sum_{j=0}^{N}\left(1-\sum_{k=N-j+1}^{N-j+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N-j,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)r_{n+1}^{j}\right)\\
&\quad -\left(\sum_{j=0}^{N}\left(1-\sum_{k=N-j+1}^{N-j+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N-j,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)r_{n+1}^{j+1}\right)\\
&= \left(1-\sum_{k=N+1}^{N+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)\\
&\quad +\left(\sum_{j=1}^{N}\left(1-\sum_{k=N-j+1}^{N-j+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N-j,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)r_{n+1}^{j}\right)\\
&\quad -\left(\sum_{j=1}^{N+1}\left(1-\sum_{k=N-(j-1)+1}^{N-(j-1)+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N-(j-1),n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)r_{n+1}^{j}\right)\\
&= 1-\left(\sum_{k=N+1}^{N+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)\\
&\quad +\left(\sum_{j=1}^{N}\left(\left(\sum_{k=N-j+2}^{N-j+1+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N-j+1,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)-\left(\sum_{k=N-j+1}^{N-j+n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{N-j,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)\right)r_{n+1}^{j}\right)\\
&\quad -\left(1-\sum_{k=1}^{n}\left(\sum_{|\boldsymbol{\alpha}|_n=k}\delta_{\boldsymbol{\alpha}}^{0,n}\mathbf{r}^{\boldsymbol{\alpha}}\right)\right)r_{n+1}^{N+1}\\
&= 1-\sum_{k=N+1}^{N+n+1}\left(\sum_{|\boldsymbol{\alpha}|_{n+1}=k}\delta_{\boldsymbol{\alpha}}^{N,n+1}(r_1,r_2,\ldots,r_n,r_{n+1})^{\boldsymbol{\alpha}}\right)=RHS,
\end{aligned}
$$

where $\delta_{\boldsymbol{\alpha}}^{N,n+1}$ is either 0, ± 1, a number from a $\delta^{*,n}$ multi-indexed sequence, or the difference of two of these numbers.

(Step 3, establishing the value of the limit.) If $v=0$, the sum claimed in the Theorem is obvious. If $v\neq 0$, and $\epsilon>0$, then, by the Cauchy property of the convergent series from Step 1, there's some $N_1 \in \mathbb{N}$ so that for all $N \geq N_1$,

$$\sum_{k=N+1}^{N+n} \left(\begin{array}{c} k+n-1 \\ n-1 \end{array} \right) \rho^k < \frac{\prod_{i=1}^{n} |1-r_i|}{2^{n-1}\|v\|} \cdot \epsilon.$$

By the equality from Step 2,

$$\begin{aligned}
& \left| \left(\prod_{i=1}^{n} (1-r_i) \right) \left(\sum_{k=1}^{N} \left(\sum_{|\boldsymbol{\alpha}|=k} \mathbf{r}^{\boldsymbol{\alpha}} \right) \right) - 1 \right| \\
= & \left| \sum_{k=N+1}^{N+n} \left(\sum_{|\boldsymbol{\alpha}|=k} \delta_{\boldsymbol{\alpha}}^{N,n} \mathbf{r}^{\boldsymbol{\alpha}} \right) \right| \\
\leq & \sum_{k=N+1}^{N+n} \left(\sum_{|\boldsymbol{\alpha}|=k} |\delta_{\boldsymbol{\alpha}}^{N,n} \mathbf{r}^{\boldsymbol{\alpha}}| \right) \\
\leq & \sum_{k=N+1}^{N+n} 2^{n-1} \left(\begin{array}{c} k+n-1 \\ n-1 \end{array} \right) \rho^k < \frac{\prod_{i=1}^{n} |1-r_i|}{\|v\|} \cdot \epsilon,
\end{aligned}$$

and this is enough to find the limit from Theorem 1.4:

$$\left\| \left(\sum_{k=1}^{N} \left(\sum_{|\boldsymbol{\alpha}|=k} v \cdot \mathbf{r}^{\boldsymbol{\alpha}} \right) \right) - v \cdot \prod_{i=1}^{n} \frac{1}{(1-r_i)} \right\| < \epsilon.$$

∎

Theorem 2.3 (Geometric series: divergence)**.** *For v, $\mathbf{r}$, as in the previous Theorem, but with $v \neq 0$ and $|r_i| \geq 1$ for some $i = 1, \ldots, n$, $v \cdot \mathbf{r}^{\boldsymbol{\alpha}}$ does not form a convergent multi-indexed series.*

Proof. Finite sets of the form

$$\Lambda = \{(0, 0, \ldots, 0, k, 0, \ldots, 0) : N_1 \leq k \leq N_2\} \subseteq \mathbb{W}^n,$$

with $\alpha_j = 0$ for $j \neq i$, give sums of the form

$$\sum_{\boldsymbol{\alpha} \in \Lambda} \|v \cdot \mathbf{r}^{\boldsymbol{\alpha}}\| = \sum_{k=N_1}^{N_2} \|v\| \cdot |r_i|^k \geq \|v\|(N_2 - N_1 + 1),$$

which are unbounded. (Here, as always, we are using the convention that $r_j^0 = 1$ for any $r_j \in \mathbb{K}$.) ∎

3 Power series

Notation 3.1. For $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$, and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{K}^n$, define the "polydisc with center $\mathbf{a}$ and polyradius $\mathbf{r}$," $\Delta(\mathbf{a}, \mathbf{r}) \subseteq \mathbb{K}^n$, by

$$\Delta(\mathbf{a}, \mathbf{r}) = \{(x_1, \ldots, x_n) \in \mathbb{K}^n : |x_i - a_i| < r_i, i = 1, \ldots, n\}.$$

Note that if some $r_i \leq 0$, then $\Delta(\mathbf{a}, \mathbf{r}) = \emptyset$.

Definition 3.2. For $c_{\boldsymbol{\alpha}}$, a multi-indexed sequence in $\mathbf{B}$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{K}^n$, and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$, denote a multi-indexed sequence in $\mathbf{B}$:

$$c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}} = c_{\boldsymbol{\alpha}} \cdot (x_1 - a_1)^{\alpha_1} \cdot (x_2 - a_2)^{\alpha_2} \cdot \ldots \cdot (x_n - a_n)^{\alpha_n}.$$

If it forms a convergent multi-indexed series, call its sum, $\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$, a "convergent ($\mathbf{B}$-valued) power series." Given $c_{\boldsymbol{\alpha}}$, and $\mathbf{a}$, call the set

$$\{\mathbf{x} : \sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}} \text{ is a convergent power series}\} \subseteq \mathbb{K}^n$$

the "set of convergence of the power series with coefficients $c_{\boldsymbol{\alpha}}$ and center $\mathbf{a}$." Such a set always contains $\mathbf{a}$. Its (possibly empty) interior is the "domain of convergence." If S is any subset of the set of convergence, we will say "the power series $\sum c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$ converges for $\mathbf{x} \in S$."

Theorem 3.3. *If $c_{\boldsymbol{\alpha}}$ is a multi-indexed sequence in $\mathbf{B}$, and $\mathbf{a}$, $\mathbf{y} \in \mathbb{K}^n$, and $\{c_{\boldsymbol{\alpha}}(y_1 - a_1)^{\alpha_1} \cdot \ldots \cdot (y_n - a_n)^{\alpha_n} : \boldsymbol{\alpha} \in \mathbb{W}^n\}$ is a bounded set in $\mathbf{B}$, then $\sum c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$, $\sum \|c_{\boldsymbol{\alpha}}\|(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$, and $\sum \|c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}\|$ all converge for $\mathbf{x} \in \Delta(\mathbf{a}, (|y_1 - a_1|, \ldots, |y_n - a_n|))$.*

Proof. By definition of "bounded," there's some $M \in \mathbb{R}$ so that for all $\boldsymbol{\alpha}$,

$$\|c_{\boldsymbol{\alpha}}(y_1 - a_1)^{\alpha_1} \cdot \ldots \cdot (y_n - a_n)^{\alpha_n}\| = \|c_{\boldsymbol{\alpha}}\| \cdot |y_1 - a_1|^{\alpha_1} \cdot \ldots \cdot |y_n - a_n|^{\alpha_n} \leq M.$$

If $\mathbf{x} \in \Delta(\mathbf{a}, (|y_1 - a_1|, \ldots, |y_n - a_n|))$, then

$$\begin{aligned} \|c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}\| &= |\|c_{\boldsymbol{\alpha}}\|(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}| \\ &= \|c_{\boldsymbol{\alpha}}\| \cdot |x_1 - a_1|^{\alpha_1} \cdot \ldots \cdot |x_n - a_n|^{\alpha_n} \\ &\leq M \cdot \left|\frac{x_1 - a_1}{y_1 - a_1}\right|^{\alpha_1} \cdot \ldots \cdot \left|\frac{x_n - a_n}{y_n - a_n}\right|^{\alpha_n}, \end{aligned}$$

so $\sum c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$, $\sum \|c_{\boldsymbol{\alpha}}\|(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$, and $\sum \|c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}\|$ converge by comparison to the geometric series. ∎

Corollary 3.4. *Given $c_{\boldsymbol{\alpha}}$, $\mathbf{a}$, and $\mathbf{y}$, if $\sum c_{\boldsymbol{\alpha}}(\mathbf{y} - \mathbf{a})^{\boldsymbol{\alpha}}$ is a convergent power series, then the polydisc $\Delta(\mathbf{a}, (|y_1 - a_1|, \ldots, |y_n - a_n|))$ is a subset of the set of convergence of the power series with coefficients $c_{\boldsymbol{\alpha}}$ and center $\mathbf{a}$. The same polydisc is also a subset of the set of convergence of the power series with* *coefficients* $\|c_{\boldsymbol{\alpha}}\|$ *and center* $\mathbf{a}$. *There exists a constant* M *such that for all* $\mathbf{x} \in \Delta(\mathbf{a}, (|y_1 - a_1|, \ldots, |y_n - a_n|))$, *the sum* $\sum c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$ *satisfies*

$$\left\|\sum c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}\right\| \leq \sum \|c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}\| \leq M \prod_{i=1}^{n} \frac{1}{1 - \frac{|x_i - a_i|}{|y_i - a_i|}}.$$

Similarly,

$$\left|\sum \|c_{\boldsymbol{\alpha}}\|(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}\right| \leq \sum \|c_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}\| \leq M \prod_{i=1}^{n} \frac{1}{1 - \frac{|x_i - a_i|}{|y_i - a_i|}}.$$

Proof. The boundedness of the terms follows immediately from the definition of convergent series. The estimates follow from Theorems 1.11 and 2.2. ∎

Notation 3.5. For a multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{W}^n$, we'll use a "prime" to denote $\boldsymbol{\alpha}' = (\alpha_1, \ldots, \alpha_{n-1})$, and then denote $\boldsymbol{\alpha} = (\boldsymbol{\alpha}', \alpha_n)$. Similarly for vectors $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{K}^n$, let $\mathbf{y}' = (y_1, \ldots, y_{n-1})$ and $\mathbf{y} = (\mathbf{y}', y_n)$.

Theorem 3.6. *Given* $n \geq 2$, *a multi-indexed sequence* c *in* $\mathbf{B}$, *a sequence* $b : \mathbb{W} \to \mathbb{K}$, *and* $\mathbf{y} \in \mathbb{K}^n$, *if*

$$\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{y}')^{\boldsymbol{\alpha}'}\right\|$$

forms a convergent multi-indexed series for each $\alpha_n \in \mathbb{W}$, *and*

$$\left\{\left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{y}')^{\boldsymbol{\alpha}'}\right\|\right) \cdot b_{\alpha_n} \cdot y_n^{\alpha_n} : \alpha_n \in \mathbb{W}\right\}$$

is a bounded subset of $\mathbb{K}$, *then, for all* $\mathbf{x} \in \Delta(\mathbf{0}, (|y_1|, \ldots, |y_n|))$,

$$\sum_{\alpha_n \in \mathbb{W}} \left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}\right) \cdot b_{\alpha_n} \cdot x_n^{\alpha_n}$$

and

$$\sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} \cdot b_{\alpha_n} \cdot \mathbf{x}^{\boldsymbol{\alpha}}$$

are both convergent, with the same sum.

Proof.

$$\mathbf{x} \in \Delta(\mathbf{0}, (|y_1|, \ldots, |y_n|)) \implies \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}\right\| \leq \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{y}')^{\boldsymbol{\alpha}'}\right\|,$$

so $\displaystyle\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}$ and $\displaystyle\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}\right\|$ converge by comparison (Theorem 1.11), and

$$\begin{aligned} \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}\right\| \quad &\leq \quad \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{y}')^{\boldsymbol{\alpha}'}\right\| \implies \\ \left|\left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}\right\|\right) b_{\alpha_n} y_n^{\alpha_n}\right| \quad &\leq \quad \left|\left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}', \alpha_n)}(\mathbf{y}')^{\boldsymbol{\alpha}'}\right\|\right) b_{\alpha_n} y_n^{\alpha_n}\right|. \end{aligned}$$

By hypothesis, the RHS is bounded by $M \geq 0$, so

$$\left|\left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}',\alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}\right\|\right) b_{\alpha_n} x_n^{\alpha_n}\right| \leq M \left|\frac{x_n}{y_n}\right|^{\alpha_n}$$

(assuming $y_n \neq 0$, since otherwise the Theorem is trivial). The convergence of the first claimed sum from the Theorem follows from comparison with the single-variable geometric series.

The convergence of

$$\left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \left\|c_{(\boldsymbol{\alpha}',\alpha_n)}(\mathbf{x}')^{\boldsymbol{\alpha}'}\right\|\right) \cdot |b_{\alpha_n} x_n^{\alpha_n}| = \sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \|c_{\boldsymbol{\alpha}} b_{\alpha_n} x^{\boldsymbol{\alpha}}\|$$

for each α_n, and the convergence of

$$\sum_{\alpha_n \in \mathbb{W}} \left(\sum_{\boldsymbol{\alpha}' \in \mathbb{W}^{n-1}} \|c_{\boldsymbol{\alpha}} b_{\alpha_n} x^{\boldsymbol{\alpha}}\|\right)$$

are enough, by Theorem 1.13, to establish the convergence of $\sum_{\boldsymbol{\alpha}} c_{\boldsymbol{\alpha}} b_{\alpha_n} \mathbf{x}^{\boldsymbol{\alpha}}$, and the claimed equality. ∎

Notation 3.7. For any $\boldsymbol{\alpha} \in \mathbb{W}^n$, there exists a multi-indexed sequence in $\mathbb{R}$,

$$\mathbb{W}^n \to \mathbb{R} : \boldsymbol{\beta} \mapsto \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}},$$

with these properties:

- $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \geq 0$,
- If for some i, $\beta_i > \alpha_i$, then $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} = 0$; otherwise, if $\beta_i \leq \alpha_i$ for all $i = 1, \ldots, n$, denote this property of $\boldsymbol{\beta}$ by "$\boldsymbol{\beta} \leq \boldsymbol{\alpha}$."
- For any $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^n$, $(\mathbf{x}+\mathbf{y})^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \mathbf{y}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}$.

We won't need any exact values for $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}}$ until Section 5. It will sometimes be convenient to write

$$\sum_{\boldsymbol{\beta} \leq \boldsymbol{\alpha}} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \mathbf{y}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} = \sum_{\boldsymbol{\beta} \in \mathbb{W}^n} \binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} \mathbf{x}^{\boldsymbol{\beta}} \mathbf{y}^{\boldsymbol{\alpha}-\boldsymbol{\beta}},$$

with the understanding that all terms where "$\boldsymbol{\beta} \leq \boldsymbol{\alpha}$" is false are zero, even though negative exponents formally appear.

Theorem 3.8. *Suppose $\Delta(\mathbf{0},\mathbf{r})$ is a subset of the set of convergence of a power series with coefficients $c_{\boldsymbol{\alpha}}$ and center $\mathbf{0}$, and $\mathbf{a}\in\Delta(\mathbf{0},\mathbf{r})$. Then, there is a multi-indexed sequence in $\mathbf{B}$, $c'_{\boldsymbol{\alpha}}$, so that for all $\mathbf{x}\in\Delta(\mathbf{a},(r_1-|a_1|,\dots,r_n-|a_n|))$, $\sum c'_{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}}$ is a convergent power series, and*

$$\sum c'_{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}}=\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}.$$

Proof. (Step 1, establishing convergence of a multi-indexed series.) Given any $\mathbf{x}\in\Delta(\mathbf{a},(r_1-|a_1|,\dots,r_n-|a_n|))$,

$$|x_i|\le|x_i-a_i|+|a_i|<(r_i-|a_i|)+|a_i|=r_i$$

implies both $\mathbf{x}$ and $(|x_1-a_1|+|a_1|,\dots,|x_n-a_n|+|a_n|)$ are elements of $\Delta(\mathbf{0},\mathbf{r})$, so $\Delta(\mathbf{a},(r_1-|a_1|,\dots,r_n-|a_n|))\subseteq\Delta(\mathbf{0},\mathbf{r})$, the RHS of the claimed equation is a convergent power series, and $\sum c_{\boldsymbol{\alpha}}(|x_1-a_1|+|a_1|,\dots,|x_n-a_n|+|a_n|)^{\boldsymbol{\alpha}}$ is also a convergent power series. By definition, there is some upper bound $U(\mathbf{x})$ for the partial sums:

$$\sum_{\textit{finite}}\|c_{\boldsymbol{\alpha}}\cdot(|x_1-a_1|+|a_1|)^{\alpha_1}\cdot\ldots\cdot(|x_n-a_n|+|a_n|)^{\alpha_n}\|\le U(\mathbf{x}).$$

For $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{W}^n$, let $(\boldsymbol{\alpha},\boldsymbol{\beta})$ denote the element $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n)\in\mathbb{W}^{2n}$. Define a multi-indexed sequence

$$\mathbb{W}^{2n}\to\mathbf{B}:(\boldsymbol{\alpha},\boldsymbol{\beta})\mapsto c_{\boldsymbol{\alpha}}\cdot\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}.$$

It forms a convergent multi-indexed series: let Λ be a finite subset of $\mathbb{W}^{2n}$, and $N=\max\{|\boldsymbol{\alpha}|:(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda\}$. Then

$$\begin{aligned}&\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda}\left\|c_{\boldsymbol{\alpha}}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\right\|\\ \le&\sum_{|\boldsymbol{\alpha}|\le N}\left(\sum_{\boldsymbol{\beta}\le\boldsymbol{\alpha}}\|c_{\boldsymbol{\alpha}}\|\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}|x_1-a_1|^{\beta_1}\cdots|x_n-a_n|^{\beta_n}|a_1|^{\alpha_1-\beta_1}\cdots|a_n|^{\alpha_n-\beta_n}\right)\\ =&\sum_{|\boldsymbol{\alpha}|\le N}\|c_{\boldsymbol{\alpha}}\|\cdot(|x_1-a_1|+|a_1|)^{\alpha_1}\cdot\ldots\cdot(|x_n-a_n|+|a_n|)^{\alpha_n}\le U(\mathbf{x}).\end{aligned}$$

(Step 2., establishing the claimed equality.) Define, as in Theorem 1.12, a map

$$\sigma_1:\mathbb{W}^n\to2^{\mathbb{W}^{2n}}:\boldsymbol{\alpha}\mapsto\{(\boldsymbol{\alpha},\boldsymbol{\beta}):\boldsymbol{\beta}\in\mathbb{W}^n\}.$$

It, and the multi-indexed series from Step 1, satisfy the hypotheses of that

Theorem, so

$$\begin{aligned}
&\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathbb{W}^{2n}} c_{\boldsymbol{\alpha}} \cdot \left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{array}\right)(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \\
= &\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n}\left(\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\sigma_1(\boldsymbol{\alpha})} c_{\boldsymbol{\alpha}} \cdot \left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{array}\right)(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\right) \\
= &\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n}\left(c_{\boldsymbol{\alpha}} \cdot \left(\sum_{\boldsymbol{\beta}\in\mathbb{W}^n} \left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{array}\right)(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\right)\right) \\
= &\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}.
\end{aligned}$$

The Theorem also applies to another map

$$\sigma_2 : \mathbb{W}^n \to 2^{\mathbb{W}^{2n}} : \boldsymbol{\beta} \mapsto \{(\boldsymbol{\alpha},\boldsymbol{\beta}) : \boldsymbol{\alpha} \in \mathbb{W}^n\},$$

to give

$$\begin{aligned}
&\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\mathbb{W}^{2n}} c_{\boldsymbol{\alpha}} \cdot \left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{array}\right)(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} \\
= &\sum_{\boldsymbol{\beta}\in\mathbb{W}^n}\left(\sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\sigma_2(\boldsymbol{\beta})} c_{\boldsymbol{\alpha}} \cdot \left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{array}\right)(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\right) \\
= &\sum_{\boldsymbol{\beta}\in\mathbb{W}^n}\left(\left(\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}} \cdot \left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{array}\right)\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}}\right)(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}}\right).
\end{aligned}$$

Technically, the last expression follows from the previous one only for the terms where $(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}} \neq 0$. Since $\Delta(\mathbf{a},(r_1-|a_1|,\ldots,r_n-|a_n|))$ is non-empty, it has some element $\mathbf{x}$ so that $(\mathbf{x}-\mathbf{a})^{\boldsymbol{\beta}} \neq 0$ for all $\boldsymbol{\beta}$, and we can use this to establish the convergence of

$$\sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}} \cdot \left(\begin{array}{c}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{array}\right)\mathbf{a}^{\boldsymbol{\alpha}-\boldsymbol{\beta}},$$

which defines $c'_{\boldsymbol{\beta}}$ not depending on $\mathbf{x}$. ∎

4 Geometry of the ball

Definition 4.1. A "positive semidefinite Hermitian form" on $\mathbb{K}^n$ is a function $g : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ such that:

- (homogeneity) For all $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^n$, $\lambda \in \mathbb{K}$, $g(\lambda \cdot \mathbf{x}, \mathbf{y}) = \lambda g(\mathbf{x}, \mathbf{y})$.
- (additivity) For all $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbb{K}^n$, $g(\mathbf{x} + \mathbf{y}, \mathbf{z}) = g(\mathbf{x}, \mathbf{z}) + g(\mathbf{y}, \mathbf{z})$.
- (Hermitian symmetry) For all $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^n$, $g(\mathbf{x}, \mathbf{y}) = \overline{g(\mathbf{y}, \mathbf{x})}$. (so, for any $\mathbf{x} \in \mathbb{K}^n$, $g(\mathbf{x}, \mathbf{x}) \in \mathbb{R}$.)
- (positivity) For all $\mathbf{x} \in \mathbb{K}^n$, $g(\mathbf{x}, \mathbf{x}) \geq 0$.

Lemma 4.2 (CBS)**.** *Given a positive semidefinite Hermitian form g, for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^n$,*

$$|g(\mathbf{x}, \mathbf{y})|^2 \leq g(\mathbf{x}, \mathbf{x}) g(\mathbf{y}, \mathbf{y}).$$

Proof. For any λ, $\mu \in \mathbb{K}$,

$$\begin{aligned} 0 &\leq g(\lambda \cdot \mathbf{x} + \mu \cdot \mathbf{y}, \lambda \cdot \mathbf{x} + \mu \cdot \mathbf{y}) \\ &= \lambda\bar{\lambda} g(\mathbf{x}, \mathbf{x}) + \mu\bar{\lambda} g(\mathbf{y}, \mathbf{x}) + \lambda\bar{\mu} g(\mathbf{x}, \mathbf{y}) + \mu\bar{\mu} g(\mathbf{y}, \mathbf{y}). \end{aligned}$$

In particular, for $\lambda = g(\mathbf{y}, \mathbf{y})$ and $\mu = -g(\mathbf{x}, \mathbf{y})$,

$$\begin{aligned} 0 &\leq \lambda\bar{\lambda} g(\mathbf{x}, \mathbf{x}) + \mu\bar{\lambda}(-\bar{\mu}) + \lambda\bar{\mu}(-\mu) + \mu\bar{\mu}\lambda \\ &= \bar{\lambda}(g(\mathbf{x}, \mathbf{x}) g(\mathbf{y}, \mathbf{y}) - |g(\mathbf{x}, \mathbf{y})|^2), \end{aligned}$$

and if $g(\mathbf{y}, \mathbf{y}) \neq 0$, this proves the claim. Similarly, for $\lambda = -g(\mathbf{y}, \mathbf{x})$ and $\mu = g(\mathbf{x}, \mathbf{x})$,

$$\begin{aligned} 0 &\leq \lambda\bar{\lambda}\mu + \mu\bar{\lambda}(-\lambda) + \lambda\bar{\mu}(-\bar{\lambda}) + \mu\bar{\mu} g(\mathbf{y}, \mathbf{y}) \\ &= \bar{\mu}(g(\mathbf{x}, \mathbf{x}) g(\mathbf{y}, \mathbf{y}) - |g(\mathbf{y}, \mathbf{x})|^2), \end{aligned}$$

and if $g(\mathbf{x}, \mathbf{x}) \neq 0$, this proves the claim. Finally, if $g(\mathbf{x}, \mathbf{x}) = g(\mathbf{y}, \mathbf{y}) = 0$, let $\lambda = 1$ and $\mu = -g(\mathbf{x}, \mathbf{y})$, so

$$\begin{aligned} 0 &\leq 0 - g(\mathbf{x}, \mathbf{y}) g(\mathbf{y}, \mathbf{x}) - g(\mathbf{y}, \mathbf{x}) g(\mathbf{x}, \mathbf{y}) + 0 \\ &= -2|g(\mathbf{x}, \mathbf{y})|^2, \end{aligned}$$

proving $g(\mathbf{x}, \mathbf{y}) = 0$, and the claim. ∎

Lemma 4.3 ($\Delta \neq$)**.** *Given a positive semidefinite Hermitian form g, the function*

$$\mathbb{K}^n \to \mathbb{R} : \mathbf{x} \mapsto \|\mathbf{x}\|_g = +\sqrt{g(\mathbf{x}, \mathbf{x})}$$

satisfies, for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^n$,

$$\|\mathbf{x} + \mathbf{y}\|_g \leq \|\mathbf{x}\|_g + \|\mathbf{y}\|_g.$$

Proof.

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|_g^2 &= g(\mathbf{x}+\mathbf{y},\mathbf{x}+\mathbf{y}) \\
&= |g(\mathbf{x},\mathbf{x})+g(\mathbf{y},\mathbf{x})+g(\mathbf{x},\mathbf{y})+g(\mathbf{y},\mathbf{y})| \\
&\leq g(\mathbf{x},\mathbf{x})+g(\mathbf{y},\mathbf{y})+2|g(\mathbf{x},\mathbf{y})| \\
&\leq g(\mathbf{x},\mathbf{x})+g(\mathbf{y},\mathbf{y})+2\sqrt{g(\mathbf{x},\mathbf{x})g(\mathbf{y},\mathbf{y})} \\
&= (\|\mathbf{x}\|_g+\|\mathbf{y}\|_g)^2,
\end{aligned}$$

using the previous Lemma. ∎

Definition 4.4. For $i = 1, \ldots, n$, denote the "reflections in the coordinate hyperplanes"

$$R_i : (x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_1, \ldots, -x_i, \ldots, x_n).$$

A positive semidefinite Hermitian form g is in "standard position" if all of the reflections satisfy the "isometry" equation: for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{K}^n$,

$$g(R_i(\mathbf{x}), R_i(\mathbf{y})) = g(\mathbf{x}, \mathbf{y}).$$

Lemma 4.5. *If g is in standard position, then it is of the form*

$$g(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n g_i x_i \bar{y}_i,$$

for nonnegative real constants $g_1, \ldots, g_n$.

Proof. First, any Hermitian form can be expressed in terms of a matrix, with respect to the usual basis of row vectors $\{\mathbf{e}^i = (0, \ldots, 0, 1, 0, \ldots, 0)\}$. For $\mathbf{x} = \sum x_i \mathbf{e}^i$ and $\mathbf{y} = \sum y_j \mathbf{e}^i$, the linearity properties give

$$g(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \left(\sum_{j=1}^n \bar{y}_j g(\mathbf{e}^i, \mathbf{e}^j) \right) = \mathbf{x} G \overline{\mathbf{y}}^T.$$

The "standard position" hypothesis, applied to the basis vectors, gives, for $j \neq i$,

$$g(\mathbf{e}^i, \mathbf{e}^j) = g(R_i(\mathbf{e}^i), R_i(\mathbf{e}^j)) = g(-\mathbf{e}^i, \mathbf{e}^j) = -g(\mathbf{e}^i, \mathbf{e}^j),$$

so G is a diagonal matrix, with diagonal entries $g_i = g(\mathbf{e}^i, \mathbf{e}^i) \geq 0$. ∎

Notation 4.6. For a positive semidefinite Hermitian form g, denote the "ball with center $\mathbf{a} \in \mathbb{K}^n$ and radius $R \in \mathbb{R}$" by

$$B_g(\mathbf{a}, R) = \{(x_1, \ldots, x_n) : \|(x_1 - a_1, \ldots, x_n - a_n)\|_g < R\} \subseteq \mathbb{K}^n.$$

Geometrically, this shape will be the interior of an ellipsoid (if g is positive definite), or of an ellipsoidal cylinder (if degenerate), or all of $\mathbb{K}^n$ (if $g = 0$).

Lemma 4.7. *If g is in standard position, then any ball $B_g(\mathbf{a}, R)$ is a union of polydiscs with center $\mathbf{a}$.*

Proof. Given $\mathbf{x} \in B_g(\mathbf{a}, R)$, pick any constant ρ such that $\|\mathbf{x} - \mathbf{a}\|_g^2 < \rho^2 < R^2$. Then, pick any $\delta_1, \ldots, \delta_n > 0$ so that $\sum_{i=1}^n g_i \delta_i^2 < R^2 - \rho^2$. Define $\mathbf{r}$ by

$$r_i = \begin{cases} \frac{|x_i - a_i|}{\|\mathbf{x}-\mathbf{a}\|_g} \cdot \rho & \text{if } x_i - a_i \neq 0 \\ \delta_i & \text{if } x_i - a_i = 0. \end{cases}$$

Then $\mathbf{x} \in \Delta(\mathbf{a}, \mathbf{r})$, and $\mathbf{a} + \mathbf{r} \in B_g(\mathbf{a}, R)$:

$$\begin{aligned} \sum_{i=1}^n g_i |a_i + r_i - a_i|^2 &= \sum_{i=1}^n g_i r_i^2 \\ &\leq \sum_{i=1}^n g_i \delta_i^2 + \sum_{i=1}^n g_i \left(\frac{|x_i - a_i|}{\|\mathbf{x} - \mathbf{a}\|_g} \cdot \rho \right)^2 \\ &\leq \sum_{i=1}^n g_i \delta_i^2 + \rho^2 < R^2. \end{aligned}$$

For any element $\mathbf{y} \in \Delta(\mathbf{a}, \mathbf{r})$,

$$\|\mathbf{y} - \mathbf{a}\|_g^2 = \sum_{i=1}^n g_i |y_i - a_i|^2 \leq \sum_{i=1}^n g_i r_i^2 < R^2.$$

So, for any $\mathbf{x} \in B_g(\mathbf{a}, R)$, there is a polydisc such that $\mathbf{x} \in \Delta(\mathbf{a}, \mathbf{r}) \subseteq B_g(\mathbf{a}, R)$. ∎

Theorem 4.8. *Given c, a multi-indexed sequence in $\mathbf{B}$, a complex Banach space, and a vector $\mathbf{a} \in \mathbb{R}^n$, if g is in standard position and $\sum c_{\boldsymbol{\alpha}} (\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$ converges for all $\mathbf{x}$ in a real ball,*

$$\{\mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n g_i (x_i - a_i)^2 < R^2\} = B_g(\mathbf{a}, R) \cap \mathbb{R}^n,$$

then $\sum c_{\boldsymbol{\alpha}} (\mathbf{z} - \mathbf{a})^{\boldsymbol{\alpha}}$ and $\sum \|c_{\boldsymbol{\alpha}}\| (\mathbf{z} - \mathbf{a})^{\boldsymbol{\alpha}}$ converge on the complex ball with the same radius,

$$B_g(\mathbf{a}, R) = \{\mathbf{z} \in \mathbb{C}^n : \sum_{i=1}^n g_i |z_i - a_i|^2 < R^2\}.$$

Proof. Given any complex vector $\mathbf{z} \in B_g(\mathbf{a}, R)$, the real vector $(|z_1 - a_1| + a_1, \ldots, |z_n - a_n| + a_n)$ is an element of $B_g(\mathbf{a}, R) \cap \mathbb{R}^n$. From the Proof of the previous Lemma, there is some $\mathbf{r}$ such that $\mathbf{a} + \mathbf{r} \in B_g(\mathbf{a}, R) \cap \mathbb{R}^n$ and $(|z_1 - a_1| + a_1, \ldots, |z_n - a_n| + a_n) \in \Delta(\mathbf{a}, \mathbf{r})$. It follows that $\mathbf{z}$ is in the complex polydisc $\Delta(\mathbf{a}, \mathbf{r})$. By hypothesis, $\sum c_{\boldsymbol{\alpha}} (\mathbf{a} + \mathbf{r} - \mathbf{a})^{\boldsymbol{\alpha}}$ is convergent, and by Corollary 3.4, $\sum c_{\boldsymbol{\alpha}} (\mathbf{z} - \mathbf{a})^{\boldsymbol{\alpha}}$ and $\sum \|c_{\boldsymbol{\alpha}}\| (\mathbf{z} - \mathbf{a})^{\boldsymbol{\alpha}}$ are also convergent. ∎

Theorem 4.9. *If g is in standard position and $\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ converges on $B_g(\mathbf{0}, R)$, and $\mathbf{a} \in B_g(\mathbf{0}, R)$, then there is some multi-indexed sequence $c'_{\boldsymbol{\alpha}}$ so that for all $\mathbf{x} \in B_g(\mathbf{a}, R - \|\mathbf{a}\|_g)$, $\sum c'_{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}}$ is a convergent power series, with sum equal to $\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$.*

Proof. By Lemma 4.3, $B_g(\mathbf{a}, R - \|\mathbf{a}\|_g) \subseteq B_g(\mathbf{0}, R)$. Given $\mathbf{x} \in B_g(\mathbf{a}, R - \|\mathbf{a}\|_g)$, there is, by the construction of the previous Lemma, some $\mathbf{r} \in \mathbb{R}^n$ such that $\|\mathbf{r}\|_g < R - \|\mathbf{a}\|_g$ and $\mathbf{x} \in \Delta(\mathbf{a}, \mathbf{r})$. The claim is that

$$\Delta(\mathbf{a}, \mathbf{r}) \subseteq \Delta(\mathbf{0}, (|a_1| + r_1, \ldots, |a_n| + r_n)) \subseteq B_g(\mathbf{0}, R).$$

For the first subset, suppose $\mathbf{y} \in \Delta(\mathbf{a}, \mathbf{r})$. Then

$$|y_i| \leq |y_i - a_i| + |a_i| < r_i + |a_i|.$$

For the second subset, suppose $\mathbf{y} \in \Delta(\mathbf{0}, (|a_1| + r_1, \ldots, |a_n| + r_n))$. Then, using the "standard position" hypothesis, and Lemmas 4.5 and 4.2 (CBS),

$$\begin{aligned}
\|y\|_g^2 &= \sum_{i=1}^{n} g_i |y_i|^2 \\
&< \sum_{i=1}^{n} g_i (|a_i| + r_i)^2 \\
&= \|\mathbf{a}\|_g^2 + \|\mathbf{r}\|_g^2 + 2g((|a_1|, \ldots, |a_n|), \mathbf{r}) \\
&\leq (\|\mathbf{a}\|_g + \|\mathbf{r}\|_g)^2 < R^2.
\end{aligned}$$

The Theorem follows from the claimed inclusion: since $\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ converges on $\Delta(\mathbf{0}, (|a_1| + r_1, \ldots, |a_n| + r_n))$, there exist coefficients $c'_{\boldsymbol{\alpha}}$, defining a power series $\sum c'_{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}}$ which converges to $\sum c_{\boldsymbol{\alpha}}\mathbf{x}^{\boldsymbol{\alpha}}$ on $\Delta(\mathbf{a}, \mathbf{r})$, by Theorem 3.8. From the Proof of that Theorem, these coefficients $c'_{\boldsymbol{\alpha}}$ do not depend on $\mathbf{x}$ or the choice of $\mathbf{r}$, so $B_g(\mathbf{a}, R - \|\mathbf{a}\|_g)$ is a subset of the set of convergence of $\sum c'_{\boldsymbol{\alpha}}(\mathbf{x} - \mathbf{a})^{\boldsymbol{\alpha}}$. ∎

5 Functions defined by power series

Theorem 5.1. *If $\sum c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ converges on some polydisc $\Delta(\mathbf{0}, \mathbf{r})$, then the function*

$$f : \Delta(\mathbf{0}, \mathbf{r}) \to \mathbf{B} : \mathbf{x} \mapsto f(\mathbf{x}) = \sum c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$$

is continuous at $\mathbf{a}$ *for all* $\mathbf{a} \in \Delta(\mathbf{0}, \mathbf{r})$.

Proof. "Continuity at the point $\mathbf{a}$" means that for any $\epsilon > 0$, there are positive numbers δ_i, $i = 1, \ldots, n$, so that if $\mathbf{x} \in \Delta(\mathbf{a}, (\delta_1, \ldots, \delta_n))$, then $\|f(\mathbf{x}) - f(\mathbf{a})\| < \epsilon$.

(Step 1, showing continuity at $\mathbf{0}$.) Fix some $\mathbf{w} \in \Delta(\mathbf{0}, \mathbf{r})$, such that $w_i > 0$ for $i = 1, \ldots, n$. Theorem 1.12 applies to the series $\sum c_{\boldsymbol{\alpha}} \mathbf{w}^{\boldsymbol{\alpha}}$ and the map

$$\sigma : \mathbb{W}^1 \to 2^{\mathbb{W}^n} : \begin{cases} (0) & \mapsto & \{\mathbf{0}\} & \\ (1) & \mapsto & \{\boldsymbol{\alpha} : \alpha_1 > 0\} & \\ (i) & \mapsto & \{\boldsymbol{\alpha} : \alpha_1 = \ldots = \alpha_{i-1} = 0, \alpha_i > 0\} & \text{if } 2 \le i \le n \\ (j) & \mapsto & \emptyset & \text{if } j > n \end{cases}$$

to give

$$\begin{aligned} \sum_{\boldsymbol{\alpha} \in \mathbb{W}^n} c_{\boldsymbol{\alpha}} \mathbf{w}^{\boldsymbol{\alpha}} &= c_{\mathbf{0}} + \sum_{i=1}^n \left(\sum_{\boldsymbol{\alpha} \in \sigma(i)} c_{\boldsymbol{\alpha}} \mathbf{w}^{\boldsymbol{\alpha}} \right) \\ &= c_{\mathbf{0}} + \sum_{i=1}^n w_i \left(\sum_{\boldsymbol{\alpha} \in \sigma(i)} c_{\boldsymbol{\alpha}} w_i^{\alpha_i - 1} w_{i+1}^{\alpha_{i+1}} \cdots w_n^{\alpha_n} \right). \end{aligned}$$

For each $i = 1, \ldots, n$, Corollary 3.4 applies to the convergent power series

$$\sum_{\boldsymbol{\alpha} \in \sigma(i)} c_{\boldsymbol{\alpha}} w_i^{\alpha_i - 1} w_{i+1}^{\alpha_{i+1}} \cdots w_n^{\alpha_n},$$

so there's some $M_i > 0$ so that for all $\mathbf{x} \in \Delta(\mathbf{0}, \mathbf{w})$,

$$\left\| \sum_{\boldsymbol{\alpha} \in \sigma(i)} c_{\boldsymbol{\alpha}} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \right\| \le M_i \prod_{i=1}^n \frac{1}{1 - \frac{|x_i|}{w_i}}.$$

Multiplying both sides by $|x_i|$ gives

$$\left\| \sum_{\boldsymbol{\alpha} \in \sigma(i)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \right\| \le |x_i| M_i \prod_{i=1}^n \frac{1}{1 - \frac{|x_i|}{w_i}}.$$

So, given $\epsilon > 0$, let $\delta_i = \min\{\frac{\epsilon}{n 2^n M_i}, \frac{w_1}{2}, \ldots, \frac{w_n}{2}\}$. Then,

$$|x_i| < \delta_i \implies 1 - \frac{|x_i|}{w_i} > \frac{1}{2} \implies \prod_{i=1}^n \frac{1}{1 - \frac{|x_i|}{w_i}} < 2^n,$$

and

$$\|f(\mathbf{x})-f(\mathbf{0})\| = \|f(\mathbf{x})-c_{\mathbf{0}}\| \; = \; \left\| \sum_{i=1}^{n} \left(\sum_{\boldsymbol{\alpha}\in\sigma(i)} c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \right) \right\|$$
$$\leq \sum_{i=1}^{n} \left(|x_i| M_i \prod_{i=1}^{n} \frac{1}{1-\frac{|x_i|}{w_i}} \right) < \epsilon.$$

(Step 2, showing continuity everywhere else.) By Theorem 3.8, for any point $\mathbf{a}\in\Delta(\mathbf{0},\mathbf{r})$, there are coefficients $c'_{\boldsymbol{\alpha}}$, and a polydisc with center $\mathbf{a}$, so that for $\mathbf{x}$ in that polydisc, $\sum c'_{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}}$ converges, with sum $f(\mathbf{x})$. By the construction from the Proof of that Theorem, and the fact that the multinomial coefficient $\begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix}$ has value 1 for all $\boldsymbol{\alpha}$,

$$c'_{\mathbf{0}} = \sum_{\boldsymbol{\alpha}\in\mathbb{W}^n} c_{\boldsymbol{\alpha}} \cdot \begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{0} \end{pmatrix} \mathbf{a}^{\boldsymbol{\alpha}} = f(\mathbf{a}).$$

So, Step 1 applies to show

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = \lim_{\mathbf{x}-\mathbf{a}\to\mathbf{0}} \sum c'_{\boldsymbol{\alpha}}(\mathbf{x}-\mathbf{a})^{\boldsymbol{\alpha}} = c'_{\mathbf{0}} = f(\mathbf{a}).$$

∎

The following Theorem is for single-indexed series, with coefficients $c:\mathbb{W}\to\mathbf{B}$, but Step 2 uses the methods of multi-indexed series (Theorem 3.8).

Theorem 5.2. *If* $\sum_{k=0}^{\infty} c_k z^k$ *converges on some disc* $\{z : |z| < r\} \subseteq \mathbb{K}^1$*, then the (*$\mathbf{B}$*-valued) function* $f(z) = \sum_{k=0}^{\infty} c_k z^k$ *is differentiable at* a *for all* a *in the disc, with* $f'(a) = \sum_{k=1}^{\infty} c_k \cdot k a^{k-1}$.

Proof. "Differentiability at the point a" means that there's an element $f'(a)\in\mathbf{B}$ so that for any $\epsilon > 0$, there is a $\delta > 0$ so that if $0 < |z-a| < \delta$, then $\left\| \frac{f(z)-f(a)}{z-a} - f'(a) \right\| < \epsilon$.

(Step 1, showing differentiability at 0.) Fix $w\in\mathbb{K}$ with $0 < |w| < r$, so

$$\frac{f(w)-f(0)}{w-0} - c_1 = \frac{c_0 + c_1 w + \left(\sum_{k=2}^{\infty} c_k w^k \right) - c_0}{w} - c_1 = w \sum_{k=2}^{\infty} c_k w^{k-1}.$$

Just as in the Proof of the previous Theorem, Corollary 3.4 applies to the convergent power series $\sum_{k=2}^{\infty} c_k w^{k-1}$, giving some M so that if $|z| < |w|$, then

$$\left\| \frac{f(z) - f(0)}{z - 0} - c_1 \right\| \leq |z| M \frac{1}{1 - \frac{|z|}{|w|}},$$

and this can be made less than any $\epsilon > 0$ by choosing $\delta = \min\{\frac{\epsilon}{2M}, \frac{|w|}{2}\}$.

(Step 2, showing differentiability everywhere else.) By Theorem 3.8, for any point a such that $|a| < r$, there are coefficients c'_k, and a disc with center a, so that for z in that disc, $\sum_{k=0}^{\infty} c'_k (z-a)^k$ converges, with sum $f(z)$. By the construction from the Proof of that Theorem, and the fact that the binomial coefficient $\left(\begin{smallmatrix} (k) \\ (1) \end{smallmatrix} \right) = \left(\begin{matrix} k \\ 1 \end{matrix} \right)$ has value k for all $k \geq 1$ (and in particular, value 0 for $k = 0$),

$$c'_1 = \sum_{k=0}^{\infty} c_k \cdot \left(\begin{matrix} k \\ 1 \end{matrix} \right) a^{k-1} = \sum_{k=1}^{\infty} c_k \cdot k a^{k-1}.$$

So, Step 1 applies to show

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{z - a \to 0} \frac{\left(\sum_{k=0}^{\infty} c'_k (z-a)^k \right) - c'_0}{z - a} = c'_1 = f'(a).$$

∎

[C] gives a proof that $\sum_{k=0}^{\infty} c_k z^k$ and $\sum_{k=1}^{\infty} c_k \cdot k z^{k-1}$ have the same radius of convergence.

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