REAL LINEAR MAPS PRESERVING SOME COMPLEX SUBSPACES

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Abstract. We find configurations of subspaces of a complex vector space such that any real linear map with sufficiently high rank that maps the subspaces into complex subspaces of the same dimension must be complex linear or antilinear.

1. Introduction

It is known that for $n > 1$, an invertible real linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ that maps all complex lines through the origin into complex lines must be complex linear or antilinear. Here we will generalize this by weakening the hypothesis in several ways: first, allowing the real linear map to be not necessarily an automorphism, but only of sufficiently high rank from one complex vector space to another; second, requiring only that some, not all, complex lines are mapped to complex lines, and third, by considering configurations of finite-dimensional subspaces, not only complex lines.

For both generality and clarity, the results and proofs will be stated using only linear algebra "over $\mathbb{R}$" — with the exception of a $\mathbb{C}$-valued cross-ratio in Section 4, we work only with real scalars, real vector spaces, and real linear maps: complex n-space is replaced by a real vector space V paired with a real linear map $J : V \to V$ satisfying $J \circ J = -Id_V$, called a complex structure operator (CSO). The main new result is Theorem 5.5, showing that, for a fixed integer ℓ, if $A : (V_1, J_1) \to (V_2, J_2)$ has rank $> 2\ell$ and each of the J_1-invariant subspaces with (real) dimension 2ℓ in some (possibly finite) configuration is mapped into a J_2-invariant subspace with dimension 2ℓ, then A is complex linear ($A \circ J_1 = J_2 \circ A$) or antilinear ($A \circ J_1 = -J_2 \circ A$). The method of using only real linear algebra has the advantage of coordinate-free statements and elementary proofs, that remain valid if $\mathbb{R}$ is replaced by any formally real field (where zero is not a sum of non-zero squares).

The original result — where A is invertible on $(V, J) = (\mathbb{C}^n, i)$ and maps all complex lines ($\ell = 1$) to complex lines — is well-known; [1] uses the term "pseudo-complex" for invertible real linear transformations of $\mathbb{C}^2$ preserving complex lines (see also Problem 2002-9 of [2]). This case has recently been used in complex differential geometry ([7] Lemma 4.4.a; [9] Lemma 1), and the general notion of line-preserving maps is also related to the Fundamental Theorem of (Projective or Affine) Geometry ([10], [8]), and Wigner's Theorem on symmetries of quantum state spaces ([6]), although again, various versions of such Theorems usually have an assumption that the map is invertible.

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In Section 3 we state a condition on $A : V_1 \to V_2$ that can be checked at finitely many points in a finite-dimensional space V_1 to establish that A is complex linear or antilinear (Theorem 3.2). In Section 5, we state a different condition that can be satisfied by A on finitely many complex lines (Theorem 5.3), which is perhaps more natural from a projective geometric point of view. We start in Section 2 with some facts about real vector spaces V with complex structure operators.

2. Preliminary Lemmas

These Lemmas are entirely elementary and confirm that J-invariant subspaces H of V (where $J(H) \subseteq H$) behave in the same way as complex subspaces of $\mathbb{C}^n$. The proofs are omitted here, but are available from notes of the author ([4]).

Lemma 2.1. *Given V with CSO J and $v \in V$, if $v \neq 0_V$, then the ordered list $(v, J(v))$ is linearly independent.* ∎

Lemma 2.2. *Given V with CSO J and $v_1, \dots, v_\ell \in V$, if the ordered list*

$$(v_1, \dots, v_{\ell-1}, v_\ell, J(v_1), \dots, J(v_{\ell-1}))$$

is linearly independent, then so is the ordered list

$$(v_1, \dots, v_{\ell-1}, v_\ell, J(v_1), \dots, J(v_{\ell-1}), J(v_\ell)).$$

∎

Lemma 2.3. *Given V with CSO J, and a J-invariant subspace H of V, if*

$$0 < \dim(H) < \infty,$$

then H admits an ordered basis of the form

$$(v_1, \dots, v_{\ell-1}, v_\ell, J(v_1), \dots, J(v_{\ell-1}), J(v_\ell)).$$

∎

Definition 2.4. Given V with CSO J, a two-dimensional J-invariant subspace $H = J(H) \subseteq V$ will be called a <u>J-complex line</u> in V.

A J-complex line H must be of the form $\mathrm{span}\{v, J(v)\}$ for some $v \in H$.

Lemma 2.5. *Given V with CSO J, a J-complex line L, and a J-invariant subspace $H \subseteq V$, if there is a non-zero element $v \in L \cap H$, then $L \subseteq H$. In particular, if H is a J-complex line, then $L = H$.* ∎

Lemma 2.6. *Given V with CSO J, if L^1, L^2 are distinct J-complex lines in V, then $\dim(\mathrm{span}(L^1 \cup L^2)) = 4$. In particular, a subspace $H \subseteq V$ with $\dim(H) \leq 3$ can contain at most one J-complex line.* ∎

Lemma 2.7. *Given V with CSO J, an integer ℓ, and a J-invariant subspace H of V, if $\dim(H) \leq 2\ell \leq \dim(V)$, then there exists a J-invariant subspace U of V with $\dim(U) = 2\ell$ and $H \subseteq U$.* ∎

Lemma 2.8. *Given a vector space V_1 with CSO J_1 and an element $v \in V_1$, another vector space V_2 with CSO J_2, and a real linear map $A : V_1 \to V_2$, the following are equivalent:*

(i) $A(J_1(v)) \in \operatorname{span}\{A(v), J_2(A(v))\}$;
(ii) *A maps the subspace* $\operatorname{span}\{v, J_1(v)\} \subseteq V_1$ *into the subspace*

$$\operatorname{span}\{A(v), J_2(A(v))\} \subseteq V_2.$$

$\blacksquare$

If property (i) is satisfied by a non-zero v, then it is satisfied at every point on the real line $\operatorname{span}\{v\}$. However, the above notion for a real linear map A is slightly stronger than the statement that A maps the J_1-complex line $\operatorname{span}\{v, J_1(v)\}$ into some J_2-complex line; if $A(v) = 0_{V_2}$, condition (ii) implies A maps $\operatorname{span}\{v, J_1(v)\}$ to the zero subspace.

3. Maps Preserving Complex Lines

Definition 3.1. Given a vector space V with CSO J, a subset $S \subseteq V$ is a J-superspanning set for V means: S contains a subset B such that:

- $\operatorname{span}(B) = V$;
- For any ordered pair $(b^1, b^2) \in B \times B$, if $(b^1, J(b^1), b^2, J(b^2))$ is an independent ordered list, then there are real numbers $c_1 \neq 0$, $c_2 \neq 0$, such that $c_1 \cdot b^1 + c_2 \cdot b^2 \in S$.

The following Theorem considers a real linear map that preserves, in the strong sense of Lemma 2.8, the J-complex lines containing a J-superspanning set.

Theorem 3.2. *Given V_1 with CSO J_1, V_2 with CSO J_2, and a real linear map $A : V_1 \to V_2$, if $\dim(A(V_1)) > 2$, then the following are equivalent:*

(i) *There exists a J_1-superspanning set S for V_1, such that for each $v \in S$,*

$$A(J_1(v)) \in \operatorname{span}\{A(v), J_2(A(v))\};$$

(ii) $A \circ J_1 = J_2 \circ A$ *or* $A \circ J_1 = -J_2 \circ A$.

Proof. $A(V_1)$ refers to the image of A as a real linear subspace of V_2, with real dimension $\dim(A(V_1))$. The direction (ii) $\implies$ (i) is trivial, since $S = B = V_1$ qualifies as a J_1-superspanning set for V_1. For the other direction, assume S is a J_1-superspanning set for V_1, with subset B as in Definition 3.1.

Assuming $\dim(A(V_1)) > 0$, there is some $v \in B$ with $A(v) \neq 0_{V_2}$, since otherwise, if $A(v) = 0_{V_2}$ for all v in a spanning set B, then A is the zero map. By Lemma 2.1, both $(v, J_1(v))$ and $(A(v), J_2(A(v)))$ are independent ordered lists.

If $\dim(A(V_1)) > 2$, then there is some $u \in B$ with

$$A(u) \notin \operatorname{span}\{A(v), J_2(A(v))\}.$$

Again, this follows from B spanning V_1, since if

$$A(b) \in \operatorname{span}\{A(v), J_2(A(v))\}$$

for all $b \in B$, then $A(V_1) \subseteq \operatorname{span}\{A(v), J_2(A(v))\}$. So,

$$(A(v), J_2(A(v)), A(u))$$

is an independent ordered list, and by Lemma 2.2, the ordered list

$$(A(v), J_2(A(v)), A(u), J_2(A(u))) \tag{3.1}$$

is independent.

Let v, u as above be fixed. From (i), for each $w \in S$, there are some coefficients $\theta(w)$, $\zeta(w) \in \mathbb{R}$ such that

$$A(J_1(w)) = \theta(w) \cdot A(w) + \zeta(w) \cdot J_2(A(w)). \tag{3.2}$$

If $A(w) \neq 0_{V_2}$, then such coefficients are unique, by the independence of $(A(w), J_2(A(w)))$ (Lemma 2.1). So, θ and ζ are well-defined functions from $S \setminus \ker(A)$ to $\mathbb{R}$.

Case 1. If w is an element of $B \setminus \ker(A)$ such that

$$(A(v), J_2(A(v)), A(w)) \tag{3.3}$$

is an independent ordered list, then

$$(A(v), J_2(A(v)), A(w), J_2(A(w))) \tag{3.4}$$

is also an independent ordered list, by Lemma 2.2. It also follows that

$$(v, J_1(v), w, J_1(w)) \tag{3.5}$$

is an independent ordered list; suppose toward a contradiction that $(v, J_1(v), w)$ is dependent, so $w = a_1 \cdot v + a_2 \cdot J_1(v)$. Then

$$\begin{aligned} A(w) &= a_1 \cdot A(v) + a_2 \cdot A(J_1(v)) \\ &= a_1 \cdot A(v) + a_2\theta(v) \cdot A(v) + a_2\zeta(v) \cdot J_2(A(v)), \end{aligned}$$

contradicting the independence of (3.3). So (3.5) is independent by Lemma 2.2. By Definition 3.1, there is an element

$$x = c_1 \cdot v + c_2 \cdot w \in S$$

with $c_1 \neq 0$ and $c_2 \neq 0$. Consider the following two expressions.

$$\begin{aligned} A(x) &= A(c_1 \cdot v + c_2 \cdot w) \\ &= c_1 \cdot A(v) + c_2 \cdot A(w), \qquad (3.6) \\ A(J_1(x)) &= A(J_1(c_1 \cdot v + c_2 \cdot w)) = c_1 \cdot A(J_1(v)) + c_2 \cdot A(J_1(w)) \\ &= c_1\theta(v) \cdot A(v) + c_1\zeta(v) \cdot J_2(A(v)) \\ &\quad + c_2\theta(w) \cdot A(w) + c_2\zeta(w) \cdot J_2(A(w)). \qquad (3.7) \end{aligned}$$

$A(x) \neq 0_{V_2}$, by (3.6) and the independence of $(A(v), A(w))$. So, (3.2) applies because $x \in S \setminus \ker(A)$, with unique values for $\theta(x)$ and $\zeta(x)$:

$$\begin{aligned} A(J_1(x)) &= \theta(x) \cdot A(x) + \zeta(x) \cdot J_2(A(x)) \\ &= \theta(x) \cdot A(c_1 \cdot v + c_2 \cdot w) + \zeta(x) \cdot J_2(A(c_1 \cdot v + c_2 \cdot w)) \\ &= c_1\theta(x) \cdot A(v) + c_1\zeta(x) \cdot J_2(A(v)) \\ &\quad + c_2\theta(x) \cdot A(w) + c_2\zeta(x) \cdot J_2(A(w)). \qquad (3.8) \end{aligned}$$

By the independence of (3.4), and using $c_1 \neq 0$, $c_2 \neq 0$, comparing (3.7) and (3.8) gives $\theta(w) = \theta(x) = \theta(v)$, and $\zeta(w) = \zeta(x) = \zeta(v)$. In particular, the fixed element $u \in V_1$ falls into this case, and $\theta(u) = \theta(v)$ and $\zeta(u) = \zeta(v)$.

Case 2. If w is an element of $B \setminus \ker(A)$ such that

$$(A(v), J_2(A(v)), A(w))$$

is a dependent ordered list, then $A(w) \notin \operatorname{span}\{A(u), J_2(A(u))\}$, by the independence of (3.1), so $(A(u), J_2(A(u)), A(w))$ is an independent ordered list, and by the same argument as Case 1., $\theta(w) = \theta(u)$ and $\zeta(w) = \zeta(u)$.

Case 3. If w is an element of $B \cap \ker(A)$, then by (i),

$$A(J_1(w)) \in \operatorname{span}\{A(w), J_2(A(w))\} = \{0_{V_2}\},$$

and $A(J_1(w)) = \theta(v) \cdot A(w) + \zeta(v) \cdot J_2(A(w))$ is a true statement.

We can conclude from Cases 1. and 2. that the functions θ and ζ are constant on $B \setminus \ker(A)$, and from Case 3. that they extend to constant functions $B \to \mathbb{R}$, such that for all $w \in B$, (3.2) is satisfied with $\theta(w) = \theta(v)$ and $\zeta(w) = \zeta(v)$:

$$A(J_1(w)) = \theta(v) \cdot A(w) + \zeta(v) \cdot J_2(A(w)). \tag{3.9}$$

Since the linear maps $A \circ J_1$ and $\theta(v) \cdot A + \zeta(v) \cdot J_2 \circ A$ agree on the spanning set B, they are equal on all of V_1.

To find the values of these constants, apply (3.9) to $J_1(v)$:

$$\begin{aligned}
A(J_1(J_1(v))) &= A(-v) = (-1) \cdot A(v) \\
&= \theta(v) \cdot A(J_1(v)) + \zeta(v) \cdot J_2(A(J_1(v))) \\
&= \theta(v) \cdot (\theta(v) \cdot A(v) + \zeta(v) \cdot J_2(A(v))) \\
&\quad +\zeta(v) \cdot J_2(\theta(v) \cdot A(v) + \zeta(v) \cdot J_2(A(v))) \\
&= ((\theta(v))^2 - (\zeta(v))^2) \cdot A(v) + 2\theta(v)\zeta(v) \cdot J_2(A(v)).
\end{aligned}$$

By the independence of $(A(v), J_2(A(v)))$,

$$(\theta(v))^2 - (\zeta(v))^2 = -1 \quad \text{and} \quad 2\theta(v)\zeta(v) = 0.$$

The only real solutions are $\theta(v) = 0$ and $\zeta(v) = \pm 1$, so (ii) holds. ∎

In the finite-dimensional case, property (ii) of Theorem 3.2 can follow from checking property (i) on a finite set of points, or a finite configuration of 1-dimensional real subspaces.

4. The Cross-Ratio

Here we briefly depart from real linear algebra to consider the complex vector space $\mathbb{C}^2 = \{(z_0, z_1) : z_0 \in \mathbb{C}, z_1 \in \mathbb{C}\}$. The set $\mathbb{C}P^1$ is the set of complex subspaces of $\mathbb{C}^2$ with complex dimension 1 (equivalently, J-complex lines in $\mathbb{R}^4$ where $J(z) = i \cdot z$). The J-complex line $L \in \mathbb{C}P^1$ containing $(w_0, w_1) \neq (0, 0)$, or equivalently $(\lambda w_0, \lambda w_1)$ for any non-zero $\lambda \in \mathbb{C}$, can be labeled with homogeneous coordinates:

$$L = \operatorname{span}_{\mathbb{C}}\{(w_0, w_1)\} = \operatorname{span}_{\mathbb{C}}\{(\lambda w_0, \lambda w_1)\} = [w_0 : w_1] = [\lambda w_0 : \lambda w_1].$$

Returning to real linear algebra, let V be a real vector space with CSO J, and suppose there are four elements v^1, v^2, v^3, v^4 so that for each ordered pair of distinct indices (j, k),

- the ordered list $(v^j, J(v^j), v^k, J(v^k))$ is independent;
- $\{v^1, v^2, v^3, v^4\} \subseteq \operatorname{span}\{v^j, J(v^j), v^k, J(v^k)\}$.

It follows from these properties that $\operatorname{span}\{v^j, J(v^j), v^k, J(v^k)\}$ does not depend on the index pair (j, k), so it can be called P, and P is a four-dimensional, J-invariant subspace of V. Let b^1, b^2 be any two elements of P such that $(b^1, J(b^1), b^2, J(b^2))$ is an independent ordered list, and as a consequence, an ordered basis for P. Then each element v^k has unique real number coordinates:

$$v^k = r_1^k \cdot b^1 + r_2^k \cdot J(b^1) + r_3^k \cdot b^2 + r_4^k \cdot J(b^2). \tag{4.1}$$

Definition 4.1. The <u>J-cross-ratio</u> of the ordered list (v^1, v^2, v^3, v^4) is this element of $\mathbb{C}P^1$:

$$\begin{aligned}\chi(v^1,v^2,v^3,v^4) &= \left[\left| \begin{array}{cc} r_1^4+ir_2^4 & r_1^1+ir_2^1 \\ r_3^4+ir_4^4 & r_3^1+ir_4^1 \end{array}\right| \left| \begin{array}{cc} r_1^3+ir_2^3 & r_1^2+ir_2^2 \\ r_3^3+ir_4^3 & r_3^2+ir_4^2 \end{array}\right| \right. \\ &\quad \left. : \left| \begin{array}{cc} r_1^3+ir_2^3 & r_1^1+ir_2^1 \\ r_3^3+ir_4^3 & r_3^1+ir_4^1 \end{array}\right| \left| \begin{array}{cc} r_1^4+ir_2^4 & r_1^2+ir_2^2 \\ r_3^4+ir_4^4 & r_3^2+ir_4^2 \end{array}\right| \right], \end{aligned} \tag{4.2}$$

where $\left| \begin{array}{cc} z_1 & z_2 \\ z_3 & z_4 \end{array}\right| = z_1z_4 - z_2z_3$ is the complex 2×2 determinant.

This determinantal expression is, of course, exactly analogous to the classical complex cross-ratio ([5]), but it depends on the CSO $J|_P$. It has the following two invariance properties. First, if v^k is replaced by $s\cdot v^k + t\cdot J(v^k)$ for some real $(s,t)\neq(0,0)$, then $r_1^k+ir_2^k$ and $r_3^k+ir_4^k$ are replaced by $(s+it)(r_1^k+ir_2^k)$ and $(s+it)(r_3^k+ir_4^k)$. The non-zero factor $s+it$ occurs in two columns in (4.2) and does not change the ratio. Second, if the vectors in the ordered pair (b^1,b^2) are replaced by:

$$\begin{aligned} b^1 &= a_1^1\cdot\tilde b^1 + a_2^1\cdot J(\tilde b^1) + a_3^1\cdot \tilde b^2 + a_4^1\cdot J(\tilde b^2) \\ b^2 &= a_1^2\cdot\tilde b^1 + a_2^2\cdot J(\tilde b^1) + a_3^2\cdot \tilde b^2 + a_4^2\cdot J(\tilde b^2), \end{aligned}$$

with

$$\left[\begin{array}{cccc} a_1^1 & -a_2^1 & a_1^2 & -a_2^2 \\ a_2^1 & a_1^1 & a_2^2 & a_1^2 \\ a_3^1 & -a_4^1 & a_3^2 & -a_4^2 \\ a_4^1 & a_3^1 & a_4^2 & a_3^2 \end{array}\right]$$

invertible, then $(\tilde b^1, J(\tilde b^1), \tilde b^2, J(\tilde b^2))$ is an ordered basis for P, and this coordinate change transforms the real coordinates of v^k in (4.1) via the following complex matrix product:

$$\left[\begin{array}{c} r_1^k+ir_2^k \\ r_3^k+ir_4^k \end{array}\right] \mapsto \left[\begin{array}{cc} a_1^1+ia_2^1 & a_1^2+ia_2^2 \\ a_3^1+ia_4^1 & a_3^2+ia_4^2 \end{array}\right]\left[\begin{array}{c} r_1^k+ir_2^k \\ r_3^k+ir_4^k \end{array}\right].$$

The 2×2 complex matrix is invertible, and by the product rule for complex determinants, each of the four determinants in (4.2) is multiplied by the same non-zero factor, so again the ratio is unchanged.

By the independence of the ordered lists $(v^j, J(v^j), v^k, J(v^k))$, all the determinants in (4.2) are non-zero, and $\chi(v^1,v^2,v^3,v^4) = [1:z]$ for a unique non-zero complex number z. As with the classical cross-ratio, various re-orderings of (v^1,v^2,v^3,v^4) result in six complex numbers: z, $\frac{1}{z}$, $1-z$, $1-\frac{1}{z}$, $\frac{1}{1-z}$, $\frac{z}{z-1}$. The independence also implies $z\neq 1$.

It follows from the above invariance properties that given pairwise distinct J-complex lines L^1, L^2, L^3, L^4 all contained in some four-dimensional J-invariant subspace $P\subseteq V$, the J-cross-ratio $\chi(L^1,L^2,L^3,L^4)$ can be defined by choosing any non-zero elements $v^k\in L^k$, and any ordered basis of the form $(b^1, J(b^1), b^2, J(b^2))$ for P; then setting

$$\chi(L^1,L^2,L^3,L^4) = \chi(v^1,v^2,v^3,v^4) = [1:z]$$

does not depend on the choices made. The following normalization is convenient. Because $v^4\in\operatorname{span}(L^1\cup L^2)$, there are coefficients so that

$$v^4 = r_1^4\cdot v^1 + r_2^4\cdot J(v^1) + r_3^4\cdot v^2 + r_4^4\cdot J(v^2)\in L^4,$$

and because L^1, L^2, L^4 are pairwise distinct, $(r^4_1, r^4_2) \neq (0,0)$ and $(r^4_3, r^4_4) \neq (0,0)$. Let $b^1 = r^4_1 \cdot v^1 + r^4_2 \cdot J(v^1) \in L^1$, and $b^2 = r^4_3 \cdot v^2 + r^4_4 \cdot J(v^2) \in L^2$, so $(b^1, J(b^1), b^2, J(b^2))$ is an ordered basis for P and $v^4 = b^1 + b^2$. There are also coefficients so that

$$v^3 = r^3_1 \cdot b^1 + r^3_2 \cdot J(b^1) + r^3_3 \cdot b^2 + r^3_4 \cdot J(b^2) \in L^3,$$

and by J-invariance, L^3 has a non-zero element of the form $b^1 + r \cdot b^2 + s \cdot J(b^2)$. The J-cross-ratio is:

$$\begin{aligned} \chi(L^1, L^2, L^3, L^4) &= \chi(v^1, v^2, v^3, v^4) \\ &= \chi(b^1, b^2, b^1 + r \cdot b^2 + s \cdot J(b^2), b^1 + b^2) \qquad (4.3) \\ &= \left[\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ r+is & 1 \end{vmatrix} : \begin{vmatrix} 1 & 1 \\ r+is & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \right] \\ &= [1 : r + is]. \end{aligned}$$

5. Configurations of Subspaces

Lemma 5.1. *Let V_1 be a vector space with CSO J_1 and $\dim(V_1) = 4$, and let (L^1, L^2, L^3, L^4) be an ordered list of pairwise distinct J_1-complex lines in V_1, such that $\chi(L^1, L^2, L^3, L^4) = [1 : z]$ and z is a non-real complex number. Given another vector space V_2 with CSO J_2, and a real linear map $A : V_1 \to V_2$ with $\operatorname{rank}(A) > 2$, the following are equivalent:*

(i) *For each $k = 1, \ldots, 4$ there is a J_2-complex line H^k with $A(L^k) \subseteq H^k$;*
(ii) $\operatorname{rank}(A) = 4$, *and $A \circ J_1 = J_2 \circ A$ or $A \circ J_1 = -J_2 \circ A$.*

Proof. (ii) $\implies$ (i) is elementary. As previously mentioned, the containment in property (i) is *a priori* a weaker assumption than property (i) of Lemma 2.8.

Suppose toward a contradiction that $\operatorname{rank}(A) = 3$. Then $\ker(A)$ is one-dimensional and, by Lemma 2.5, can be contained in at most one of the J_1-complex lines $L^1, \ldots, L^4$. So, there are distinct L^j and L^k which are each mapped one-to-one onto J_2-complex lines H^j, H^k. By Lemma 2.6, the three-dimensional subspace $A(V_1)$ can contain at most one J_2-complex line, so $H^j = H^k$. However, since $L^j \cup L^k$ spans V_1, $A(V_1) \subseteq H^j$, contradicting $\operatorname{rank}(A) > 2$. The conclusion is that A has rank 4 and is one-to-one.

By hypothesis, for each line L^k there is some H^k with $A(L^k) \subseteq H^k$, and because A is one-to-one on each L^k, if $0_{V_1} \neq x \in L^k$, then

$$A(J_1(x)) \in A(L^k) = H^k = \operatorname{span}\{A(x), J_2(A(x))\}.$$

This is property (i) from Lemma 2.8; in a case where the set $L^1 \cup L^2 \cup L^3 \cup L^4$ contains some J_1-superspanning set, Theorem 3.2 would apply to establish this Lemma, but there may be no such containment, so we continue with a direct proof.

Let $v^1 = b^1$, $v^2 = b^2$, $v^3 = b^1 + r \cdot b^2 + s \cdot J_1(b^2)$, $v^4 = b^1 + b^2$ be normalized representatives for $L^1, \ldots, L^4$ as in (4.3).

Because A has rank 4, for any non-zero $x \in L^1 \cup L^2 \cup L^3 \cup L^4$, $(A(x), J_2(A(x)))$ is an independent ordered list by Lemma 2.1, and there are unique real coefficients as in (3.2):

$$A(J_1(x)) = \theta(x) \cdot A(x) + \zeta(x) \cdot J_2(A(x)).$$

In particular,

$$\begin{aligned}
A(J_1(v^4)) &= \theta(v^4)\cdot A(v^4) + \zeta(v^4)\cdot J_2(A(v^4)) \\
&= \theta(v^4)\cdot (A(v^1)+A(v^2)) + \zeta(v^4)\cdot J_2(A(v^1)+A(v^2)) \\
&= A(J_1(v^1)) + A(J_1(v^2)) \\
&= \big(\theta(v^1)\cdot A(v^1) + \zeta(v^1)\cdot J_2(A(v^1))\big) \\
&\quad + \big(\theta(v^2)\cdot A(v^2) + \zeta(v^2)\cdot J_2(A(v^2))\big).
\end{aligned}$$

By the independence of $(A(v^1), J_2(A(v^1)), A(v^2), J_2(A(v^2)))$ (using $A(v^2)\notin A(L^1) = \operatorname{span}\{A(v^1), J_2(A(v^1))\}$ as in (3.1)), comparing coefficients gives $\theta(v^1) = \theta(v^4) = \theta(v^2)$ and $\zeta(v^1) = \zeta(v^4) = \zeta(v^2)$.

$$\begin{aligned}
& A(J_1(v^3)) \\
= \;& \theta(v^3)\cdot A(v^3) + \zeta(v^3)\cdot J_2(A(v^3)) \\
= \;& \theta(v^3)\cdot (A(v^1) + r\cdot A(v^2) + s\cdot A(J_1(v^2))) \\
& +\zeta(v^3)\cdot J_2(A(v^1) + r\cdot A(v^2) + s\cdot A(J_1(v^2))) \\
= \;& \theta(v^3)\cdot (A(v^1) + (r+s\theta(v^1))\cdot A(v^2) + s\zeta(v^1)\cdot J_2(A(v^2))) \\
& +\zeta(v^3)\cdot (J_2(A(v^1)) + (r+s\theta(v^1))\cdot J_2(A(v^2)) - s\zeta(v^1)\cdot A(v^2)) \\
= \;& A(J_1(v^1 + r\cdot v^2 + s\cdot J_1(v^2))) \\
= \;& A(J_1(v^1)) + r\cdot A(J_1(v^2)) - s\cdot A(v^2) \\
= \;& \theta(v^1)\cdot A(v^1) + \zeta(v^1)\cdot J_2(A(v^1)) \\
& +(r\theta(v^1) - s)\cdot A(v^2) + r\zeta(v^1)\cdot J_2(A(v^2)).
\end{aligned}$$

Again, comparing coefficients gives this system of equations:

$$\begin{aligned}
\theta(v^1) &= \theta(v^3) \\
\zeta(v^1) &= \zeta(v^3) \\
r\theta(v^1) - s &= \theta(v^3)r + \theta(v^3)s\theta(v^1) - \zeta(v^3)s\zeta(v^1) \\
r\zeta(v^1) &= \theta(v^3)s\zeta(v^1) + \zeta(v^3)r + \zeta(v^3)s\theta(v^1).
\end{aligned}$$

Using the hypothesis $s \neq 0$, the only real solutions are $\theta(v^3) = \theta(v^1) = 0$ and $\zeta(v^3) = \zeta(v^1) = \pm 1$. So, $A(J_1(x)) = \zeta(v^1)\cdot J_2(A(x))$ for $x = v^1$ and $x = v^2$. Since these expressions are equal:

$$\begin{aligned}
A(J_1(J_1(v^1))) &= (-1)\cdot A(v^1) \\
\zeta(v^1)\cdot J_2(A(J_1(v^1))) &= \zeta(v^1)\cdot J_2(\zeta(v^1)\cdot J_2(A(v^1))) = (-1)\cdot A(v^1),
\end{aligned}$$

and similarly $A(J_1(J_1(v^2))) = \zeta(v^1)\cdot J_2(A(J_1(v^2)))$, the composites $A\circ J_1$ and $\zeta(v^1)\cdot J_2\circ A$ agree on all four of the vectors in the ordered basis $(v^1, J_1(v^1), v^2, J_1(v^2))$ of V_1. The conclusion is that either $A\circ J_1 = J_2\circ A$ or $A\circ J_1 = -J_2\circ A$. ∎

Definition 5.2. Given a vector space V with CSO J, let $\mathcal{S}$ be a set of J-complex lines in V. The set $\mathcal{S}$ is a J-superspanning configuration means: $\mathcal{S}$ contains a subset $\mathcal{B}$ such that:

- $\operatorname{span}\left(\bigcup_{L\in\mathcal{B}} L\right) = V$;
- If $L^1 \in \mathcal{B}$, $L^2 \in \mathcal{B}$, and $L^1 \neq L^2$, then there are some $L^3, L^4 \in \mathcal{S}$ such that L^1, L^2, L^3, L^4 are pairwise distinct, $L^3 \cup L^4 \subseteq \operatorname{span}(L^1 \cup L^2)$, and the J-cross-ratio is
$$\chi(L^1, L^2, L^3, L^4) = [1 : z],$$
where z is a non-real complex number.

The property that the cross-ratio is non-real does not depend on any ordering of the four J-complex lines.

Theorem 5.3. *Given V_1 with CSO J_1, V_2 with CSO J_2, and a real linear map $A : V_1 \to V_2$, if $\dim(A(V_1)) > 2$, then the following are equivalent:*

(i) *There exists some J_1-superspanning configuration $\mathcal{S}$ for V_1, such that for each $L \in \mathcal{S}$, there is some J_2-complex line H of V_2 with $A(L) \subseteq H$;*
(ii) *$A \circ J_1 = J_2 \circ A$ or $A \circ J_1 = -J_2 \circ A$.*

Proof. As in Theorem 3.2, the direction (ii) $\implies$ (i) is trivial, since $\mathcal{S} = \mathcal{B} = \{\text{all } J_1\text{-complex lines}\}$ qualifies as a J_1-superspanning configuration for V_1. For the other direction, assume $\mathcal{S}$ is a J_1-superspanning configuration for V_1, with subset $\mathcal{B}$ as in Definition 5.2.

Assuming $\dim(A(V_1)) > 2$, there is some $v \in \bigcup_{L\in\mathcal{B}} L$ with $A(v) \neq 0_{V_2}$, and there is some $u \in \bigcup_{L\in\mathcal{B}} L$ with
$$A(u) \notin \operatorname{span}\{A(v), J_2(A(v))\}.$$
Elements of $\mathcal{B}$ can be labeled so that $v \in L^1$ and $u \in L^2$, with $L^1 \neq L^2$, and then as in the Proof of Theorem 3.2,
$$(A(v), J_2(A(v)), A(u), J_2(A(u))) \tag{5.1}$$
is an independent ordered list.

Let v, u as above be fixed, and let $P = \operatorname{span}(L^1 \cup L^2)$, so $\dim(P) = 4$. The restriction of A to P has $\operatorname{rank}(A|_P) \geq 2$. By hypothesis, P also contains J_1-complex lines L^3, L^4 in $\mathcal{S}$ satisfying the J_1-cross-ratio condition from Definition 5.2.

Suppose, toward a contradiction, that $\operatorname{rank}(A|_P) = 2$. Then the image of $A|_P$ is $A(P) = \operatorname{span}\{A(v), A(u)\}$, and the kernel $\ker(A|_P)$ is a two-dimensional subspace of P.

Case 1. Each of the subspaces $L^1, \ldots, L^4$ meets $\ker(A|_P)$ in a non-zero point: $x^k \in L^k$, $A(x^k) = 0_{V_2}$, $k = 1, \ldots, 4$. Because x^1 and x^2 are independent and $\ker(A|_P)$ is two-dimensional, there are real coefficients so that $x^3 = r_1 \cdot x^1 + r_2 \cdot x^2$ and $x^4 = r_3 \cdot x^1 + r_4 \cdot x^2$. The x^k vectors can be used to calculate the J_1-cross-ratio:
$$\begin{aligned}\chi(L^1, L^2, L^3, L^4) &= \left[\left| \begin{matrix} r_3 & 1 \\ r_4 & 0 \end{matrix} \right| \left| \begin{matrix} r_1 & 0 \\ r_2 & 1 \end{matrix} \right| : \left| \begin{matrix} r_1 & 1 \\ r_2 & 0 \end{matrix} \right| \left| \begin{matrix} r_3 & 0 \\ r_4 & 1 \end{matrix} \right| \right], \\ &= \left[1 : \frac{r_2 r_3}{r_1 r_4} \right].\end{aligned}$$

This contradicts the assumed non-real property of the J_1-cross-ratio.

Case 2. For some $k = 1, \ldots, 4$, $L^k \cap \ker(A|_P) = \{0_{V_1}\}$. Because $L^k \in \mathcal{S}$, $A(L^k) \subseteq H$ for some J_2-complex line H, and because A is one-to-one on L^k, $A(L^k) = H$. However, $A(L^k) \subseteq A(P) = \operatorname{span}\{A(v), A(u)\}$, which is not J_2-invariant, a contradiction.

Since either Case leads to a contradiction, the conclusion is that $\operatorname{rank}(A|_P) > 2$. Lemma 5.1 applies to $A|_P$ and the ordered list (L^1, L^2, L^3, L^4), so A is one-to-one on P, and either $A|_P \circ J_1|_P = J_2 \circ A|_P$ or $A|_P \circ J_1|_P = -J_2 \circ A|_P$.

We will consider the first case, where $A|_P \circ J_1|_P = J_2 \circ A|_P$ holds, and show that the same identity holds on all of V_1. The antilinear case is analogous.

Continuing with v, u, L^1, L^2, P fixed as above, if $V_1 = P$, we are done; otherwise, consider any J_1-complex line $L^5 \in \mathcal{B}$ with $L^5 \not\subseteq P$. Let $P' = \operatorname{span}(L^1 \cup L^5)$, so P' is a four-dimensional J_1-invariant subspace of V_1. By Definition 5.2, there are J_1-complex lines L^6, L^7 contained in P', satisfying the J_1-cross-ratio condition from Definition 5.2. Because A is one-to-one on $L^1 \subseteq P'$, $A|_{P'}$ has rank ≥ 2.

Case 1. If $\operatorname{rank}(A|_{P'}) > 2$, then Lemma 5.1 applies to $A|_{P'}$ and (L^1, L^5, L^6, L^7), so it is either complex linear or antilinear, and it has already been assumed that $A(J_1(v)) = J_2(A(v)) \neq 0_{V_2}$, so $A|_{P'} \circ J_1|_{P'} = J_2 \circ A|_{P'}$.

Case 2. Suppose $\operatorname{rank}(A|_{P'}) = 2$ and $\ker(A|_{P'}) = L^5$. The quantities $A(J_1(w))$ and $J_2(A(w))$ are both 0_{V_2} for any $w \in L^5$.

Case 3. Suppose $\operatorname{rank}(A|_{P'}) = 2$ and $\ker(A|_{P'}) \neq L^5$. Then there is some $w \in L^5$ with $A(w) \neq 0_{V_2}$. Because A is one-to-one on L^1, $A(P') = A(L^1)$, so $A(w)$ is a non-zero element of $\operatorname{span}\{A(v), J_2(A(v))\}$. Consider $P'' = \operatorname{span}(L^2 \cup L^5)$, which contains two more J_1-complex lines L^8, L^9, satisfying the J_1-cross-ratio condition from Definition 5.2. By the independence of (5.1),

$$(A(w), A(u), J_2(A(u)) = A(J_1(u)))$$

is an independent ordered list, so $\operatorname{rank}(A|_{P''}) \geq 3$. Applying Lemma 5.1 to $A|_{P''}$ and (L^2, L^5, L^8, L^9), and using the assumption $A(J_1(u)) = J_2(A(u)) \neq 0_{V_2}$, it follows that $A(J_1(x)) = J_2(A(x))$ for all $x \in P''$.

The conclusion is that the identity $A \circ J_1 = J_2 \circ A$ holds on every line $L \in \mathcal{B}$; since these lines span V_1, the maps are equal: $A \circ J_1 = J_2 \circ A$. ∎

It is possible that condition (i) in Theorem 5.3 can be changed to allow configurations that have fewer J-complex lines, or that satisfy some other condition. However, the following example shows that a real linear map can preserve infinitely many J-complex lines, but is neither complex linear nor antilinear, so there is no version of Lemma 5.1 or Theorem 5.3 where (i) is replaced by some purely quantitative lower bound on the number of J-complex lines.

Example 5.4. Consider the invertible linear transformations of $\mathbb{R}^4$ with coordinates $\{(x_1, x_2, x_3, x_4)\}$, and the following matrix representations acting on column vectors:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Any subspace of the form $\{(x_1, x_2, rx_1, rx_2)\}$, for a real constant r, is invariant under both J and A. The union of two such J-complex lines can contain the vectors in an ordered basis of $\mathbb{R}^4$, but any four of these J-complex lines will have a real J-cross-ratio so no set of these J-complex lines forms a J-superspanning configuration. The only vector v where $A(J(v)) = \pm J(A(v))$ is $v = (0,0,0,0)$.

Theorem 5.5. *Given V_1 with CSO J_1, let $\mathcal{S}$ be a J_1-superspanning configuration, with a subset $\mathcal{B}$ as in Definition 5.2, and let $(\mathcal{S}^1, \mathcal{S}^2, \ldots)$ be a (finite or infinite) ordered list of sets of subspaces of V_1 defined by*

- $\mathcal{S}^1 = \mathcal{S}$;
- *For* $k > 1$, $\mathcal{S}^k = \{\operatorname{span}(H \cup L) : H \in \mathcal{S}^{k-1},\ L \in \mathcal{B},\ L \not\subseteq H\}$.

Given V_2 with CSO J_2, and $\ell \geq 1$, if A is a real linear map $A : V_1 \to V_2$ and $\dim(A(V_1)) > 2\ell$, then the following are equivalent:

I(ℓ). *For every $H \in \mathcal{S}^\ell$ there exists a J_2-invariant subspace $K \subseteq V_2$ with $\dim(K) = 2\ell$ and $A(H) \subseteq K$;*

II. *$A \circ J_1 = J_2 \circ A$ or $A \circ J_1 = -J_2 \circ A$.*

Proof. Theorem 5.3 is the $\ell = 1$ case: II $\iff$ I(1). For $\ell > 1$, the direction II $\implies$ I(ℓ) is easy to check; the strategy for equivalence is to prove I(ℓ) $\implies$ I(1). Specifically, the following argument shows that for $j \geq 2$, if $\dim(A(V_1)) > 2j$, then I(j) $\implies$ I($j-1$).

By construction, every $H \in \mathcal{S}^k$ has dimension $2k$ and is J_1-invariant. Assuming $\dim(A(V_1)) > 2j > 2$ and that I(j) holds, take any $H_1 \in \mathcal{S}^{j-1}$. Since $\dim(A(H_1)) \leq 2(j-1) < \dim(A(V_1))$ and $\operatorname{span}\left(\bigcup L\right) = V_1$, there is some $L_1 \in \mathcal{B}$ and $0_{V_1} \neq v^1 \in L_1$ with $A(v^1) \notin A(H_1)$, so $v^1 \notin H_1$. Let H_2 be the subspace $\operatorname{span}(H_1 \cup L_1)$, so by construction, $H_2 \in \mathcal{S}^j$. By the assumption I(j), there is some J_2-invariant $2j$-dimensional subspace K_1 of V_2 with $A(H_2) \subseteq K_1$.

Since $\dim(K_1) = 2j < \dim(A(V_1))$, there is some $L_2 \in \mathcal{B}$ and $0_{V_1} \neq v^2 \in L_2$ with $A(v^2) \notin K_1$. Let $H_3 = \operatorname{span}(H_1 \cup L_2)$. As previously, $H_3 \in \mathcal{S}^j$ and there is some J_2-invariant $2j$-dimensional subspace K_2 of V_2 with $A(H_3) \subseteq K_2$.

The intersection $K_1 \cap K_2$ is a J_2-invariant subspace of V_2, and is not all of K_2, since $A(v^2) \in K_2$ and $A(v^2) \notin K_1$. So, $\dim(K_1 \cap K_2) < 2j$, and by Lemma 2.3, $\dim(K_1 \cap K_2) \leq 2j - 2$. By Lemma 2.7, there is some J_2-invariant $2(j-1)$-dimensional subspace K_3 containing $K_1 \cap K_2$. So,

$$A(H_1) \subseteq A(H_2 \cap H_3) \subseteq A(H_2) \cap A(H_3) \subseteq K_1 \cap K_2 \subseteq K_3,$$

which means the condition I($j-1$) is satisfied, establishing the required step. ∎

The inequality for the rank in the hypothesis is easily seen to be sharp; there exist real linear, onto maps $(\mathbb{R}^{2n}, J_1) \to (\mathbb{R}^{2\ell}, J_2)$ that satisfy I(ℓ) but not II. The strategy of induction on the dimension is similar to a step sketched in [8] (Theorems 3.7, 4.6).

Corollary 5.6. *Given a real vector space V with two complex structure operators, J_1 and J_2, let $\mathcal{S}$ be a J_1-superspanning configuration and let $(\mathcal{S}^1, \mathcal{S}^2, \ldots)$ be a ordered list of sets of subspaces as constructed in Theorem 5.5. If there is an integer ℓ such that $0 < 2\ell < \dim(V)$ and every $H \in \mathcal{S}^\ell$ is a J_2-invariant subspace, then $J_2 = \pm J_1$.*

Proof. Theorem 5.5 applies to $A = Id_V$. ∎

6. Applications

The above linear algebra results can be applied in geometric situations, although at this point we require the genuine $\mathbb{R}$, not just any formally real field. The following two applications involve Lemma 5.1 on four J-complex lines in $(\mathbb{R}^4, J)$, but could be generalized to other configurations in higher dimensions.

Let $\mathbb{R}P^3$ be the set of real 1-dimensional subspaces of $\mathbb{R}^4$; then the 2-dimensional subspaces of $\mathbb{R}^4$ correspond to real projective lines in the real projective 3-space $\mathbb{R}P^3$. Given a CSO J on $\mathbb{R}^4$, some of these 2-dimensional subspaces are J-complex lines. Let $\mathbb{C}P^1$ denote the set of J-complex lines in $(\mathbb{R}^4, J)$; then the map $\pi : \mathbb{R}P^3 \to \mathbb{C}P^1$ taking an element of $\mathbb{R}P^3$ to the unique J-complex line containing it gives a well-known $\mathbb{R}P^1$-bundle. Given a J-complex line H, the fiber $\pi^{-1}(H)$ is the real projective line in $\mathbb{R}P^3$ whose elements are the real 1-dimensional subspaces of H.

Corollary 6.1. *Let $\Gamma : \mathbb{R}P^3 \to \mathbb{R}P^3$ be a collineation of real projective space. If there are four elements L^1, L^2, L^3, $L^4 \in \mathbb{C}P^1$ with non-real J-cross-ratio such that for each $k = 1, \dots, 4$, Γ maps the real projective line $\pi^{-1}(L^k)$ into a real projective line which is also a fiber of the form $\pi^{-1}(H^k)$, $H^k \in \mathbb{C}P^1$, then Γ maps every fiber of π to a fiber.*

Proof. A collineation is an invertible map taking real projective lines into real projective lines. By the Fundamental Theorem of Real Projective Geometry, Γ is induced by some real linear map $A : \mathbb{R}^4 \to \mathbb{R}^4$, to which Lemma 5.1 applies. ∎

This last application is in the setting of differential topology. Let U be a connected open subset of $\mathbb{R}^4$, with an almost complex structure: a CSO $J_1(x)$ depending continuously on $x \in U$. Consider four different foliations of U by real differentiable surfaces, so each point $x \in U$ lies on exactly one surface from each of the four foliations, and the four tangent planes are pairwise distinct $J_1(x)$-complex lines. This structure is an almost complex version of a 4-web on U ([3]). Suppose further that at each point x, the four complex tangents have a non-real $J_1(x)$-cross-ratio.

If M is another manifold with almost complex structure $J_2(y)$, $\alpha : U \to M$ is a $\mathcal{C}^1$ map with differential $D\alpha(x)$ of rank > 2 at each x, and each surface in the four foliations is mapped by α into a surface with tangent planes that are $J_2(y)$-complex lines, then Lemma 5.1 applies, and $D\alpha(x)$ has rank 4 and satisfies $D\alpha(x) \circ J_1(x) = \pm J_2(\alpha(x)) \circ D\alpha(x)$. So, α is an immersion, and by continuity, the $\pm$ sign is consistent on U, so α is either (J_1, J_2)-holomorphic or $(J_1, -J_2)$-holomorphic.

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