ADDENDUM TO: CONSTRUCTING DISCONTINUOUS BUT LOCALLY BOUNDED RATIONAL FUNCTIONS USING ŁOJASIEWICZ INEQUALITIES

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8. Supplemental notes

The following Sections of this unpublished Addendum sketch some proofs omitted from our preprint [CP], "Constructing Discontinuous but Locally Bounded Rational Functions using Łojasiewicz Inequalities". Section 9 and Section 10 give some background leading to the statement of Proposition 3.2 in [CP].

9. Elementary algebraic geometry

We recall but do not prove a few well-known ideas leading up to a definition of "dimension" in algebraic geometry. We refer to Coffman's course notes [C] for some basic background in commutative algebra and proofs of some of the following statements labeled "Lemma" or "Theorem".

Definition 9.1. An ideal I of the commutative ring R is a prime ideal means: (1) $I \neq R$, and (2) if $a, b \in R$ and $a \cdot b \in I$, then $a \in I$ or $b \in I$.

Proposition 9.2. *Given a commutative ring R, if I is a prime ideal and there are N elements $a_1, \ldots, a_N \in R$ so that $a_1 \cdot \ldots \cdot a_N \in I$, then at least one of the elements satisfies $a_k \in I$.* ∎

Notation 9.3. Given a commutative ring R, let p be any element of R. Define $\langle p \rangle \subseteq R$ to be the intersection of all ideals I such that $p \in I$.

This set $\langle p \rangle$ is an ideal in R and is equal to the set of all multiples of p, $\{r \cdot p : r \in R\}$.

Proposition 9.4. *For a field $\mathbb{F}$, the polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$ is a unique factorization domain.* ∎

This means that any polynomial $f \in \mathbb{F}[\vec{x}]$ can be factored as a product of irreducible polynomials, which are polynomials that cannot be factored as products of non-constant polynomials, and that for $f \neq 0$, the factorization is unique up to re-ordering and scalar multiplication.

Lemma 9.5. *For $f = f(x_1, \ldots, x_n) \in \mathbb{F}[\vec{x}]$, the following are equivalent: f is a non-constant irreducible polynomial $\iff$ $\langle f \rangle$ is a prime ideal in $\mathbb{F}[\vec{x}]$.* ∎

Notation 9.6. For a field $\mathbb{F}$ and a subset $B \subseteq \mathbb{F}^n$, the symbol $\mathcal{I}(B)$ refers to the set of polynomials in $\mathbb{F}[\vec{x}]$ which have value 0 at all points in B. $\mathcal{I}(B)$ is an ideal in the polynomial ring $\mathbb{F}[\vec{x}]$.

Definition 9.7. A subset $V \subseteq \mathbb{F}^n$ is an algebraic set means that there are finitely many polynomials $f_1(\vec{x}), \ldots, f_N(\vec{x}) \in \mathbb{F}[\vec{x}]$ so that

$$V = \{\vec{x} \in \mathbb{F}^n : f_1(\vec{x}) = \cdots = f_N(\vec{x}) = 0\}. \tag{9.1}$$

In the case $\mathbb{F} = \mathbb{R}$, an algebraic set can always be defined by just one polynomial, $P = f_1^2 + \cdots + f_N^2$. This is consistent with the notation $\mathcal{Z}(P)$ for a real algebraic set in [CP].

Proposition 9.8. *The intersection of two algebraic sets is an algebraic set.* ∎

Theorem 9.9. *For a set $V \subseteq F^n$, and a algebraic set $W \subseteq F^n$, $V \subseteq W \iff \mathcal{I}(W) \subseteq \mathcal{I}(V)$.* ∎

Corollary 9.10. *For two algebraic sets V, $W \subseteq F^n$, $V = W \iff \mathcal{I}(W) = \mathcal{I}(V)$.* ∎

Definition 9.11. A algebraic set $V \subseteq \mathbb{F}^n$ is irreducible means: if $V = V_1 \cup V_2$ for algebraic sets V_1, V_2, then $V_1 = V$ or $V_2 = V$.

Theorem 9.12. *For an algebraic set V, V is irreducible $\iff$ $\mathcal{I}(V)$ is a prime ideal in $\mathbb{F}[\vec{x}]$.* ∎

Definition 9.13. Given a non-empty subset $V \subseteq \mathbb{F}^n$, the dimension of V is $m-1 = \dim(V)$, where m is the largest number so that there exist distinct prime ideals I_1, $I_2, \ldots, I_m$ in $\mathbb{F}[\vec{x}]$ such that:

$$\mathcal{I}(V) \subseteq I_m \subsetneq \ldots I_2 \subsetneq I_1 \subsetneq \mathbb{F}[\vec{x}].$$

For example, if V is a union of distinct irreducible surfaces S and T in $\mathbb{R}^n$ and V does not contain any three-dimensional neighborhood, then $\mathcal{I}(V) \subseteq \mathcal{I}(S)$ and S contains some irreducible curve C through a point P, so $\mathcal{I}(S) \subsetneq \mathcal{I}(C) \subsetneq \mathcal{I}(P) \subsetneq \mathbb{F}[\vec{x}]$.

Proposition 9.14. *If $\emptyset \neq V \subseteq W \subseteq F^n$ then $\dim(V) \leq \dim(W)$.* ∎

Proposition 9.15. *For any algebraic set V in $\mathbb{F}^n$, there exists a unique finite set of irreducible algebraic sets $\{V_1, \ldots, V_k\}$ so that $V = V_1 \cup V_2 \cup \ldots V_k$ and if $k > 1$ then each V_j is not contained in the union of the other $k-1$ sets. The dimension of $V \neq \emptyset$ is equal to the maximum of $\{\dim(V_1), \ldots, \dim(V_k)\}$.*

Sketch. The statement holds for any field $\mathbb{F}$. We refer to [BCR] Theorem 2.8.3.i. ∎

Proposition 9.16. *For an algebraic set $V \subseteq \mathbb{F}^n$, the following are equivalent.*

(1) $\dim(V) = n$.
(2) $V = \mathbb{F}^n$.

Sketch. The statement holds for any field $\mathbb{F}$. The (2) $\implies$ (1) statement $\dim(\mathbb{F}^n) = n$ is exhibited by the chain of prime ideals

$$\mathcal{I}(\mathbb{F}^n) = \langle 0 \rangle \subsetneq J_n \subsetneq \ldots J_2 \subsetneq J_1 \subsetneq \mathbb{F}[\vec{x}], \tag{9.2}$$

where J_k is the ideal generated by the monomials $x_1, \ldots, x_{n-k+1}$, and the claim is that there does not exist a longer chain of prime ideals in $\mathbb{F}[\vec{x}]$.

Conversely for (1) $\implies$ (2), let V_1 be an irreducible component of V with $\dim(V_1) = n$ as in Proposition 9.15, so that $V_1 \subseteq V \subseteq \mathbb{F}^n \implies \langle 0 \rangle \subseteq \mathcal{I}(V) \subseteq \mathcal{I}(V_1)$ and $\mathcal{I}(V_1)$ is a prime ideal by Theorem 9.12. From Definition 9.13 applied to $\dim(V_1) = n$, we get a chain of prime ideals

$$\langle 0\rangle \subseteq \mathcal{I}(V_1) \subsetneq I_n \subsetneq \dots I_2 \subsetneq I_1 \subsetneq \mathbb{F}[\vec{x}],$$

which would contradict the claim that (9.2) is the longest possible such chain, unless $\langle 0\rangle = \mathcal{I}(V_1)$. The conclusion is that $\mathcal{I}(V) = \langle 0\rangle$ and $V = \mathbb{F}^n$ by Corollary 9.10. ∎

In the case $\mathbb{F} = \mathbb{R}$, $\dim = n-1$, the dimension as defined above using just algebra matches the differential-topological notion of dimension; the connection between them is …derivatives!

Proposition 9.17. *Given an irreducible polynomial $P \in \mathbb{R}[\vec{x}]$, the following are equivalent.*

1. $\dim(\mathcal{Z}(P)) = n-1$.
2. *There exists a point $x \in \mathcal{Z}(P)$ so that $\vec{\nabla}P \neq \vec{0}$.*
3. *There exists a point $x \in \mathcal{Z}(P)$ and a neighborhood U of x in $\mathbb{R}^n$ so that $U \cap \mathcal{Z}(P)$ is a $\mathcal{C}^\infty$-smooth submanifold of $\mathbb{R}^n$ with topological dimension $n-1$.*

Sketch. The equivalence (1) $\iff$ (2) is the (*iii*) $\iff$ (*v*) implication from Theorem 4.5.1 of [BCR]. The proof uses a technical notion of x being a nonsingular point of $\mathcal{Z}(P)$, and some ideas about nonsingular points that work for $\mathbb{R}$ but may not hold over all fields. The machinery developed in Chapter 2 of [BCR] about the topology of semialgebraic sets leads to a conclusion about the dimension being $n-1$. In particular, [BCR] Lemma 4.5.2 shows how $\mathcal{Z}(P)$ separates the set $\{P > 0\}$ from $\{P < 0\}$ near a point where $\vec{\nabla}P \neq \vec{0}$. This does not appear to need the full strength of the Real Nullstellensatz or other results from [BCR] Chapter 4.

For (2) $\iff$ (3), the gradient $\vec{\nabla}P$ is continuous at x and non-zero in some neighborhood U (Euclidean or Zariski) and $\mathcal{Z}(P) \cap U = (P|_U)^{-1}(\{0\})$ is a regular submanifold as in ([B] Theorem 5.8). ∎

10. Theorems on common factors

The first Theorem and Corollary work for any field $\mathbb{F}$.

Theorem 10.1. *Given polynomials f, $g \in \mathbb{F}[x_1, \dots, x_n]$, with f, g not both $\equiv 0$, if the algebraic set $\mathcal{Z}(f) \cap \mathcal{Z}(g)$ has dimension $n-1$, then there exist polynomials q, m_1, m_2, so that q is non-constant and irreducible, and $f = m_1 q$, and $g = m_2 q$.*

Proof. Let W be an irreducible component of the algebraic set $\mathcal{Z}(f) \cap \mathcal{Z}(g)$, with $\dim(W) = n-1$, as in Proposition 9.8 and Proposition 9.15. By Theorem 9.12, $\mathcal{I}(W)$ is a prime ideal, and by Definition 9.13, there is a longest chain of prime ideals such that $\langle 0\rangle \neq \mathcal{I}(W)$ and $I_1 \neq \mathbb{F}[\vec{x}]$:

$$\mathcal{I}(W) = I_n \subsetneq I_{n-1} \subsetneq \dots I_2 \subsetneq I_1. \tag{10.1}$$

Let p be any non-zero element of $\mathcal{I}(W)$. By Proposition 9.4, the polynomial p has a unique factorization $p = p_1 \cdot \ldots \cdot p_k$ with each factor a non-constant irreducible polynomial. Because $\mathcal{I}(W)$ is a prime ideal, one of these factors, say p_1, satisfies $p_1 \in \mathcal{I}(W)$ by Proposition 9.2.

As in Notation 9.3, the ideal generated by p_1 is $\langle p_1\rangle$, equal to the set of all polynomial multiples of p_1, and $\langle p_1\rangle \subseteq \mathcal{I}(W)$. By Lemma 9.5, p_1 being irreducible implies that $\langle p_1\rangle$ is a prime ideal. So we have a chain of prime ideals extending the chain from (10.1):

$$\langle 0\rangle \subsetneq \langle p_1\rangle \subseteq \mathcal{I}(W) = I_n \subsetneq I_{n-1} \subsetneq \dots I_2 \subsetneq I_1. \tag{10.2}$$

If $\langle p_1\rangle \neq \mathcal{I}(W)$ then this chain would be longer than the chain from (9.2), contradicting Proposition 9.16. We can conclude $\langle p_1\rangle = \mathcal{I}(W)$, meaning that any function vanishing on W is a polynomial multiple of p_1 — this applies to both polynomials f and g, so $q = p_1$ is the claimed common factor. ∎

Corollary 10.2. *Given polynomials $f, g \in \mathbb{F}[x_1, \dots, x_n]$, with f, g not both $\equiv 0$, if f and g do not have any non-constant common factor, then the algebraic set $\mathcal{Z}(f) \cap \mathcal{Z}(g)$ has dimension $< n-1$.* ∎

Theorem 10.3. *Given polynomials $P, Q \in \mathbb{R}[x_1, \dots, x_n]$, with P, Q not both $\equiv 0$, the following are equivalent.*

1. *The algebraic set $\mathcal{Z}(P) \cap \mathcal{Z}(Q)$ has dimension $n-1$.*
2. *There exist polynomials M_1, M_2, an irreducible polynomial F, and a point $x \in \mathbb{R}^n$, so that $F(x) = 0$, $\vec{\nabla}F(x) \neq \vec{0}$, $P = M_1F$, and $Q = M_2F$.*

Proof. The (1) $\implies$ (2) direction begins by choosing W in the same way as in Theorem 10.1. Then, instead of choosing any $p \in \mathcal{I}(W)$, we pick p more specifically as follows.

By Proposition 3.3.14. of [BCR], there exists a nonsingular point x of W. [Comment: As in Proposition 9.17, this proposition on the nonsingular point x works for $\mathbb{R}$ but not for all fields $\mathbb{F}$.] By Proposition 3.3.8.ii. of [BCR], there exists a polynomial $p \in \mathcal{I}(W)$ so that $\vec{\nabla}p(x) \neq \vec{0}$. [Comment: The existence of p is a non-trivial result from commutative algebra.]

By Proposition 9.4, the polynomial p has a unique factorization $p = p_1 \cdots p_k$ with each factor an irreducible polynomial. For the above point x, there is at least one factor, say p_1, so that $p_1(x) = 0$. If there were another factor, say p_2, with $p_2(x) = 0$, then

$$\begin{aligned} \vec{\nabla}p &= (\vec{\nabla}p_1)p_2 \cdots p_k + p_1(\vec{\nabla}p_2) \cdots p_k + p_1p_2\vec{\nabla}(p_3 \cdots p_k) \\ \implies \vec{\nabla}p(x) &= \vec{0} + \vec{0} + \vec{0}, \end{aligned}$$

contradicting the construction of p. So $p_2(x) \cdots p_k(x) \neq 0$, and the above product rule calculation also shows that $\vec{\nabla}p_1(x) \neq \vec{0}$. Also, $p_2 \cdots p_k \notin \mathcal{I}(W)$, and because $\mathcal{I}(W)$ is a prime ideal, $p_1 \in \mathcal{I}(W)$.

This irreducible polynomial p_1 is the claimed F, by showing $\langle p_1\rangle = \mathcal{I}(W)$ exactly as in the last paragraph in the proof of Theorem 10.1.

For the (2) $\implies$ (1) direction, the properties of F assumed in statement (2) imply that $\dim(\mathcal{Z}(F)) = n-1$, by Proposition 9.17. By Notation 9.3, $P \in \langle F\rangle \implies \langle P\rangle \subseteq \langle F\rangle$, and it follows that $\mathcal{Z}(F) \subseteq \mathcal{Z}(P)$. Similarly for Q, so $\mathcal{Z}(F) \subseteq \mathcal{Z}(P) \cap \mathcal{Z}(Q)$. By Proposition 9.14, $\dim \mathcal{Z}(P) \cap \mathcal{Z}(Q) \geq n-1$, and by Proposition 9.16, P, Q not both $\equiv 0$ implies (1). ∎

Corollary 10.4. *Given a rational function $\frac{P(x_1,\ldots,x_n)}{Q(x_1,\ldots,x_n)}$ with $P,\ Q \in \mathbb{R}[x_1,\ldots,x_n]$, the following are equivalent.*

1. *The indeterminacy set $\{(x_1,\ldots,x_n) : \frac{P(x_1,\ldots,x_n)}{Q(x_1,\ldots,x_n)} = \frac{0}{0}\}$ has dimension $n-1$.*
2. *There exist polynomials M_1, M_2, an irreducible polynomial F, and a point $x \in \mathbb{R}^n$, so that $F(x) = 0$, $\vec{\nabla}F(x) \neq \vec{0}$, $P = M_1 F$, and $Q = M_2 F$.*

∎

11. More properties of rational functions

Here we give a complete and elementary proof of [CP] Proposition 2.7.

Proposition 11.1. *For an open set $\Delta \subseteq \mathbb{R}^n$, if there is a point $\vec{x}_0 \in \Delta$ and a polynomial $Q \not\equiv 0$ with $Q(\vec{x}_0) = 0$, then for any polynomial $P \not\equiv 0$ there exists a number $\xi_0 > 0$ so that the function $\dfrac{P(\vec{x})}{|Q(\vec{x})|^{\xi}}$ is not bounded on $\Delta \setminus \mathcal{Z}(Q)$ for any $\xi \geq \xi_0$.*

Proof. If $\Delta \cap \mathcal{Z}(Q) \nsubseteq \Delta \cap \mathcal{Z}(P)$ then P/Q is not bounded on Δ, by [CP] Lemma 2.5, and $\xi_0 = 1$. So we continue under the assumption that $\vec{x}_0 \in \Delta \cap \mathcal{Z}(Q) \subseteq \Delta \cap \mathcal{Z}(P)$. Because $P \not\equiv 0$, there is some real line $\vec{\ell}(t) = \vec{x}_0 + t\vec{v}$ so that $P(\vec{\ell}(t)) \not\equiv 0$. This composite is a single-variable polynomial of the form $P(\vec{\ell}(t)) = t^{\mu}p(t)$ for some multiplicity $\mu \geq 1$ and polynomial factor with $p(0) \neq 0$. Similarly, because $Q(\vec{\ell}(t)) \equiv 0$ would contradict $\Delta \cap \mathcal{Z}(Q) \subseteq \Delta \cap \mathcal{Z}(P)$, $Q(\vec{\ell}(t)) = t^{\nu}q(t)$ for some $\nu \geq 1$ and $q(0) \neq 0$. For any $\xi_0 > \mu/\nu$, $\lim_{t\to 0} \dfrac{|P(\vec{\ell}(t))|}{|Q(\vec{\ell}(t))|^{\xi_0}} = \lim_{t\to 0} \dfrac{|t^{\mu}p(t)|}{|t|^{\nu\xi_0}|q(t)|^{\xi_0}} = +\infty$.

For $\xi > \xi_0$, the unboundedness of $\dfrac{P(\vec{x})}{|Q(\vec{x})|^{\xi}}$ follows from the unboundedness of $\dfrac{P(\vec{x})}{|Q(\vec{x})|^{\xi_0}}$, by [CP] Lemma 2.6 applied to the continuous (not necessarily polynomial) functions $P(\vec{x})$ and $|Q(\vec{x})|^{\xi}$. ∎

The following Lemma shows how, for a rational function P/Q, to choose coordinates so that the indeterminacy locus does not contain a specific line. It is a real variable version (with the same calculation) of an Observation on the Weierstrass Preparation Theorem in Section 2.1 of [GR].

Lemma 11.2. *For polynomials $P_1(\vec{x}),\ldots,P_m(\vec{x})$ in $\vec{x} = (x_1,\ldots,x_n) = (x',x_n)$, all $\not\equiv 0$, there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ so that for every $k = 1,\ldots,m$, the composite polynomial $P_k \circ T$ satisfies*

$$P_k(T(0,\ldots,0,x_n)) \not\equiv 0.$$

Proof. For each polynomial P_k, choose any degree d_k so that there is some term of degree d_k with a non-zero coefficient, and let P_k^H be the homogeneous part of P_k with all of its degree d_k terms. Let P_k^R be all the other terms, so that:

$$\begin{aligned} P_k &= P_k^H + P_k^R \\ P_k^H &= p_{k,n}x_n^{d_k} + \sum_{j=1}^{d_k}\sum_{|I|=j} p_{k,I}(x')^I x_n^{d_k-j}. \end{aligned}$$

If the coefficient $p_{k,n}$ is non-zero, then $P_k(0,\ldots,0,x_n) = p_{k,n}x_n^{d_k} + P_k^R(0,\ldots,0,x_n) \not\equiv 0$, and if $p_{k,n} \neq 0$ for all k, then we can let T be the identity map. Otherwise, consider an upper triangular linear transformation of the form

$$T(x_1,\ldots,x_n) = (x_1+\epsilon_1 x_n, x_2+\epsilon_2 x_n, \ldots, x_{n-1}+\epsilon_{n-1}x_n, x_n),$$

for small real constants $\epsilon_1,\ldots,\epsilon_{n-1}$. T is invertible and composing with P_k does not change the total degree of any term.

$$\begin{aligned} P_k^H(T(\vec{x})) &= p_{k,n}x_n^{d_k} + \sum_{j=1}^{d_k}\sum_{|I|=j} p_{k,I}(x_1+\epsilon_1 x_n)^{i_1}\cdots(x_{n-1}+\epsilon_{n-1}x_n)^{i_{n-1}}x_n^{d_k-j} \\ &= p_{k,n}x_n^{d_k} + \sum_{j=1}^{d_k}\sum_{|I|=j} p_{k,I}\epsilon_1^{i_1}\cdots\epsilon_{n-1}^{i_{n-1}}x_n^{d_k} + \sum_{j=1}^{d_k}\sum_{|I|=j}\tilde{p}_{k,I}(x')^I x_n^{d_k-j} \\ &= P_k^H(\epsilon_1,\ldots,\epsilon_{n-1},1)x_n^{d_k} + \sum_{j=1}^{d_k}\sum_{|I|=j}\tilde{p}_{k,I}(x')^I x_n^{d_k-j}, \end{aligned}$$

where there are some new coefficients $\tilde{p}$ from collecting like terms in the expanded composite. If $P_k^H(x',1) \equiv 0$, then because $P_k^H(\vec{x})$ is homogeneous, $P_k^H(x',x_n) = 0$ for any (x',x_n) with $x_n \neq 0$, and so $P_k^H(\vec{x})$ would $\equiv 0$, contradicting the construction. So $P_k^H(x',1) \not\equiv 0$. The claim of the Lemma follows from showing there exist $\epsilon_1,\ldots,\epsilon_{n-1}$ so that $P_k^H(\epsilon_1,\ldots,\epsilon_{n-1},1)$ is simultaneously $\neq 0$ for $k=1,\ldots,m$. The union of zero sets:

$$\bigcup_{k=1}^m \{x' : P_k^H(x',1) = 0\}$$

is a real algebraic set with empty interior in $\mathbb{R}^{n-1}$ by [BCR] Proposition 2.8.4. So, there exists some $x' = (\epsilon_1,\ldots,\epsilon_{n-1})$ in the complement, which can be taken arbitrarily close to $(0,\ldots,0)$. ∎

12. Real algebraic curves

Here we give a complete proof of [CP] Proposition 3.4.

Proposition 12.1. *Given a semi-algebraic set $X \subseteq \mathbb{R}^2$ with dimension 1, if X contains $(0,0)$ then there exist $\delta > 0$, $\eta > 0$, $K \geq 0$ so that the intersection of X with the open box $(0,\delta)\times(0,\eta)$ is a disjoint, finite union of graphs of increasing functions, $b_1,\ldots,b_K$ so that for each k, $\lim_{u\to 0^+} b_k(u) = 0$ and $b_k : (0,\delta) \to (0,\eta)$ is of the form $\tilde{b}_k(u^{1/e_k})$ where e_k is a positive integer and $\tilde{b}_k$ is real analytic on $(-\delta^{1/e_k},\delta^{1/e_k})$.*

Proof. Let (u,v) denote the coordinates of $\mathbb{R}^2$. The one-dimensional semi-algebraic set X is contained in (but possibly not equal to) some real algebraic curve $\mathcal{Z}(F(u,v))$, with $F((0,0)) = 0$, by [BCR] Prop. 2.1.8. $\mathcal{Z}(F)$ can contain only finitely many circles centered at $(0,0)$, so any circle contained in the interior of the smallest such circle will meet $\mathcal{Z}(F)$ in finitely many points.

It is possible that $(0,0)$ is an isolated point in X, in which case the claim trivially holds with X meeting a small box in $K = 0$ graphs.

Otherwise, by [BCR] Prop. 9.3.6., there is some small closed disk $\overline{B}_{r_1}(\vec{0})$, whose boundary circle meets X at finitely many points $\{\vec{w}_1, \dots, \vec{w}_J\} \subseteq \mathcal{Z}(F)$ as described above, and so that $X \cap (\overline{B}_{r_1} \setminus \{(0,0)\})$ is a finite union of disjoint connected components, called half-branches, each the homeomorphic image $\phi_k((0,1])$ of a continuous semi-algebraic parametrization $\phi_k : [0,1] \to \overline{B}_{r_1}$ with coordinates $\phi_k(t) = (u_k(t), v_k(t))$, $\phi_k(0) = (0,0)$, $\phi_k(1) = \vec{w}_k$. (In particular, $\overline{B}_{r_1}$ is small enough to exclude any isolated points in X.) By [BCR] Prop. 8.1.12., for each k there exist $\delta_{1,k} > 0$, a positive integer p_k, and a real analytic function $\tilde{u}_k : (-\delta_{1,k}, \delta_{1,k}) \to \mathbb{R}$ so that $\tilde{u}_k(t) = u_k(t^{p_k})$ for $t \in [0, \delta_{1,k})$. Similarly, there exists $\tilde{v}_k : (-\delta'_{1,k}, \delta'_{1,k}) \to \mathbb{R}$ so that $\tilde{v}_k(t) = v_k(t^{q_k})$. Because the zeroes of $d\tilde{u}_k/dt$ do not accumulate at $t = 0$ unless $\tilde{u}_k$ is constant, there is some $\delta_{2,k} \le \delta_{1,k}$ where $\tilde{u}_k$ is either increasing, decreasing, or constant on $[0, \delta_{2,k}]$, and similarly for $\tilde{v}_k$ on $[0, \delta'_{2,k}]$. It follows that there is some interval $[0, \delta_{3,k}]$ on which u_k is increasing, decreasing, or constant, and similarly for v_k on $[0, \delta'_{3,k}]$. So for $\delta_{4,k} = \min\{\delta_{3,k}, \delta'_{3,k}\}$, the parametric map $\phi_k = (u_k, v_k)$ restricted to $(0, \delta_{4,k}]$ is contained in exactly one open quadrant or on the u-axis or v-axis, and still a homeomorphism onto its image arc. To exclude everything except these arcs from the claimed box, the complementary arc $\phi_k([\delta_{4,k}, 1])$ is compact and in the exterior of a small disk $B_{\epsilon_k}(\vec{0})$. Corresponding to $\epsilon = \min\{\epsilon_k\} > 0$, there is for each k some $\delta_{5,k} \le \delta_{4,k}$ so that if $0 \le t < \delta_{5,k}$ then $\phi_k(t)$ is in the open disk $B_\epsilon(\vec{0})$, and $X \cap B_\epsilon$ contains only points of the form $\phi_k(t)$ with $t < \delta_{4,k}$.

At this point we select the arcs ϕ_k contained in the positive first quadrant, where u_k and v_k are continuous, semi-algebraic, and increasing on $[0, \delta_{5,k}]$, and (unless $K = 0$, which is possible here, too) re-label the indices k from 1 to K. The inverse of $u_k(t)$ is also continuous, semi-algebraic, and increasing, denoted $t_k(u)$ for $0 \le u \le \delta_{6,k}$. The composite $v_k(t_k(u))$ is continuous, semi-algebraic, and increasing on $[0, \delta_{6,k}]$, and $u \mapsto (u, v_k(t_k(u)))$ parametrizes the same arc as (u_k, v_k). By [BCR] Prop. 8.1.12. again, for each k there exist $\delta_{7,k} > 0$, a positive integer e_k, and a real analytic function $\tilde{b}_k : (-\delta_{7,k}, \delta_{7,k}) \to \mathbb{R}$ so that $\tilde{b}_k(x) = (v_k \circ t_k)(x^{e_k})$ for $x \in [0, \delta_{7,k})$. Equivalently, $\tilde{b}_k(u^{1/e_k}) = (v_k \circ t_k)(u)$ for $u \in [0, \delta_{7,k}^{e_k})$. Then $\delta = \min\{\delta_{7,k}^{e_k}\}/2$ is a width for the box as claimed, the claimed functions $b_k(u)$ are the restrictions of $v_k \circ t_k$ to $0 < u < \delta$, and the claimed box contains all the b_k graphs and fits in the disk B_ϵ when the height η is $\max\{b_k(\delta)\}$. ∎

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