FORMAL STABILITY OF THE CR CROSS-CAP

Adam Coffman

For $m < n$, any real analytic m-submanifold of complex n-space with a non-degenerate CR singularity is shown to be locally equivalent, under a formal holomorphic coordinate change, to a fixed real algebraic variety defined by linear and quadratic polynomials. The situation is analogous to Whitney's stability theorem for cross-cap singularities of smooth maps. The formal coordinate change is obtained from a sequence of approximations whose norms are controlled only on polydiscs shrinking to a point, so the analytic classification remains an open question.

1. Introduction

In [**Wh**$_{43}$], H. Whitney described the parametrization $(u, v) \mapsto (u^2, v, uv)$ of the "cross-cap" as an example of a differentiable map from $\mathbb{R}^2$ to $\mathbb{R}^3$ with a singular point at the origin. He further demonstrated that for any sufficiently generic singularity of a map f from $\mathbb{R}^2$ to $\mathbb{R}^3$, there exists a local change of coordinates with respect to which f has exactly that normal form: $f(u, v) = (u^2, v, uv)$. The coordinate change is C^r, smooth, or real analytic in a neighborhood, as f is C^{4r+8}, smooth, or real analytic. This "stability theorem" was later generalized ([**H**]) to generic singularities of maps $f : \mathbb{R}^m \to \mathbb{R}^n$ for $m \leq n$, with an analogous normal form. A crucial step in Whitney's argument was the use of his famous lemma that a smooth, even function $F(x) = F(-x)$ of one variable can be written as a smooth function of x^2: $F(x) = g(x^2)$.

The objects of study in this paper are real m-submanifolds M of complex n-space $\mathbb{C}^n$. The occurence of a complex line in a tangent space T_xM when $m \leq n$ is called a "CR singularity," or "complex tangent," and some of the local and global geometric properties of these objects have analogues in the singularity theory of maps; the $m = n$ case has been studied since [**Bishop**]. Here the $m < n$ case is examined and compared with Whitney's normal form theorem.

Specifically, the local geometry of M near the complex tangent is analyzed by establishing some non-degeneracy conditions and arriving at a normal form for the quadratic part of the defining equations. The main result is the following Proposition:

Proposition 1.1. *Given m and n such that $\frac{2}{3}(n+1) \leq m < n$, there exists a real algebraic variety $\widetilde{M}^m$ in $\mathbb{C}^n$ defined by quadratic and linear polynomials such that a germ of any real analytic m-submanifold M^m of $\mathbb{C}^n$ at a suitably non-degenerate complex tangent is equivalent to $\widetilde{M}^m$ under a formal holomorphic coordinate change of $\mathbb{C}^n$.*

The simplest case of the Proposition is $m = 4$, $n = 5$, which follows as a consequence of Theorem 3.16. The generalization to higher dimensions is outlined in Subsection 2.4. The result contrasts with the $m = n$ case, where there can be higher order invariants of real analytic embeddings ([**MW**]). For 4-manifolds in $\mathbb{C}^5$, the corresponding variety $\widetilde{M}^4$ is defined by these equations in the variables $z_1 = x_1 + iy_1, \ldots, z_5 = x_5 + iy_5$:

$$\widetilde{M}^4 = \{\vec{z} : y_2 = y_3 = 0,\ z_4 = (\bar{z}_1 + x_2 + ix_3)^2,\ z_5 = z_1(\bar{z}_1 + x_2 + ix_3)\}.$$

Note there are no continuous invariants (again, unlike the $m = n$ case, where the normal form depends on the value of Bishop's invariant $\beta \geq 0$), and the two quantities $\bar{L}^2$ and $z_1\bar{L}$, where $\bar{L} = \bar{z}_1 + x_2 + ix_3$, cannot be simultaneously transformed into monomials by a holomorphic coordinate change (unlike Whitney's normal form).

A strict analogy with Whitney's result would suggest that a change of coordinates can be found which converges in a neighborhood of $\vec{0}$, but there are holomorphic normal form problems where the formal and analytic classifications are different — [**V**], for example. Here, the existence of formal power series defining a coordinate change bringing the singularity to normal form is established by an iterative procedure, where the main step is to solve a linearized equation. If the defining functions are quadratic plus degree d and higher terms, a coordinate change defined using the solution of the linear equation eliminates terms up to about twice the degree. However, control over a certain norm of this solution will only be established on a polydisc significantly smaller than that on which the defining equations were given. Iterating the linearization makes this polydisc arbitrarily small, and no conclusion can be drawn regarding the convergence of the formal solution. Contributing to this shrinking is a rearrangement phenomenon specific to the calculus of several variables. As in Whitney's calculations, some real analytic functions are split into even and odd parts, but the Proof of Theorem 3.16 will first rearrange series in z_1, $\bar{z}_1$, x_2, x_3 into series in z_1, $\bar{L}$, x_2, x_3, and then find the (even + odd) decomposition in the $\bar{L}$ variable. These even and odd parts may be real analytic on a smaller domain.

The effect of such a rearrangement on the polyradius of convergence of a power series will be analyzed in Section 3, to the extent that the details relate to the linearized normal form problem. Since the 4-manifold is not a uniqueness set for holomorphic functions in $\mathbb{C}^5$, there are divergent solutions of the linear equation, but, after a choice of normalization, a unique solution is constructed which converges on a small polydisc.

This paper is based on the main result of the author's 1997 dissertation, [**C**], supervised by S. Webster.

Note: after this paper was accepted for publication, the author learned that the $m = 4$, $n = 5$ case of Proposition 1.1 has also been considered by [**Beloshapka**].

2. Coordinate changes, and normal forms for complex tangents

Attention will be focused on the $m = 4$, $n = 5$ case of a real m-submanifold of a complex n-manifold. These are the lowest dimensions $m < n$ where complex tangents are "topologically stable"; for example, if a real surface in $\mathbb{C}^3$ has a complex tangent plane at some point, this property is not shared with most nearby surfaces. When $m < n$, the real m-planes T such that $\dim_{\mathbb{C}} T \cap iT \geq j$ form a subvariety D_j of real codimension $2j(n - m + j)$ inside the grassmannian of real m-planes in $\mathbb{C}^n \cong \mathbb{R}^{2n}$. The occurence of complex tangents of an immersion corresponds to the intersection of the Gauss map with D_j, and the immersion could be called "generic" if the immersed manifold's Gauss map meets each stratum $D_j \setminus D_{j+1}$ transversely. So, generic immersions of M in $\mathbb{C}^n$ are "totally real" outside a codimension $2(n-m+1)$ subset of M, and if $m < \frac{2}{3}(n+1)$, then M is generically totally real everywhere. (This resembles the bounds in Whitney's embedding and immersion theorems; see [$\mathbf{Wh}_{44}$].) The case addressed by this paper is $\frac{2}{3}(n + 1) \leq m < n$, and $j = 1$; only points where the tangent space contains exactly one complex line will be considered. In the $m = 4$, $n = 5$ case, the generic singularity is isolated. This Section establishes non-degeneracy conditions for complex tangents and proposes quadratic normal forms in the above dimension range. The initial assumption is that $M = M^4$ is a 4-dimensional submanifold containing the origin of $\mathbb{C}^5$, defined as the zero set of real analytic functions in a neighborhood of the origin.

2.1. Holomorphic coordinate changes. It can be assumed that the tangent plane at the origin in $\mathbb{C}^5$, $T = T_{\vec{0}}M$, has four real dimensions and contains the complex line with coordinate $z_1 = x_1 + iy_1$. The remaining coordinates of $\mathbb{C}^5$ can be chosen as $z_2, \ldots, z_5$, so that T is defined by $y_2 = y_3 = z_4 = z_5 = 0$ and has coordinates z_1, x_2, and x_3. The complex linear transformations of $\mathbb{C}^5$ that preserve T, written as 5×5 matrices acting on column vectors, are of the form

$$\mathbf{A} = \begin{pmatrix} a_1^1 & a_1^2 & a_1^3 & a_1^4 & a_1^5 \\ 0 & r_2^2 & r_2^3 & a_2^4 & a_2^5 \\ 0 & r_3^2 & r_3^3 & a_3^4 & a_3^5 \\ 0 & 0 & 0 & a_4^4 & a_4^5 \\ 0 & 0 & 0 & a_5^4 & a_5^5 \end{pmatrix},$$

where the a entries are complex, and the entries of the block $\mathbf{R} = \begin{pmatrix} r_2^2 & r_2^3 \\ r_3^2 & r_3^3 \end{pmatrix}$ are real.

Since these entries will be used in a coordinate change transforming the quadratic and cubic parts of the defining equations for M into a normal form, $\det(\mathbf{A}) \neq 0$. If T is given some orientation, then the transformations preserving that orientation also satisfy $\det(\mathbf{R}) > 0$. For example, the map $z_3 \mapsto -z_3$ is complex linear, but reverses the orientation of the real subspace.

Once the real submanifold M is placed with its tangent space at the origin as described, the geometry of interest is its interaction with the ambient complex analytic structure. To find features which are invariant under biholomorphic coordinate changes of $\mathbb{C}^5$, it suffices to consider local biholomorphisms with linear part preserving T. The transformation from z_s to $\tilde{z}_r$ coordinates, r, $s = 1, \ldots, 5$, and its holomorphic inverse, will be written as multivariable power series, using the summation convention:

$$\tilde{z}_r = a_r^s z_s + p_r^{st} z_s z_t + p_r^{stu} z_s z_t z_u + \ldots, \tag{1}$$
$$z_r = b_r^s \tilde{z}_s + q_r^{st} \tilde{z}_s \tilde{z}_t + q_r^{stu} \tilde{z}_s \tilde{z}_t \tilde{z}_u + \ldots, \tag{2}$$

where the linear coefficients form block matrices $\mathbf{A} = (a_r^s)$ and $\mathbf{A}^{-1} = (b_r^s)$, fixing the tangent space as described above.

The six real defining equations of M in $\mathbb{C}^5$ can be expressed in the z_s coordinates as a graph, over a region in the tangent space:

$$M = \{\vec{z} : y_2 = H_2, y_3 = H_3, z_4 = h_4, z_5 = h_5\}.$$

The functions $H_2(z_1, \bar{z}_1, x_2, x_3)$ and $H_3(z_1, \bar{z}_1, x_2, x_3)$ are real-valued, the functions $h_4(z_1, \bar{z}_1, x_2, x_3)$ and $h_5(z_1, \bar{z}_1, x_2, x_3)$ are complex-valued, and all are real analytic, vanishing to second order at the origin. For example, h_5 begins with quadratic terms:

$$h_5 = \alpha z_1^2 + \beta \bar{z}_1^2 + \gamma z_1 \bar{z}_1 + c^{ab} x_a x_b + d^a x_a z_1 + e^a x_a \bar{z}_1 + O(3),$$

where the summations are over $2 \leq a \leq b \leq 3$ and $O(3)$ denotes terms of degree three or higher. This expression can be simplified by a linear change of coordinates.

Lemma 2.1. *If $|\beta| \neq |\gamma|/2$, then there exist complex numbers a_1^2 and a_1^3, such that in new variables $\tilde{z}_1 = z_1 + a_1^2 z_2 + a_1^3 z_3$, $\tilde{z}_s = z_s$ for $s = 2, \ldots, 5$, the above defining function h_5 is transformed to:*

$$\tilde{z}_5 = \alpha \tilde{z}_1^2 + \beta \bar{\tilde{z}}_1^2 + \gamma \tilde{z}_1 \bar{\tilde{z}}_1 + \tilde{c}^{ab} \tilde{x}_a \tilde{x}_b + \tilde{d}^a \tilde{x}_a \tilde{z}_1 + O(3).$$

∎

This Lemma appears as part of the calculations of [**Bishop**], so a proof is not given here. Note that the coefficients c^{ab} and d^a may change, the $e^a x_a \bar{z}_1$ terms are eliminated, and the coefficients α, β, and γ are unchanged. The coordinate change may also change some coefficients in h_5 of the higher-order terms, and of some of the terms appearing in the other three defining equations. Also note that no holomorphic coordinate change of the form $\tilde{z}_5 = z_5 + p_5^{st} z_s z_t$ could eliminate the $e^a x_a \bar{z}_1$ terms; transforming the z_1 variable is necessary, and under the hypothesis of the Lemma, the e^2, e^3 coefficients can be transformed to any pair of complex numbers. The idea of the Lemma, using a holomorphic transformation of z_1 to eliminate terms involving $\bar{z}_1$, will be generalized to be the key idea for the solution of the normal form problem (Theorem 3.16).

2.2. The quadratic normal form. The general form of the real functions H_2, H_3, and complex functions h_4, h_5, which define M in $\mathbb{C}^5$, is as follows:

$$\begin{aligned}
z_2 &= x_2 + iH_2 = x_2 + i(\mathrm{Re}(k_2^a x_a z_1 + k_2 z_1^2) + r_2^{ab} x_a x_b + r_2 z_1 \bar{z}_1) + O(3)\\
z_3 &= x_3 + iH_3 = x_3 + i(\mathrm{Re}(k_3^a x_a z_1 + k_3 z_1^2) + r_3^{ab} x_a x_b + r_3 z_1 \bar{z}_1) + O(3)\\
z_4 &= h_4 = \alpha_4 z_1^2 + \beta_4 \bar{z}_1^2 + \gamma_4 z_1 \bar{z}_1 + c_4^{ab} x_a x_b + d_4^a x_a z_1 + e_4^a x_a \bar{z}_1 + O(3)\\
z_5 &= h_5 = \alpha_5 z_1^2 + \beta_5 \bar{z}_1^2 + \gamma_5 z_1 \bar{z}_1 + c_5^{ab} x_a x_b + d_5^a x_a z_1 + e_5^a x_a \bar{z}_1 + O(3).
\end{aligned}$$

The coefficients are all complex except for the real coefficients r on the "conjugation-invariant" terms in H_2 and H_3.

The plan is to find a coordinate change $(1-2)$ to convert these quadratic quantities into a simpler "normal form." The linear and second-order coordinate changes that affect the quadratic terms are of the form $\tilde{z}_r = a_r^d z_d + p_r^{st} z_s z_t$ for $1 \le d \le 5$ and $1 \le s \le t \le 3$. The linear coefficients a_1^4, a_1^5 and the nonlinear coefficients p_1^{st} do not contribute quadratic terms.

The "first non-degeneracy condition" is that the matrix of coefficients $\begin{pmatrix} \beta_4 & \gamma_4 \\ \beta_5 & \gamma_5 \end{pmatrix}$ is nonsingular. If the matrix has rank 1 or 0, this degeneracy is preserved under any change of coordinates, and the degenerate normal forms will be listed in the next Subsection. The case of main interest is that the matrix has rank 2, so that the block $\begin{pmatrix} a_4^4 & a_4^5 \\ a_5^4 & a_5^5 \end{pmatrix}$ can be used to normalize these coefficients to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then Lemma 2.1 applies to h_5 to eliminate the $e_5^a x_a \bar{z}_1$ terms, using the linear coefficients a_1^2, a_1^3.

Lemma 2.1 cannot be used again, on the h_4 terms, without possibly re-introducing $x_a \bar{z}_1$ terms into h_5. (This problem will occur later, in the elimination of higher-order terms from both h_4 and h_5; the matter will be resolved by transforming the x_a variables to normalize h_4.) The current state of the quadratic terms is:

$$\begin{aligned}
h_4 &= \bar{z}_1^2 + \alpha_4 z_1^2 + c_4^{ab} x_a x_b + d_4^a x_a z_1 + e_4^a x_a \bar{z}_1 + O(3)\\
h_5 &= z_1 \bar{z}_1 + \alpha_5 z_1^2 + c_5^{ab} x_a x_b + d_5^a x_a z_1 + O(3).
\end{aligned}$$

The sum $e_4^a x_a \bar{z}_1$ can be rewritten as a matrix product:

$$\left(\begin{array}{cc} 1 & i \end{array} \right) \left(\begin{array}{cc} \mathrm{Re}(e_4^2) & \mathrm{Re}(e_4^3) \\ \mathrm{Im}(e_4^2) & \mathrm{Im}(e_4^3) \end{array} \right) \left(\begin{array}{c} x_2 \\ x_3 \end{array} \right) \bar{z}_1,$$

and the "second non-degeneracy condition" is that this real 2×2 coefficient matrix is non-singular. In this case, the real block $\mathbf{R}$ can transform the x_2, x_3 coordinates so that these terms have the form $x_2\bar{z}_1 + ix_3\bar{z}_1$. The remaining quadratic terms, for example, $d_4^2 x_2 z_1$, do not involve $\bar{z}_1$, and their coefficients can be made to vanish, or to attain any complex value, by the non-linear transformations of the form $\tilde{z}_4 = z_4 + p_4^{12} z_1 z_2$, etc. Quadratic transformations of z_2 and z_3 can eliminate most of the terms of H_2 and H_3, except the $z_1\bar{z}_1$ terms. These terms, assuming the first non-degeneracy condition holds, can be eliminated by linear transformations of the form $\tilde{z}_2 = z_2 + a_2^5 z_5$, $\tilde{z}_3 = z_3 + a_3^5 z_5$.

With as many terms as possible eliminated, the non-degenerate quadratic normal form for M is:

$$\{y_2 = O(3),\ y_3 = O(3),\ z_4 = \bar{z}_1^2 + x_2\bar{z}_1 + ix_3\bar{z}_1 + O(3),\ z_5 = |z_1|^2 + O(3)\}.$$

To make a normal form more convenient for later calculations, "complete the square" for h_4, and use the same linear factor $\bar{L} = \bar{z}_1 + x_2 + ix_3$ in h_5:

$$\begin{aligned} h_4(z_1, \bar{z}_1, x_2, x_3) &= (\bar{z}_1 + x_2 + ix_3)^2 + O(3) = \bar{L}^2 + O(3) \\ h_5(z_1, \bar{z}_1, x_2, x_3) &= z_1(\bar{z}_1 + x_2 + ix_3) + O(3) = z_1\bar{L} + O(3). \end{aligned}$$

If the linear coordinate change $\mathbf{A}$ is required to fix an orientation of the real tangent plane, then the second non-degeneracy condition leads to two alternatives for the new coefficients on $x_a\bar{z}_1$ terms: either $x_2\bar{z}_1 + ix_3\bar{z}_1$ or $x_2\bar{z}_1 - ix_3\bar{z}_1$ in the first set of equations, or $\bar{L} = \bar{z}_1 + x_2 \pm ix_3$ in the second set. The $\pm$ sign is not a biholomorphic invariant since it can change under the previously mentioned transformation $\tilde{z}_3 = -z_3$. The two normal forms are also related by complex conjugation of all five coordinates.

2.3. Degenerate cases of the normal form. Complex tangents where the non-degeneracy conditions fail can still be put into a quadratic normal form, with the possibility of continuous invariants. The quadratic terms of the real-valued functions H_2 and H_3 can be made to vanish, or, if there is no $z_1\bar{z}_1$ term in h_4 or h_5, then $H_2 = r_2 z_1\bar{z}_1 + O(3)$ and $H_3 = r_3 z_1\bar{z}_1 + O(3)$. The quadratic normal forms (in the unoriented case) for h_4 and h_5, including the full rank case, are as follows:

$$\begin{aligned} h_4 &= (\bar{z}_1 + e_4^2 x_2 + e_4^3 x_3)^2 + O(3), \\ h_5 &= z_1(\bar{z}_1 + e_4^2 x_2 + e_4^3 x_3) + O(3), \ (e_4^2, e_4^3) \in \{(1,i),(1,0),(0,0)\}; \end{aligned}$$

$$\begin{aligned} h_4 &= \beta(z_1^2 + \bar{z}_1^2) + |z_1|^2 + O(3), \ \beta \in [0, \tfrac{1}{2}) \cup (\tfrac{1}{2}, \infty], \\ h_5 &= e_5^2 x_2 \bar{z}_1 + e_5^3 x_3 \bar{z}_1 + O(3), \ (e_5^2, e_5^3) \in \{(1,i),(1,0),(0,0)\}; \end{aligned}$$

$$\begin{aligned} h_4 &= e_4^2 x_2 \bar{z}_1 + \frac{1}{2}(z_1 + \bar{z}_1)^2 + O(3), \\ h_5 &= e_5^2 x_2 \bar{z}_1 + e_5^3 x_3 \bar{z}_1 + O(3), \\ & \quad (e_4^2, e_5^2, e_5^3) \in \{(1,0,1),(1,0,0),(0,1,i),(0,1,0),(0,0,0)\}; \end{aligned}$$

$$\begin{aligned} h_4 &= e_4^2 x_2 \bar{z}_1 + O(3), \\ h_5 &= e_5^3 x_3 \bar{z}_1 + O(3), \ (e_4^2, e_5^3) \in \{(1,1),(1,0),(0,0)\}. \end{aligned}$$

For example, the defining functions $z_4 = \bar{L}^2$, $z_5 = L\bar{L} = (z_1 + x_2 - ix_3)(\bar{z}_1 + x_2 + ix_3)$ satisfy the first non-degeneracy condition, but not the second, and can be transformed into the $(0,0)$ case of the first group.

The h_4 functions in the second and third groups resemble the "elliptic", "hyperbolic", and "parabolic" cases when $m = n$. If $M^2 = \{z_2 = h(z_1, \bar{z}_1)\}$ is a surface in $\mathbb{C}^2$ with a complex tangent, then $M^2 \times \mathbb{R}^2 \times \{0\} = \{y_2 = y_3 = z_5 = 0, \ z_4 = h(z_1, \bar{z}_1)\}$, contained inside $\mathbb{C}^4 \subseteq \mathbb{C}^5$ is not only degenerate (in the sense that the first non-degeneracy condition is not satisfied), but the locus of complex tangents is two-dimensional.

More generally, it can be shown that if a real manifold M has order of contact higher than 2 with any smooth complex hypersurface, then the first non-degeneracy condition fails.

However, in a neighborhood of a non-degenerate CR singularity, M is contained in a *singular* complex hypersurface in $\mathbb{C}^5$, which could be considered as a complexification of M. The normal form variety $\widetilde{M}^4$, defined by $y_2 = y_3 = 0$, $z_4 = \bar{L}^2$, $z_5 = z_1\bar{L}$, is contained in the hypersurface $\mathcal{H}$ defined by $z_1^2 z_4 - z_5^2 = 0$. Algebraically, this equation is obtained by eliminating the $\bar{z}_1$ variable. Geometrically, this hypersurface can be obtained by replacing the $\bar{z}$ variables in the defining equations for $\widetilde{M}^4$ by new variables w (see Section 5, and [**W**$_{84}$]), to get a complex 4-submanifold of $\mathbb{C}^{10}$, and then projecting this smooth manifold into the original $\mathbb{C}^5$ so that its image is the singular variety $\mathcal{H}$. The containment of $\widetilde{M}^4$ inside $\mathcal{H}$ exhibits some interesting geometry (briefly considered in [**C**] and possibly also in some future paper), and resembles the parametrized cross-cap (u^2, v, uv) inside the Whitney umbrella variety $xy^2 - z^2 = 0$.

2.4. m-manifolds in $\mathbb{C}^n$, $m < n$. Neglecting the orientation issue, and using the $\bar{L} = \bar{z}_1 + x_2 + ix_3$ abbreviation, the generic complex tangent for m-manifolds M in $\mathbb{C}^n$, for $\frac{2}{3}(n+1) \leq m < n$, has a normal form generalizing the $m = 4$, $n = 5$ case. The tangent space has exactly one complex direction, with coordinate z_1, and $m-2$ real coordinates $x_2, \ldots, x_{m-1}$. The $2n-m$-normal space has exactly $n-m+1$ complex directions, $z_m, \ldots, z_n$, and $m-2$ remaining real coordinates, $y_2, \ldots, y_{m-1}$. The graphing functions (H, h) can be put into the form:

$$\begin{aligned} y_{a=2,\ldots,m-1} &= O(3) \\ z_{t=m,\ldots,n-2} &= (x_{4+2(t-m)} + ix_{5+2(t-m)})\bar{L} + O(3) \\ z_{n-1} &= \bar{L}^2 + O(3) \\ z_n &= z_1\bar{L} + O(3). \end{aligned}$$

This normal form again assumes that two non-degeneracy conditions on the h functions are satisfied, the first being a full rank condition on the $\bar{z}_1^2$ and $z_1\bar{z}_1$ coefficients, and the second being a full rank condition on the real and imaginary parts of the $x^a\bar{z}_1$ coefficients, $a = 2, \ldots, m-1$, in the equations h_t, $t = m, \ldots, n-2$, and h_{n-1}. The indices on the z_t terms range from $z_m = (x_4 + ix_5)\bar{L} + O(3)$ to $z_{n-2} = (x_{2(n-m)} + ix_{2(n-m)+1})\bar{L} + O(3)$. This indexing scheme breaks down when $m < \frac{2}{3}(n+1)$, but this is outside the topological stability range under consideration. When $m = n-1$, the functions $z_t = h_t$ in this normal form do not appear. When $m = \frac{2}{3}(n+1)$, the complex tangents are isolated, but if $m > \frac{2}{3}(n+1)$, the $3m-2n-2$ variables $x_{2(n-m+1)}, \ldots, x_{m-1}$ do not appear in the above quadratic terms, and in fact $3m-2n-2$ is the real dimension of the locus $N \subseteq M$ of complex tangents. When N has positive dimension, the quadratic normal form has the structure of a product of a totally real plane and a submanifold with an isolated complex tangent. This geometry is comparable to the normal forms of [**Wh**$_{58}$] and [**H**].

3. A linearized CR normal form problem

3.1. The nonlinear functional equation. Finding a formal or analytic transformation bringing a CR singularity of M in $\mathbb{C}^5$ to normal form requires solving a system of functional equations. Starting with the non-degenerate normal form,

$$\{y_2 = H_2, y_3 = H_3, z_4 = \bar{L}^2 + e_4(z_1, \bar{z}_1, x_2, x_3), z_5 = z_1\bar{L} + e_5(z_1, \bar{z}_1, x_2, x_3)\},$$

where H_2, H_3, e_4, e_5 are vanishing to third order, the goal is to find a (formal) holomorphic coordinate transformation of the form (1), with identity linear part:

$$\tilde{z}_s = z_s + p_s(z_1, z_2, z_3, z_4, z_5) \tag{3}$$

for $s = 1, \dots, 5$, so that the defining functions in the new coordinates are

$$\{\tilde{y}_2 = \tilde{y}_3 = 0, \tilde{z}_4 = \overline{\tilde{L}}^2 = (\bar{\tilde{z}}_1 + \tilde{x}_2 + i\tilde{x}_3)^2, \tilde{z}_5 = \tilde{z}_1\overline{\tilde{L}} = \tilde{z}_1(\bar{\tilde{z}}_1 + \tilde{x}_2 + i\tilde{x}_3)\}.$$

Notation 3.1. Define E to be the real vector space of quadruples of formal power series $\vec{e} = (H_2, H_3, e_4, e_5)$, where H_2, H_3 are formally real-valued, with terms of degree at least 2, and e_4, e_5 are complex-valued, with terms of degree at least 3. Define P to be the complex vector space of quintuples $\vec{p} = (p_1, \dots, p_5)$ of complex-valued formal power series in $z_1, \dots, z_5$ without constant terms.

A normalizing transformation exists, if given $\vec{e} = (H_2, H_3, e_4, e_5) \in E$, there exists a solution $\vec{p} \in P$ of the following four equations:

$$\begin{aligned}
0 &= \mathrm{Im}(\tilde{z}_2) = \mathrm{Im}(z_2 + p_2(\vec{z})) \\
&= H_2(z_1, \bar{z}_1, x_2, x_3) + \mathrm{Im}(p_2) && (4)\\
0 &= \mathrm{Im}(\tilde{z}_3) = \mathrm{Im}(z_3 + p_3(\vec{z})) \\
&= H_3(z_1, \bar{z}_1, x_2, x_3) + \mathrm{Im}(p_3) && (5)\\
0 &= \tilde{z}_4 - \overline{\tilde{L}}^2 \\
&= h_4(z_1, \bar{z}_1, x_2, x_3) + p_4(\vec{z}) \\
&\quad -(\bar{z}_1 + x_2 + ix_3 + \overline{p_1(\vec{z})} + \mathrm{Re}(p_2(\vec{z})) + i\mathrm{Re}(p_3(\vec{z})))^2 \\
&= e_4(z_1, \bar{z}_1, x_2, x_3) + p_4 - 2\bar{L}(\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3)) && (6)\\
&\quad -(\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3))^2 \\
0 &= \tilde{z}_5 - \tilde{z}_1\overline{\tilde{L}} \\
&= h_5(z_1, \bar{z}_1, x_2, x_3) + p_5(\vec{z}) \\
&\quad -(z_1 + p_1(\vec{z}))(\bar{L} + \overline{p_1(\vec{z})} + \mathrm{Re}(p_2(\vec{z})) + i\mathrm{Re}(p_3(\vec{z}))) \\
&= e_5(z_1, \bar{z}_1, x_2, x_3) + p_5 - \bar{L}p_1 - z_1(\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3)) && (7)\\
&\quad -p_1(\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3)),
\end{aligned}$$

where the functions $p_1, \dots p_5$ appearing in Equations $(4-7)$ are restricted to the points $\vec{z} = (z_1, x_2 + iH_2, x_3 + iH_3, h_4, h_5)$ on M. The action of the coordinate change (3) on the series $\vec{e}$ can be described as a non-linear map $F : E \times P \to E$, taking the pair $(\vec{e}, \vec{p})$ to the quantities $(4-7)$.

The analytic normal form problem is to start with $\vec{e}$ convergent on some polydisc $\{|z_1| < r_1, |x_2| < r_2, |x_3| < r_3\}$, and to find $\vec{p}$ convergent on some polydisc $\{|z_1| < \rho_1, |z_2| < \rho_2, \dots, |z_5| < \rho_5\}$ so that $F(\vec{e}, \vec{p}) = 0_E$. If $\vec{p}$ is not an exact solution, $F(\vec{e}, \vec{p})$ will have to be converted into the $\tilde{z}$ variables, using the inverse transformation (2), $z = \mathbf{A}^{-1}\tilde{z} + q(\tilde{z})$, to get the higher-order terms in the defining functions in the $\tilde{z}$ coordinate system.

3.2. The stabilizer of the polynomial normal form. Recall $\widetilde{M}^4$ denotes the 4-manifold already exactly defined by quadratic and linear polynomials, with H_2, H_3, e_4, e_5 all identically zero. There are some coordinate transformations (3) which preserve the form of the defining equations, for example, if the five functions p_s all vanish on $\widetilde{M}^4$. However, these are not the only such transformations in the "stabilizer" of the variety, defined to be the set of formal transformations (3) with $\vec{p} \in P$ such that $F(0_E, \vec{p}) = 0_E$.

Substituting $\vec{e} = 0_E$ into Equations $(4-7)$ gives

$$
\begin{aligned}
0 &= \mathrm{Im}(p_2) = \mathrm{Im}(p_3) \\
0 &= p_4 - 2\bar{L}(\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3)) - (\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3))^2 \\
0 &= p_5 - \bar{L}p_1 - z_1(\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3)) - p_1(\overline{p_1} + \mathrm{Re}(p_2) + i\mathrm{Re}(p_3)),
\end{aligned}
$$

where all the functions p_s are restricted to $\widetilde{M}^4$. Since $z_4 = \bar{L}^2$ and $z_5 = z_1\bar{L}$ are related by the expression $z_1^2 z_4 = z_5^2$, any formal power series $p_s(\vec{z})$ in the ideal $\mathcal{J} \subseteq \mathbb{C}[[z_1, \ldots, z_5]]$ generated by $z_1^2 z_4 - z_5^2$ is zero when restricted to $z_4 = \bar{L}^2$ and $z_5 = z_1\bar{L}$. Recall $\mathcal{J}$ defines a singular complex hypersurface $\mathcal{H}$ in $\mathbb{C}^5$ containing $\widetilde{M}^4$.

The first two equations imply p_2 and p_3 must be real-valued on $\widetilde{M}^4$, and the second two,

$$
\begin{aligned}
&(8) \qquad p_4(z_1, x_2, x_3, \bar{L}^2, z_1\bar{L}) = (2\bar{L} + \overline{p_1} + p_2 + ip_3)(\overline{p_1} + p_2 + ip_3) \\
&(9) \qquad p_5(z_1, x_2, x_3, \bar{L}^2, z_1\bar{L}) = \bar{L}p_1 + (z_1 + p_1)(\overline{p_1} + p_2 + ip_3),
\end{aligned}
$$

show that p_4 and p_5 are determined by p_1, p_2, p_3 in $\mathbb{C}[[z_1, \ldots, z_5]]/\mathcal{J}$, but p_1, p_2, p_3 cannot be arbitrary. For example, if $p_2 = r x_2^a x_3^b$, with r real, then $2r\bar{L}x_2^a x_3^b + r^2 x_2^{2a} x_3^{2b}$ is not a function of the arguments $z_1, x_2, x_3, \bar{L}^2, z_1\bar{L}$, and Equation (8) cannot hold for any p_4 unless $r = 0$.

Notation 3.2. Let $\mathcal{Q} \subseteq \mathbb{C}[[z_1, \bar{L}, x_2, x_3]]$ denote the subalgebra of formal power series which can be rewritten as series in z_1, x_2, x_3, $\bar{L}^2$, $z_1\bar{L}$.

The only monomials not in $\mathcal{Q}$ are those of the form $x_2^a x_3^b \bar{L}^{odd}$. In particular, any expression f in z_1, $\bar{z}_1$, x_2, x_3, satisfies $z_1 \cdot f \in \mathcal{Q}$. This simplifies the conditions $(8-9)$, so that p_1, p_2, $p_3 \in \mathbb{C}[[z_1, z_2, z_3, z_4, z_5]]$ are components of a stabilizing transformation if their restrictions to $y_2 = y_3 = 0$, $z_4 = \bar{L}^2$, $z_5 = z_1\bar{L}$ satisfy:

$$\{2\bar{L}(\overline{p_1} + p_2 + ip_3) + \overline{p_1}(\overline{p_1} + 2(p_2 + ip_3)),\ (\bar{L} + \overline{p_1})p_1\} \subseteq \mathcal{Q}.$$

It should be remarked that there are some linear transformations of $\mathbb{C}^5$ which preserve the normal form variety, for example, scaling z_1, z_2, z_3 by $\sigma > 0$ and z_4 and z_5 by σ^2, but these will not be considered here.

3.3. The linear problem. A formal solution of the non-linear problem $F(\vec{e}, \vec{p}) = 0_E$ can be approximated by finding a solution $\vec{p}$ of a related linear equation. Such a $\vec{p}$ is an approximation in the sense that if the higher-order terms of the defining equations, $\vec{e}$, have lowest degree $d \geq 3$, then the coordinate change (3) transforms the equations so that the lowest degree is approximately doubled. Iterating this procedure, and composing the coordinate transformations, determines as many terms in an exact formal solution as desired.

Equations $(10-13)$ define a real linear map $dF : P \to E$, formally a derivative of F (restricted to $\{0_E\} \times P$) at the point $(0_E, 0_P)$, acting on $\vec{p} \in P$ to give a quadruple (H_2, H_3, e_4, e_5) of power series in z_1, $\bar{z}_1$, x_2, and x_3:

$$
\begin{aligned}
(10)\; H_2 &= \mathrm{Im}(p_2(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})),\\
(11)\; H_3 &= \mathrm{Im}(p_3(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})),\\
e_4 &= p_4(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}) - 2\bar{L}\overline{p_1(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})}\\
(12)\qquad &\quad -2\bar{L}(\mathrm{Re}(p_2(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})) + i\mathrm{Re}(p_3(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}))),\\
e_5 &= p_5(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}) - \bar{L}p_1(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})\\
(13)\qquad &\quad -z_1\overline{p_1(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})}\\
&\quad -z_1(\mathrm{Re}(p_2(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})) + i\mathrm{Re}(p_3(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}))).
\end{aligned}
$$

This linearization idea was used by [**M**] in another CR normal form problem, but with one equation instead of a system of equations. Note that the output involves conjugation of functions of z_1 and $\bar{L}$, so some rearrangement may be needed to express the output as power series in E (variables z_1, $\bar{z}_1$, x_2, x_3) or as power series in z_1, $\bar{L}$, x_2, x_3. Also, the restriction of all the functions p_s to the values $z_2 = x_2$, $z_3 = x_3$, $z_4 = \bar{L}^2$, $z_5 = z_1\bar{L}$ motivates a definition of the weight of their terms:

Definition 3.3. The *degree* of a monomial $z_1^{a_1}\bar{z}_1^{a_2}x_2^{a_3}x_3^{a_4}$ or $z_1^{a_1}\bar{L}^{a_2}x_2^{a_3}x_3^{a_4}$ is the integer $a_1 + a_2 + a_3 + a_4$, and the *order* of a power series refers to the smallest among the degrees of its monomials (with non-zero coefficients). The *weight* of a monomial $z_1^{a_1}z_2^{a_2}z_3^{a_3}z_4^{a_4}z_5^{a_5}$ is the integer $a_1 + a_2 + a_3 + 2a_4 + 2a_5$, and the *weight* of a power series is the smallest of its monomials' weights.

If (H_2, H_3, e_4, e_5) are power series of order $d-1$, $d-1$, d, d, with $d \geq 3$, then the power series $(p_1, p_2, p_3, p_4, p_5)$ in a solution of $(10-13)$ are of weight $d-1$, $d-1$, $d-1$, d, and d. If $\vec{p}$ is a solution of the linear problem, then comparing $(10-13)$ with $(4-7)$ shows that the output $F(\vec{e}, \vec{p})$ of the non-linear function is four power series with order $2d-3$ for the new H_2 and H_3, and with order $2d-2$ for the new e_4 and e_5.

The existence of stabilizing transformations corresponds to a non-trivial kernel $S \subseteq P$, defined by equations similar to those of the stabilizer: p_2 and p_3 must be real-valued on $\widetilde{M}^4$, and

$$
\begin{aligned}
p_4(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}) &= 2\bar{L}(\overline{p_1} + p_2 + ip_3)\\
p_5(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}) &= \bar{L}p_1 + z_1(\overline{p_1} + p_2 + ip_3),
\end{aligned}
$$

which determines p_4 and p_5 mod $\mathcal{J}$, and imposes on p_1, p_2, p_3 the condition that their restrictions to $\widetilde{M}^4$ satisfy $\{\bar{L}(\overline{p_1}+p_2+ip_3),\ \bar{L}p_1\}\subseteq\mathcal{Q}$.

3.4. Normalized transformations. Choosing a "normalization" for the solution of the problem $dF(\vec{p})=\vec{e}$ corresponds to choosing a subspace C of P complementary to the kernel S. To prove the claim that dF is surjective, the approach will be to start with $\vec{e}$ and construct an element $\vec{p}\in C$.

Motivated by the defining conditions for the kernel, the following normalization is considered for the transformation functions p_s. The idea is to decompose each p_s into a sum of functions of fewer than five variables:

$$
\begin{aligned}
&(14) & p_1 &= z_1^2 p_1^H((z_1+z_2-iz_3)^2, z_2, z_3) + p_1^A(z_2, z_3, z_4) \\
&(15) & p_a &= p_a^R(z_2, z_3) + p_a^I(z_1, z_2, z_3, z_4, z_5) \\
&(16) & p_4 &= p_4^E(z_1, z_2, z_3, z_4) + z_5 p_4^O(z_1, z_2, z_3, z_4) \\
&(17) & p_5 &= p_5^E(z_1, z_2, z_3, z_4) + z_5 p_5^O(z_1, z_2, z_3, z_4),
\end{aligned}
$$

where for $a=2,3$, $p_a^R(z_2,z_3)$ has real coefficients. The terms p_a^I are of the following form:

$$
(18)\qquad p_a^I = p_a^E(z_1, z_2, z_3, z_4) + z_5 p_a^O(z_1, z_2, z_3, z_4) + p_a'(z_1, z_2, z_3)
$$

with $p_a^E+z_5p_a^O$ being (formally) purely imaginary when restricted to $\widetilde{M}^4$, and $p_a'(z_1,x_2,x_3)$ equal to a sum of monomials $L^{odd}x_2^ax_3^b$ with complex coefficients. In some sense, the conditions on the functions p_1, p_2, p_3 are complementary to the conditions defining the kernel. The $p^E(z_1,z_2,z_3,z_4)+z_5p^O(z_1,z_2,z_3,z_4)$ decomposition in $(16-18)$ defines a vector space complement of $\mathcal{J}$ in $\mathbb{C}[[z_1,\ldots,z_5]]$, formed by remainders r in applying the Division Theorem to $p=(z_5^2+z_1^2z_4)q+r$.

The capital superscripts are not multiindices but rather denote the roles of each term in the solution. The p_a^R is a function of "real" variables, and the "imaginary" part of p_a^I will eliminate the H_a functions. The p_4 and p_5 functions, when decomposed into parts p^E and p^O, have "even" and "odd" powers of $\bar{L}$, and can eliminate all the $\mathcal{Q}$ monomials in the h_4 and h_5 functions. The remaining monomials are of the form $x_2^ax_3^b\bar{L}^{odd}$ in h_4 and h_5, and can be handled in the same way as in Lemma 2.1. p_1^H, a "holomorphic" function in z_1, eliminates non-$\mathcal{Q}$ monomials when conjugated in (12), and p_1^A becomes an "antiholomorphic" expression when z_4 is equal to $\bar{L}^2$, eliminating non-$\mathcal{Q}$ monomials in (13). The z_1^2 factor in $z_1^2p_1^H$ means it has no monomials in common with p_1^A, and it is in $\mathcal{Q}$ when restricted to $\widetilde{M}^4$. Without this z_1^2, p_1^H and p_1^A would both introduce $x_2^ax_3^b\bar{L}^1$ terms in h_4 and h_5; this is why Lemma 2.1 could not be used twice on the quadratic terms. Although p_1, as normalized, can eliminate $x_2^ax_3^b\bar{L}^1$ terms only from h_5, these terms can be eliminated from h_4 by the $\bar{L}(p_2^R+ip_3^R)$ appearing in (12), without re-introducing any in (13).

Supposing that $\vec{e}$ is of the form where H_2 and H_3 are homogeneous of degree $d-1$, and e_4 and e_5 are homogeneous of degree d, the normalized solution of $dF(\vec{p}) = \vec{e}$ should have p_1^H with degree $d-3$, p_1^A, p_2^R, p_3^R, p_2^I, p_3^I with weight $d-1$, p_4^E, and p_5^E with weight d, and p_4^O and p_5^O with weight $d-2$. It can be checked that such polynomials form a real vector subspace of P exactly equal in (real) dimension to the subspace of E defined by homogeneous polynomials with the stated degrees.

It is convenient to think of the transformation (3) having "identity linear part," but if the function h_4 contains a cubic term $\bar{z}_1^3$, it can only be eliminated by a linear transformation of the form $\tilde{z}_1 = z_1 + a_1^4 z_4$, which could be considered as a term in p_1^A of weight 2. This is the only exception that needs to be made when thinking of the functions p_s as order ≥ 2.

Inspecting the form of Equations $(10-13)$, each p_s function appears with terms involving $\bar{L}$, but $\bar{z}_1$ also appears by itself (not collected into $\bar{L}$ factors), when the p_1, p_2, and p_3 functions undergo complex conjugation. The normalization for p_1^H, as a power series in z_2, z_3, and L^2, is chosen so that $\bar{L}^2$ appears in $\overline{p_1^H}$. At this point, it seems natural to rearrange the $\vec{e}$ components as power series in z_1, $\bar{L}$, x_2, x_3, so that like terms can be compared on both sides of Equations $(10-13)$. However, this rearrangement has serious consequences; if the rearranged series is decomposed into a sum, and each subseries expanded back into the original variables, the domain can shrink significantly. Nonetheless, the strategy for constructing an inverse of $dF : C \to E$ will be to rearrange $\vec{e}$, and define the p_s series using the coefficients of the rearranged $\vec{e}$. The p_s series will be shown to converge on a small polydisc, using a Banach space norm suited to this type of construction.

3.5. Preliminary lemmas on power series and change of variables. Some standard notions ([**GF**], [**Walter**]) about power series in real and complex variables will be used in analyzing the functional equation.

Notation 3.4. For $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$, with all $r_k > 0$, define a polydisc in $\mathbb{C}^n$ by $\mathbb{D}_\mathbf{r} = \{(z_1, \ldots, z_n) : |z_k| < r_k\}$.

Notation 3.5. For multi-indexed formal power series centered at $\vec{0} \in \mathbb{C}^n$, $c(z_1, \ldots, z_n) = \Sigma c^\alpha z^\alpha = \Sigma c^{\alpha_1 \cdots \alpha_n} z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, $c^\alpha \in \mathbb{C}$, define

$$|c|_\mathbf{r} = \Sigma |c^\alpha| r^\alpha = \Sigma |c^{\alpha_1 \cdots \alpha_n}| r_1^{\alpha_1} \ldots r_n^{\alpha_n}.$$

Lemma 3.6. *If $|c|_\mathbf{r} < \infty$ then $\Sigma c^\alpha z^\alpha$ converges absolutely for all $z \in \mathbb{D}_\mathbf{r}$, and $c(z_1, \ldots, z_n)$ defines a bounded, holomorphic function on $\mathbb{D}_\mathbf{r}$. The set of formal power series c such that $|c|_\mathbf{r} < \infty$ is a complex (and real) Banach space.* ∎

Lemma 3.7. *If $|c|_\mathbf{r} < \infty$, then any power series $d = \Sigma d^\alpha z^\alpha$ with $|d^\alpha| \leq |c^\alpha|$ for each α satisfies $|d|_\mathbf{r} \leq |c|_\mathbf{r}$.* ∎

For example, this Lemma includes subseries d of c, meaning $d^\alpha = c^\alpha$ for some indices and $d^\alpha = 0$ for the rest.

Lemma 3.8. *$|z_1 \cdot c|_{\mathbf{r}} = r_1 |c|_{\mathbf{r}}$, and in general $|b \cdot c|_{\mathbf{r}} \le |b|_{\mathbf{r}} \cdot |c|_{\mathbf{r}}$.* ∎

Notation 3.9. In $\mathbb{C} \times \mathbb{R}^2$, define the polydisc $D_R = \{(z_1, x_2, x_3) : |z_1| < R, |x_2| < R, |x_3| < R\}$, and let A be the space of formal power series in z_1, $\bar{z}_1$, x_2, x_3 with complex coefficients. Define A_R to be the subspace of power series c such that $|c|_R := \Sigma |c^{abcd}| R^{a+b+c+d} < \infty$.

Using the coefficients of $c = \Sigma c^{abcd} z_1^a \bar{z}_1^b x_2^c x_3^d \in A_R$ as coefficients in a power series $c(z_1, w, z_2, z_3)$ defines a complex analytic function on $\mathbb{D}_{(R,R,R,R)}$ in $\mathbb{C}^4$, by Lemma 3.6. The restriction of this function to the real 4-plane $\{w = \bar{z}_1,\ z_2 = \bar{z}_2,\ z_3 = \bar{z}_3\}$ is a function whose power series $c(z_1, \bar{z}_1, x_2, x_3)$ is convergent on D_R. In particular, $(A_R, |\ |_R)$ is a Banach algebra.

Lemma 3.10. *If $c = \Sigma c^{abcd} z_1^a \bar{z}_1^b x_2^c x_3^d \in A_R$, then $c(z_1, \bar{z}_1, x_2, x_3)$ can be rearranged into a power series $k(z_1, \bar{L}, x_2, x_3) = \Sigma k^{abcd} z_1^a (\bar{z}_1 + x_2 + i x_3)^b x_2^c x_3^d$, where the coefficients k^{abcd} satisfy*

$$\Sigma |k^{abcd}| 3^b (R/9)^{a+b+c+d} \le |c(z_1, \bar{z}_1, x_2, x_3)|_R, \tag{19}$$

and the sum $\Sigma k^{abcd} z_1^a \bar{L}^b x_2^c x_3^d$ converges on the polydisc $D_{R/9}$.

Proof. The transformation from w to $W = w + z_2 + i z_3$ "shears" the polydisc $\mathbb{D}_{(R,R,R,R)}$ in $\mathbb{C}^4$. The norm $|\Sigma k^{abcd} z_1^a W^b x_2^c x_3^d|_{(R/9, R/3, R/9, R/9)}$ can be estimated in terms of the norm $|c(z_1, \bar{z}_1, x_2, x_3)|_R$ by comparing the coefficients k^{abcd} to the coefficients c^{abcd}. Some trinomial coefficients appear, having the following properties ([**Aigner**]) for whole numbers b, j, k, l:

$$\begin{pmatrix} b \\ jkl \end{pmatrix} = \left\{ \begin{array}{ll} \frac{b!}{j!k!l!} & \text{if } b = j + k + l \\ 0 & \text{otherwise} \end{array} \right\}; \quad \sum_{j,k,l} \begin{pmatrix} b \\ jkl \end{pmatrix} = 3^b.$$

The degree n terms of the power series c can be expanded:

$$\begin{aligned} \sum_{a+b+c+d=n} c^{abcd} z_1^a \bar{z}_1^b x_2^c x_3^d &= \sum_{a+b+c+d=n} c^{abcd} z_1^a (\bar{L} - x_2 - i x_3)^b x_2^c x_3^d \\ &= \sum_{a+b+c+d=n} c^{abcd} z_1^a \left(\sum_{j,k,l} (-1)^{j+k} i^k \begin{pmatrix} b \\ jkl \end{pmatrix} \bar{L}^l x_2^{c+j} x_3^{d+k} \right) \\ &= \sum_{a'+b'+c'+d'=n} k^{a'b'c'd'} z_1^{a'} \bar{L}^{b'} x_2^{c'} x_3^{d'}, \end{aligned}$$

where the coefficients k are formed by collecting like terms:

$$k^{a'b'c'd'} = \sum_{\substack{b' \le b \\ c \le c' \\ d \le d'}} c^{a'bcd} (-1)^{b-b'} i^{d'-d} \begin{pmatrix} b \\ c'-c\ d'-d\ b' \end{pmatrix}.$$

By the triangle inequality, and then an interchange of summation, the degree n terms of the LHS of Inequality (19) sum to

$$\begin{aligned}
\sum_{a'+b'+c'+d'=n}|k^{a'b'c'd'}|\frac{3^{b'}R^n}{9^n} &\leq \sum_{a'+b'+c'+d'=n}\Big(\sum_{\substack{b'\leq b\\ c\leq c'\\ d\leq d'}}|c^{a'bcd}|\binom{b}{c'-c\ d'-d\ b'}\Big)\frac{3^{b'}R^n}{9^n}\\
&\leq \sum_{a+b+c+d=n}|c^{abcd}|\Big(\sum_{\substack{b'\leq b\\ c\leq c'\\ d\leq d'}}\binom{b}{c'-c\ d'-d\ b'}3^{b'}\Big)\frac{R^n}{9^n}\\
&\leq \sum_{a+b+c+d=n}|c^{abcd}|\Big(\sum_{b'+c'+d'=n-a}\binom{b}{c'-c\ d'-d\ b'}\Big)\frac{3^bR^n}{9^n}\\
&\leq \sum_{a+b+c+d=n}|c^{abcd}|3^b\cdot 3^b\frac{R^n}{9^n}\leq \sum_{a+b+c+d=n}|c^{abcd}|R^n,
\end{aligned}$$

and the absolutely convergent sum of these collections over all n is independent of the order of terms and gives the inequality. This argument actually shows that $\Sigma k^{abcd}z_1^aW^bz_2^cz_3^d$ has finite $(R,R/3,R/9,R/9)$ norm, but the intersection of the polydisc $\mathbb{D}_{(R,R/3,R/9,R/9)}$ and the real 4-plane $\{W=\bar{z}_1+x_2+ix_3,\ z_2=x_2,\ z_3=x_3\}$ contains $D_{R/9}$, so the $(R/9,R/3,R/9,R/9)$ norm is considered instead. ∎

Notation 3.11. For formal power series in the variables z_1, $\bar{L}$, x_2, x_3, define $|\ |'_r$ by

$$|\Sigma k^{abcd}z_1^a\bar{L}^bx_2^cx_3^d|'_r=\Sigma|k^{abcd}|3^br^{a+b+c+d}.$$

This is the $\mathbf{r}=(r,3r,r,r)$ norm on the complex polydisc $\mathbb{D}_{\mathbf{r}}$.

If $|k|'_r<\infty$, then $k(z_1,\bar{L},x_2,x_3)$ converges for $(z_1,x_2,x_3)\in D_r$, because $|\bar{L}|\leq|\bar{z}_1|+|x_2|+|x_3|<3r$. Lemma 3.10 can be restated for the rearrangement $k(z_1,\bar{L},x_2,x_3)$ of $c(z_1,\bar{z}_1,x_2,x_3)$ in terms of this norm: $|k|'_{R/9}\leq|c|_R$. The polyradius $(R/9,R/3,R/9,R/9)$ chosen in Lemma 3.10 is convenient for calculations, but also is small enough so that using the Weierstrass Division Theorem as a step in the main Theorem will not further shrink the domain.

Note that the "prime" notation will always signify the presence of the "rearranged" variable $\bar{L}$. A disadvantage of the $|\ |'_r$ norm, and with the z_1, $\bar{L}$, x_2, x_3 coordinates in general, is the bad behavior with respect to complex conjugation. The conjugate of a power series $k(z_1,\bar{L},x_2,x_3)$ in general requires a rearrangement to be expressed as another power series in the same variables. However, if a power series is independent of the holomorphic variable z_1, then its conjugate does not involve $\bar{z}_1$. The following Lemma shows that the rearrangement of the conjugate does not increase the primed norm.

Lemma 3.12. *If the power series $k(z_1, \bar{L}, x_2, x_3) = \sum k^{0abc}\bar{L}^a x_2^b x_3^c$ has finite $|\ |_r'$ norm, and $\sum q^{a'0b'c'} z_1^{a'} x_2^{b'} x_3^{c'}$ is obtained by expanding $L = z_1 + x_2 - ix_3$ in the conjugate expression $\bar{k} = \sum \overline{k^{0abc}} L^a x_2^b x_3^c$, then $|q|_r' \leq |k|_r'$.*

Proof.

$$q^{a'0b'c'} = \sum_{\substack{a' \leq a \\ b \leq b' \\ c \leq c'}} \overline{k^{0abc}}(-i)^{c'-c} \begin{pmatrix} a \\ b'-b \ c'-c \ a' \end{pmatrix}$$

$$\sum_{a'+b'+c'=n} |q^{a'0b'c'}| r^n \leq \sum_{a+b+c=n} |k^{0abc}| \Big(\sum_{\substack{a' \leq a \\ b \leq b' \\ c \leq c'}} \begin{pmatrix} a \\ b'-b \ c'-c \ a' \end{pmatrix} \Big) r^n$$

$$\leq \sum_{a+b+c=n} |k^{0abc}| 3^a r^n.$$

∎

Notation 3.13. In analogy with Notation 3.1, let E' denote the space of formally real-valued and complex-valued power series (H_2, H_3, e_4, e_5) in the variables z_1, $\bar{L}$, x_2, x_3. E and E' are isomorphic (as real vector spaces) by the rearrangement. Define E_r (and E_r') to be the real vector subspace of E (E') where all four components have order at least 3, and finite $|\ |_r$ ($|\ |_r'$) norms, defining real-valued functions H_2 and H_3 and complex-valued functions e_4 and e_5 on D_r. The maximum norms are denoted

$$|\vec{e}|_r = \max\{|H_2|_r, |H_3|_r, |e_4|_r, |e_5|_r\}; \quad |\vec{e}|_r' = \max\{|H_2|_r', |H_3|_r', |e_4|_r', |e_5|_r'\},$$

so that $(E_r,\ |\ |_r)$ and $(E_r',\ |\ |_r')$ are Banach spaces.

By Lemma 3.10, rearrangement is a continuous, one-to-one map $E_R \to E'_{R/9}$.

Notation 3.14. Recall P is the complex vector space of quintuples $\vec{p}$ of formal power series $(p_1, p_2, p_3, p_4, p_5)$ in the variables $z_1, \ldots, z_5$. Let Δ_r denote the polydisc $\mathbb{D}_{(r,r,r,9r^2,3r^2)} \subseteq \mathbb{C}^5$. The $\mathbf{r} = (r, r, r, 9r^2, 3r^2)$ norm for power series $p_s(\vec{z})$ and the maximum norm for quintuples $\vec{p} = (p_1, \ldots, p_5)$ are denoted

$$\begin{aligned} ||p_s||_r &= |\Sigma p_s^{abcde} z_1^a z_2^b z_3^c z_4^d z_5^e|_{(r,r,r,9r^2,3r^2)} = \Sigma |p_s^{abcde}| 9^d 3^e r^{a+b+c+2d+2e}, \\ ||\vec{p}||_r &= \max_{s=1,\ldots,5} \{||p_s||_r\}. \end{aligned}$$

This superscript notation for the coefficients of p_s is different from that used in Section 2.

This polyradius is chosen so that a power series p_s, when restricted to $\widetilde{M}^4$, defines a function of z_1, $\bar{L}$, x_2, x_3 with controlled norm:

$$|p_s(z_1, x_2, x_3, \bar{L}^2, z_1\bar{L})|'_r = ||p_s(z_1, z_2, z_3, z_4, z_5)||_r.$$

Notation 3.15. Define P_r as the subspace of P composed of quintuples $\vec{p} = (p_1, p_2, p_3, p_4, p_5)$ so that each p_s has weight at least 2, and the norm $||\vec{p}||_r$ is finite, so that $(P_r, \ || \ ||_r)$ is a Banach space, and $\vec{p} \in P_r$ defines a complex analytic map $\Delta_r \to \mathbb{C}^5$.

3.6. A convergent solution, after the rearrangement. Recall the submanifold M was defined as the graph of real analytic functions H and h in the variables z_1, $\bar{z}_1$, x_2, x_3, over a region in $\mathbb{C} \times \mathbb{R}^2$, which can be assumed to contain a polydisc D_R. The following Theorem constructs the functions $\vec{p}$ approximating a solution of the normal form problem, and by estimating the norm of $\vec{p}$ in terms of the norm of $\vec{e}$, shows that $\vec{p}$ is convergent. However, $\vec{p}$ is defined in terms of the rearranged functions $e(z_1, \bar{L}, x_2, x_3)$, so its polyradius of convergence in $\mathbb{C}^5$ is small compared to R.

Theorem 3.16. *If $\vec{e} \in E_R$, then there is a solution $\vec{p}$ of the equation $dF(\vec{p}) = \vec{e}$ such that $\vec{p} \in P_{R/9}$. This $\vec{p}$ is the unique solution satisfying the normalization conditions.*

Proof. Let $r = R/9$. By Lemma 3.10, the components of $\vec{e}$ can be rearranged to power series $\vec{e} = (H_2, H_3, e_4, e_5) \in E'_r$, vanishing to third order in the rearranged variables, and converging on $D_r = D_{R/9} \subseteq \mathbb{C} \times \mathbb{R}^2$. The idea is to find $\vec{p} = (p_1, p_2, p_3, p_4, p_5)$ by constructing the components p_1^H, ..., in terms of $\vec{e}$. At one point, the Weierstrass Division Theorem is required, with estimates on quotients and remainders ([**GF**] §III.2.). The point of calculating these estimates is primarily to show that there is no further shrinking after the initial rearrangement.

The real-valued function $H_2(z_1, \bar{L}, x_2, x_3)$ can be decomposed into its $\mathcal{Q}$ and non-$\mathcal{Q}$ monomials:

$$H_2 = \sum_{b \text{ even}} H_2^{abcd} z_1^a \bar{L}^b x_2^c x_3^d + \sum_{\substack{a>0 \\ b \text{ odd}}} H_2^{abcd} z_1^a \bar{L}^b x_2^c x_3^d + \sum_{b \text{ odd}} H_2^{0bcd} \bar{L}^b x_2^c x_3^d.$$

Adding and subtracting the conjugate of the non-$\mathcal{Q}$ part gives two real-valued sums:

$$\begin{aligned} H_2 &= \sum_{b \text{ even}} H_2^{abcd} z_1^a \bar{L}^b x_2^c x_3^d + \sum_{\substack{a>0 \\ b \text{ odd}}} H_2^{abcd} z_1^a \bar{L}^b x_2^c x_3^d - \sum_{b \text{ odd}} \overline{H_2^{0bcd}} L^b x_2^c x_3^d \\ &\quad + \sum_{b \text{ odd}} \overline{H_2^{0bcd}} L^b x_2^c x_3^d + \sum_{b \text{ odd}} H_2^{0bcd} \bar{L}^b x_2^c x_3^d. \end{aligned}$$

By Lemma 3.7, $|\sum_{b\text{ odd}} H_2^{0bcd}\bar{L}^b x_2^c x_3^d|_r' \le |H_2|_r' \le |\vec{e}|_r'$, and by Lemma 3.12, if $\sum_{b\text{ odd}} \overline{H_2^{0bcd}} L^b x_2^c x_3^d$ is formally expanded as $q_2 = \sum q_2^{bcd} z_1^b x_2^c x_3^d$, then

$$|q_2(z_1, x_2, x_3)|_r' \le |\sum_{b\text{ odd}} H_2^{0bcd}\bar{L}^b x_2^c x_3^d|_r'.$$

Define $p_2^I = p_2' + p_2''$ on Δ_r, by

$$p_2'(z_1, z_2, z_3) = 2i\sum q_2^{bcd} z_1^b z_2^c z_3^d,$$

$$p_2'' = i(\sum_{b\text{ even}} H_2^{abcd} z_1^a z_4^{b/2} z_2^c z_3^d + z_5 \sum_{\substack{a>0\\ b\text{ odd}}} H_2^{abcd} z_1^{a-1} z_4^{(b-1)/2} z_2^c z_3^d - \sum q_2^{bcd} z_1^b z_2^c z_3^d).$$

Then, by Lemma 3.7 applied to the $\mathcal{Q}$ part of H_2, $||p_2''||_r \le |H_2|_r' + |q_2|_r'$. By construction, $H_2 = \mathrm{Im}(p_2^I(z_1, x_2, x_3, \bar{L}^2, z_1\bar{L}))$, and $||p_2^I||_r \le 4|H_2|_r' \le 4|\vec{e}|_r'$.

p_3^I similarly has bounded norm and is convergent on Δ_r. These p^I functions are normalized as in (18). The other functions p_1, p_4, p_5, p_a^R will also, by construction, be of the form $(14-17)$.

Equation (13), when using the normalized $\vec{p}$,

$$\begin{aligned} e_5 &= p_5^E(z_1, x_2, x_3, \bar{L}^2) + z_1\bar{L}p_5^O(z_1, x_2, x_3, \bar{L}^2) \\ &\quad -\bar{L}(z_1^2 p_1^H(L^2, x_2, x_3) + p_1^A(x_2, x_3, \bar{L}^2)) \\ &\quad -z_1(\bar{z}_1^2\overline{p_1^H} + \overline{p_1^A} + p_2^R + ip_3^R + \mathrm{Re}(p_2^I) + i\mathrm{Re}(p_3^I)), \end{aligned}$$

can be solved next, by comparing coefficients of $e_5(z_1, \bar{L}, x_2, x_3)$ on the LHS:

$$e_5 = \sum_{b=2r} e_5^{abcd} z_1^a \bar{L}^{2r} x_2^c x_3^d + z_1\bar{L}\cdot \sum_{\substack{a>0\\ b=2s+1}} e_5^{abcd} z_1^{a-1}\bar{L}^{2s} x_2^c x_3^d + \bar{L}\sum_{b=2t+1} e_5^{0bcd}\bar{L}^{2t} x_2^c x_3^d,$$

to the RHS, where the only non-$\mathcal{Q}$ terms are those of $\bar{L}p_1^A$. This decomposition of e_5 determines the coefficients of the series for $p_1^A(z_2, z_3, z_4) = \sum p^{\alpha\beta\gamma} z_2^\alpha z_3^\beta z_4^\gamma$, with $p^{\alpha\beta\gamma} = -e_5^{0,2\gamma+1,\alpha,\beta}$. The estimate follows:

$$\begin{aligned} 3r||p_1^A||_r &= 3r\sum |p^{\alpha\beta\gamma}| 9^\gamma r^{\alpha+\beta+2\gamma} = 3r\sum_{b=2t+1} |-e_5^{0bcd}| 9^t r^{b-1+c+d} \\ &= \sum |e_5^{0bcd}| 3^b r^{b+c+d} \le |e_5|_r' \le |\vec{e}|_r' \\ \implies ||p_1^A||_r &\le \frac{1}{3r}|\vec{e}|_r'. \end{aligned}$$

An estimate for the conjugate $\overline{p_1^A(x_2, x_3, \bar{L}^2)}$ will also be necessary. Formally expanding

$$\overline{p_1^A} = \sum \overline{p^{\alpha\beta\gamma}} L^{2\gamma} x_2^\alpha x_3^\beta = \sum q_1^{abc} z_1^a x_2^b x_3^c$$

gives, by Lemma 3.12, $|q_1|'_r \leq |p_1^A(x_2, x_3, \bar{L}^2)|'_r \leq \frac{1}{3r}|\vec{e}|'_r$.

Equation (12), again using the normalization, and expanding $\overline{z_1^2 p_1^H}$ as $(\bar{L} - (x_2 + ix_3))^2 \overline{p_1^H}$, becomes

$$\begin{aligned} e_4 &= p_4^E(z_1, x_2, x_3, \bar{L}^2) + z_1 \bar{L} p_4^O(z_1, x_2, x_3, \bar{L}^2) \\ &\quad -2\bar{L}((\bar{L}^2 + (x_2 + ix_3)^2)\overline{p_1^H(L^2, x_2, x_3)} + p_2^R(x_2, x_3) + ip_3^R(x_2, x_3)) \\ &\quad +4\bar{L}^2(x_2 + ix_3)\overline{p_1^H(L^2, x_2, x_3)} - 2\bar{L}\overline{p_1^A(x_2, x_3, \bar{L}^2)} \\ &\quad -2\bar{L}(\mathrm{Re}(p_2^I(z_1, x_2, x_3, \bar{L}^2, z_1\bar{L})) + i\mathrm{Re}(p_3^I(z_1, x_2, x_3, \bar{L}^2, z_1\bar{L}))). \end{aligned}$$

This is handled by adding the known quantity $2\bar{L}(\overline{p_1^A} + \mathrm{Re}(p_2^I) + i\mathrm{Re}(p_3^I))$ to both sides, and collecting the resulting LHS into

$$\sum_{b=2r} k^{abcd} z_1^a \bar{L}^{2r} x_2^c x_3^d + z_1 \bar{L} \cdot \sum_{\substack{a \geq 1 \\ b = 2s+1}} k^{abcd} z_1^{a-1} \bar{L}^{2s} x_2^c x_3^d + \bar{L} \sum_{b=2t+1} k^{0bcd} \bar{L}^{2t} x_2^c x_3^d.$$

Expressing LHS in terms of z_1, $\bar{L}$, x_2, x_3 means rearranging $\overline{p_1^A(x_2, x_3, \bar{L}^2)}$ as a function of z_1, x_2, x_3, so that the previous estimate applies. Then p_2^I (and similarly p_3^I) can be expressed in terms of p_2', whose norm was already bounded, and p_2'', which is imaginary and does not contribute to $\mathrm{Re}(p_2^I)$:

$$\begin{aligned} e_4 + 2\bar{L}(\overline{p_1^A} + \mathrm{Re}(p_2^I) + i\mathrm{Re}(p_3^I)) &= e_4 + 2\bar{L}(\overline{p_1^A} + \frac{1}{2}(p_2' - 2i \sum_{b \text{ odd}} H_2^{0bcd} \bar{L}^b x_2^c x_3^d) \\ &\quad + \frac{i}{2}(p_3' - 2i \sum_{b \text{ odd}} H_3^{0bcd} \bar{L}^b x_2^c x_3^d)) \end{aligned}$$

$$\begin{aligned} |e_4 + 2\bar{L}(\bar{p}_1^A + \mathrm{Re}(p_2^I) + i\mathrm{Re}(p_3^I))|'_r &\leq |e_4|'_r + 2 \cdot 3r(\frac{1}{3r}|\vec{e}|'_r + 4|\vec{e}|'_r) \\ &\leq (3 + 24r)|\vec{e}|'_r. \end{aligned}$$

Considering $\sum_{b=2t+1} k^{0bcd} \bar{L}^{2t} x_2^c x_3^d$ as a function of variables $\bar{L}^2$, x_2, and x_3, by the Weierstrass Division Theorem there exist unique analytic functions $\mathfrak{q}(\bar{L}^2, x_2, x_3)$ and $\mathfrak{r}(x_2, x_3)$ such that

$$\sum_{b=2t+1} k^{0bcd} \bar{L}^{2t} x_2^c x_3^d = (\bar{L}^2 + (x_2 + ix_3)^2)\mathfrak{q}(\bar{L}^2, x_2, x_3) + \mathfrak{r}(x_2, x_3),$$

with $\mathfrak{q} = \sum \mathfrak{q}^{rst} \bar{L}^{2r} x_2^s x_3^t$ and $\mathfrak{r} = \sum \mathfrak{r}^{st} x_2^s x_3^t$. Let

$$p_1^H(z_1, z_2, z_3) = -\frac{1}{2} \sum \overline{\mathfrak{q}^{rst}} (z_1 + z_2 - iz_3)^{2r} z_2^s z_3^t,$$

and then the coefficients of p_2^R and p_3^R are the real and imaginary parts of the coefficients of $-\frac{1}{2}\mathfrak{r}(x_2, x_3)$. This is exactly where the second non-degeneracy condition is used; without both the $x_2\bar{z}_1$ and $ix_3\bar{z}_1$ terms in h_4, not all the higher-order terms could be eliminated.

The estimates from the Division Theorem are

$$\begin{aligned}
|(x_2+ix_3)^2|'_r = 4r^2 &< \frac{1}{2}|\bar{L}^2|'_r = \frac{1}{2}\cdot 9r^2, \\
|\bar{L}\sum_{b=2t+1} k^{0bcd}\bar{L}^{2t}x_2^c x_3^d|'_r &\leq (3+24r)|\vec{e}|'_r,
\end{aligned}$$

$$\begin{aligned}
|\mathfrak{q}(\bar{L},x_2,x_3)|'_r &\leq \frac{1}{9r^2}|\sum_{b=2t+1} k^{0bcd}\bar{L}^{2t}x_2^c x_3^d|'_r \cdot \frac{1}{1-\frac{1}{2}} \leq \frac{2}{9r^2}\frac{3+24r}{3r}|\vec{e}|'_r, \\
|\mathfrak{r}(x_2,x_3)|'_r &\leq 2\frac{3+24r}{3r}|\vec{e}|'_r.
\end{aligned}$$

The remaining functions p_4^E, p_4^O can be estimated by solving (12):

$$\begin{aligned}
||p_4||_r &= |p_4^E(z_1,x_2,x_3,\bar{L}^2)+z_1\bar{L}p_4^O(z_1,x_2,x_3,\bar{L}^2)|'_r \\
&= |e_4+2\bar{L}(\overline{p_1^A}+\mathrm{Re}(p_2^I)+i\mathrm{Re}(p_3^I)) \\
&\quad +2\bar{L}((\bar{L}^2+(x_2+ix_3)^2)\overline{p_1^H}+p_2^R+ip_3^R)-4\bar{L}^2(x_2+ix_3)\overline{p_1^H}|'_r \\
&\leq ((3+24r)+(3+24r)+4\cdot 9r^2\cdot 2r\cdot\frac{1}{9r^2}\cdot\frac{3+24r}{3r})|\vec{e}|'_r \\
&= (14+112r)|\vec{e}|'_r.
\end{aligned}$$

The estimates for the other components of $\vec{p}$ follow from Lemma 3.12 and the Banach algebra properties:

$$\begin{aligned}
||p_1^H(z_1,z_2,z_3)||_r &\leq \frac{1}{9r^2}\frac{3+24r}{3r}|\vec{e}|'_r = \frac{1+8r}{9r^3}|\vec{e}|'_r, \\
||p_1||_r &= ||z_1^2p_1^H+p_1^A||_r \leq \frac{1+8r}{9r}|\vec{e}|'_r+\frac{1}{3r}|\vec{e}|'_r = (\frac{8}{9}+\frac{4}{9r})|\vec{e}|'_r, \\
||p_2||_r &= ||p_2^I+p_2^R||_r \leq (12+\frac{1}{r})|\vec{e}|'_r, \\
||p_3||_r &= ||p_3^I+p_3^R||_r \leq (12+\frac{1}{r})|\vec{e}|'_r.
\end{aligned}$$

Returning to Equation (13) to control p_5,

$$\begin{aligned}
||p_5||_r &= |p_5^E(z_1,x_2,x_3,\bar{L}^2)+z_1\bar{L}p_5^O(z_1,x_2,x_3,\bar{L}^2)|'_r \\
&= |e_5+\bar{L}p_1-2z_1\bar{L}(x_2+ix_3)\overline{p_1^H} \\
&\quad +z_1(\overline{p_1^A}+\mathrm{Re}(p_2^I)+i\mathrm{Re}(p_3^I)+(\bar{L}^2+(x_2+ix_3)^2)\overline{p_1^H}+p_2^R+ip_3^R)|'_r \\
&\leq (1+3r(\frac{8}{9}+\frac{4}{9r})+12r^3(\frac{1+8r}{9r^3})+r(\frac{1}{3r}+4)+\frac{r}{3r}(3+24r))|\vec{e}|'_r \\
&= (5+\frac{76}{3}r)|\vec{e}|'_r.
\end{aligned}$$

∎

Similar calculations, which appear in [**C**], solve the analogous linear problem for the $m < n$ normal form.

3.7. The new defining equations. If the solution $\vec{p}$ to the linear problem, obtained in Theorem 3.16, is substituted into the nonlinear expressions $(4-7)$, the resulting quadruple of formal series $F(\vec{e},\vec{p})$ has its first two components with order at least $2d-3$, and the last two components with order at least $2d-2$. If $\vec{p}$ converges on a small polydisc Δ_r, then $F(\vec{e},\vec{p})$ must be considered on even smaller polydisc, D_ρ, $\rho < r$, so that the composition of $\vec{p}$ and $x_2 + iH_2$, etc., in Equations $(4-7)$ is well-defined. Further, $F(\vec{e},\vec{p})$ involves z_1, $\bar{z}_1$, L and $\bar{L}$, and even if these are rearranged to series in E or E', the new defining equations are ready for another approximate coordinate change only after conversion to the $\tilde{z}_1$, $\tilde{x}_2$, $\tilde{x}_3$ coordinates using the inverse transformation (2). Since there was also some flexibility in the normalization of the solution $\vec{p}$, the existence of a convergent solution of $F(\vec{e},\vec{p}) = 0_E$ remains an open question.

4. An explicit example of the solution

As a demonstration of the small radius of the the normalized solution of the linearized Equations $(10-13)$, consider the following real algebraic variety in $\mathbb{C}^5$, with $R > 0$ and odd degree $d \geq 5$:

$$M = \{\vec{z} : y_2 = y_3 = 0, z_4 = \bar{L}^2 + \frac{\bar{L}^d}{R-\bar{z}_1}, z_5 = z_1\bar{L} + \frac{\bar{L}x_2^{d-1}}{R-x_2}\}.$$

Clearly, for any $\rho < R$ the variety is a graph over D_ρ, with e_4 and e_5 expressible as power series in A_ρ. The linearized functional equation for a coordinate change $\vec{z} + \vec{p}(\vec{z})$ taking this variety to the normal form variety $\widetilde{M}^4$ becomes:

$$\begin{aligned}
0 &= \operatorname{Im}(p_2(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})) = \operatorname{Im}(p_3(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})),\\
\frac{\bar{L}^d}{R-\bar{z}_1} &= p_4(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}) - 2\bar{L}\overline{p_1(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})}\\
&\quad -2\bar{L}(\operatorname{Re}(p_2(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})) + i\operatorname{Re}(p_3(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}))),\\
\frac{\bar{L}x_2^{d-1}}{R-x_2} &= p_5(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})\\
&\quad -\bar{L}p_1(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}) - z_1\overline{p_1(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})}\\
&\quad -z_1(\operatorname{Re}(p_2(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L})) + i\operatorname{Re}(p_3(z_1,x_2,x_3,\bar{L}^2,z_1\bar{L}))).
\end{aligned}$$

In terms of the expressions $(14-18)$, the normalized solution is:

$$
\begin{aligned}
p_1^A &= \frac{-z_2^{d-1}}{R-z_2}\\
p_1^H &= \frac{-(R+z_2-iz_3)L^2}{2((R+z_2-iz_3)^2-L^2)}\cdot\left(\sum_{k=0}^{(d-5)/2}(-1)^k L^{d-5-2k}(z_2-iz_3)^{2k}\right)\\
&\quad -\frac{(-1)^{(d-3)/2}(R+z_2-iz_3)^3(z_2-iz_3)^{d-3}}{2((R+z_2-iz_3)^2-L^2)((R+z_2-iz_3)^2+(z_2-iz_3)^2)}
\end{aligned}
$$

$$
\begin{aligned}
p_2^R &= \frac{x_2^{d-1}}{R-x_2}-\frac{1}{2}\mathrm{Re}\Big(\frac{(-1)^{(d-1)/2}(x_2+ix_3)^{d-1}(R+x_2+ix_3)}{(R+x_2+ix_3)^2+(x_2+ix_3)^2}\Big)\\
p_3^R &= -\frac{1}{2}\mathrm{Im}\Big(\frac{(-1)^{(d-1)/2}(x_2+ix_3)^{d-1}(R+x_2+ix_3)}{(R+x_2+ix_3)^2+(x_2+ix_3)^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
p_4^E &= \frac{2(z_2+iz_3)(R+z_2+iz_3)z_4^2}{(R+z_2+iz_3)^2-z_4}\cdot\left(\sum_{k=0}^{(d-5)/2}(-1)^k z_4^{(d-5)/2-k}(z_2+iz_3)^{2k}\right)\\
&\quad +\frac{(-1)^{(d-3)/2}2(R+z_2+iz_3)^3(z_2+iz_3)^{d-2}z_4}{((R+z_2+iz_3)^2-z_4)((R+z_2+iz_3)^2+(z_2+iz_3)^2)}\\
&\quad +\frac{z_4^{(d+1)/2}}{(R+z_2+iz_3)^2-z_4}
\end{aligned}
$$

$$
\begin{aligned}
p_5^E &= \frac{-z_1(R+z_2+iz_3)z_4^{(d-1)/2}}{2((R+z_2+iz_3)^2-z_4)}\\
p_5^O &= \frac{(z_2+iz_3)(R+z_2+iz_3)z_4}{(R+z_2+iz_3)^2-z_4}\cdot\left(\sum_{k=0}^{(d-5)/2}(-1)^k z_4^{(d-5)/2-k}(z_2+iz_3)^{2k}\right)\\
&\quad +\frac{(-1)^{(d-3)/2}(R+z_2+iz_3)^3(z_2+iz_3)^{d-2}}{((R+z_2+iz_3)^2-z_4)((R+z_2+iz_3)^2+(z_2+iz_3)^2)}+z_1\cdot p_1^H.
\end{aligned}
$$

The quantities p_2^I, p_3^I, and p_4^O are identically zero. All the p components should be functions of the z variables; in p_2^R and p_3^R, the x_a variables should be replaced by z_a, after calculating the real and imaginary parts. Also, the "L" appearing in p_1^H should be considered as $L=z_1+z_2-iz_3$. The functions are evidently holomorphic in a region around the origin of $\mathbb{C}^5$, outside the union of complex hypersurfaces where the denominators vanish. The linear factors in the denominator of p_1^H arising from the rearrangement are:

$$(R+z_2-iz_3)^2-L^2=(R-z_1)(R+z_1+2(z_2-iz_3)).$$

The factors in the denominators occurring from applying the Weierstrass Theorem to find $\mathfrak{q}$, $\mathfrak{r}$, and their conjugates, are:

$$(R+z_2+iz_3)^2+(z_2+iz_3)^2 = (R+(1+i)z_2-(1-i)z_3)(R+(1-i)z_2+(1+i)z_3),$$

$$(R+z_2-iz_3)^2+(z_2-iz_3)^2 = (R+(1-i)z_2-(1+i)z_3)(R+(1+i)z_2+(1-i)z_3).$$

(The zero locus of the last two linear factors is the same pair of hyperplanes as defined by the first two.) Finally, the $(R+z_2+iz_3)^2 - z_4$ appearing in the denominators of p_4 and p_5 is irreducible, and if z_2 and z_3 are bounded on the order of magnitude of R, then z_4 must be bounded by some fraction of R^2. By inspection, all five components of $\vec{p}$ are convergent on $\Delta_{R/9} = \mathbb{D}_{(R/9,R/9,R/9,R^2/9,R^2/27)}$, as predicted by the estimates, and also on any other polydisc $\mathbb{D}_{\mathbf{r}}$ not intersecting these complex hypersurfaces.

5. Analogy with singularity theory

If the equations given as a graph over the tangent space are considered as a real analytic map $\mathbb{R}^m \to \mathbb{R}^{2n}$, they can be complexified by replacing each $\bar{z}_s$ by an independent holomorphic variable w_s. The graphing equations then become a holomorphic parametrization of a complex submanifold: $\mathbb{C}^m \to \mathbb{C}^{2n}$. If the real manifold has complex tangents, then the composition of the embedding with the projection $\mathbb{C}^{2n} \to \mathbb{C}^n$ which forgets the new w variables is a singular map $\mathbb{C}^m \to \mathbb{C}^n$ (cf [$\mathbf{W}_{84}$]). (To be more precise, these graphing equations and parametrizations should be considered as germs of mappings.)

Example 5.1. In the $m = n = 2$ case, Bishop ([**Bishop**]) normalized the defining equation to $z_2 = \beta(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1 + O(3)$ with $\beta \geq 0$, where the parameter β is a holomorphic invariant, and describes the geometry of a complex tangent. The complexification and then projection of the quadratic terms is a map $(z, w) \mapsto (z, \beta(z^2 + w^2) + zw)$. For $\beta > 0$, this is a ramified two-to-one map ([$\mathbf{W}_{84}$]), and is analogous to Whitney's "fold" singularity $(z, w) \mapsto (z, w^2)$.

Example 5.2. The $\beta = 0$ case, where a cubic normal form is $z_2 = z_1\bar{z}_1 + \bar{z}_1^3$ ([**M**]), is similarly analogous to Whitney's "cusp", $(x, y) \mapsto (x, xy + y^3)$.

Example 5.3. For the normal form variety $\widetilde{M}^4$ in $\mathbb{C}^5$, a parametrization $\mathbb{C}^4 \to \mathbb{C}^{10} \to \mathbb{C}^5$ of the complexification $\mathcal{H}$ is

$$(z_1, w_1, z_2, z_3) \mapsto (z_1, z_2, z_3, (w_1 + z_2 + iz_3)^2, z_1(w_1 + z_2 + iz_3)).$$

A normal form more like Whitney's would be possible using a larger group, where the z and w variables could be transformed independently. Under the subgroup used to normalize the CR singularity, one expects equivalence classes of maps to be smaller, and continuous parameters ("moduli") to appear "sooner" (for more and for lower-order terms). However, invariants which distinguish maps under the larger group will still distinguish them under the smaller group. Invariants of the complexification, such as the intrinsic derivative, the Boardman sequence, Jacobian extensions, etc., could provide a coarse but general beginning to a classification of CR singularities ([**GG**], [**Porteous**]).

Under the even smaller subgroup of transformations which preserve a given orientation of the tangent space at an isolated CR singularity, the $+$ or $-$ sign appearing in the quadratic normal form (Subsection 2.2) indicates a chirality near the complex tangent that is reversed by a change of orientation, and which corresponds to a notion of a topological intersection number, an "index" ± 1. From a global viewpoint, isolated complex tangents of generic immersions of compact manifolds in $\mathbb{C}^n$ can be enumerated by characteristic class formulas which relate the sum of indices to topological invariants. For example, when $\mathbb{C}P^2$ is smoothly embedded in $\mathbb{C}^5$, the expected sum of the indices of the complex tangents is $p_1\mathbb{C}P^2 = 3$. This is reminiscent of the appearance of cross-caps in maps from $\mathbb{R}P^2$ to $\mathbb{R}^3$, except that the real cross-caps can be eliminated in pairs to give a smooth immersion, and the CR cross-caps cannot be eliminated to give a totally real immersion. (For illustrations and explicit parametrizations of surfaces with real cross-cap singularities, see [**Apéry**], [**CSS**].) More generally, enumerative formulas hold for a real subbundle T of a complex vector bundle (F, J) over a real manifold M ([**Wells**], [**L**], [**HL**$_{93}$], [**HL**$_{95}$], [**D**$_1$], [**D**$_2$]). If the locus of complex tangents $N_j = \{x \in M : \dim_{\mathbb{C}} T_x \cap J_x T_x = j\}$ forms a codimension $2j(n-m+j)$ real submanifold, $N_{j+1} = \emptyset$, and $H^j_x = T_x \cap J_x T_x$ forms a bundle over N_j, then characteristic numbers of $H^j \to N_j$ are related to the pontrjagin classes of T and the chern classes of (F, J). Formulas of [**W**$_{85}$], [**W**$_{86}$] were shown in [**C**] to be special cases of degeneracy loci formulas of [**Pragacz**].

Formulas are also known which describe obstructions to the absence of complex tangent points with $j > 1$. There may also be topological obstructions for the non-degeneracy conditions (Subsection 2.2), so that degenerate points occur as a subset of the locus of complex tangents $N_1 \subseteq M$, generically of lower dimension. The assumptions on the rank, made in arriving at the normal form, represent the most common and most important case; both non-degeneracy conditions were used in the formal stability Theorem, but there may be similar stability results in some degenerate cases.

Another phenomenon related to complex tangents generalizes the "coincidence of complex structures" of [**HL**$_{95}$]. A smooth map $f : M_1 \to M_2$ between complex manifolds (M_1, J_1), (M_2, J_2), may respect the complex structures, that is, $df(J_1(v_x)) = J_2(df(v_x))$, for some tangent vectors v_x at some points $x \in M_1$. The generic behavior of this construction, which more generally applies to real linear maps between complex vector bundles, can be described by enumerative formulas in chern classes; see [**EW**], [**HL**$_{95}$], [**C**]. The graph of a smooth map f inside $M_1 \times M_2$ (or the graph of a bundle map inside the direct sum) has complex tangents exactly at points of coincidence of the complex structures. For example, the differential of a smooth map from a complex surface to a complex 3-manifold generically will commute with the complex structure only at isolated points x, and only for a complex line of vectors in T_x. The graph is a real 4-manifold with isolated complex tangents inside a product space of five complex dimensions, and again the cross-cap singularity of real surfaces in 3-space comes to mind.

References

[Aigner] M. Aigner, *Combinatorial Theory*, GMW **234**, Springer, New York, 1979.

[Apéry] F. Apéry, *Models of the Real Projective Plane*, Vieweg, Braunschweig, 1987.

[Beloshapka] V. K. Beloshapka, *The normal form of germs of four-dimensional real submanifolds in $\mathbb{C}^5$ at generic $\mathbb{RC}$-singular points*, Mat. Zametki (no. 6) **61** (1997), 931–934; translation in Math. Notes (no. 5-6) **61** (1997), 777–779.

[Bishop] E. Bishop, *Differentiable manifolds in complex euclidean space*, Duke Math. J. (1) **32** (1965), 1–21.

[C] A. Coffman, *Enumeration and normal forms of singularities in Cauchy-Riemann structures*, Ph.D. dissertation, University of Chicago, 1997.

[CSS] A. Coffman, A. Schwartz and C. Stanton, *The algebra and geometry of Steiner and other quadratically parametrizable surfaces*, Computer Aided Geometric Design (3) **13** (1996), 257–286.

[D_1] A. V. Domrin, *The number of $\mathbb{RC}$-singular points on a 4-dimensional submanifold in a 5-dimensional complex manifold*, Mathematical Notes (1–2) **57** (1995), 167–170.

[D_2] A. V. Domrin, *A description of characteristic classes of real submanifolds in complex manifolds via RC-singularities*, Izv. Ross. Akad. Nauk Ser. Mat. (English translation) (5) **59** (1995), 899–918.

[EW] J. Eells and J. C. Wood, *Restrictions on harmonic maps of surfaces*, Topology (3) **15** (1976), 263–266.

[GG] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Springer, New York, GTM **14**, 1973.

[GF] H. Grauert and K. Fritzsche, *Several Complex Variables*, GTM **38**, Springer, New York, 1976.

[H] A. Haefliger, *Plongements différentiables de variétés dans variétés*, Comment. Math. Helv. **36** (1961), 47–82.

[HL_{93}] F. R. Harvey and H. B. Lawson, Jr., *A theory of characteristic currents associated with a singular connection*, Astérisque **213**, 1993.

[HL_{95}] F. R. Harvey and H. B. Lawson, Jr., *Geometric residue theorems*, American Journal of Mathematics (4) **117** (1995), 829–873.

[L] H. F. Lai, *Characteristic classes of real manifolds immersed in complex manifolds*, Trans. AMS **172** (1972), 1–33.

[M] J. Moser, *Analytic surfaces in $\mathbb{C}^2$ and their local hull of holomorphy*, Ann. Acad. Scient. Fenn., Ser. A. I. **10** (1985), 397–410.

[MW] J. Moser and S. Webster, *Normal forms for real surfaces in $\mathbb{C}^2$ near complex tangents and hyperbolic surface transformations*, Acta Mathematica (3–4) **150** (1983), 255–296.

[Porteous] I. Porteous, *Simple singularities of maps*, Springer Lecture Notes **192** (1971), 286–307.

[Pragacz] P. Pragacz, *Enumerative geometry of degeneracy loci*, Ann. scient. Éc. Norm. Sup., 4^e, **21** (1988), 413–454.

[V] S. M. Voronin, *Analytic classification of germs of conformal mappings $(\mathbb{C}, 0) \to (\mathbb{C}, 0)$ with identity linear part*, Functional Anal. Appl. (1) **15** (1981), 1–13.

[Walter] W. Walter, *A useful Banach algebra*, Elem. Math. (1) **47** (1992), 27–32.

[W_{84}] S. Webster, *Real submanifolds of $\mathbb{C}^n$ and their complexifications*, in Topics in Several Complex Variables, Ramírez de Arellano & Sundararaman, eds., Pitman, Boston, RNM **112**, 1985.

[W_{85}] S. Webster, *The Euler and Pontrjagin numbers of an n-manifold in $\mathbb{C}^n$*, Comment. Math. Helv. **60** (1985), 193–216.

[W_{86}] S. Webster, *On the relation between Chern and Pontrjagin numbers*, Contemp. Math. **49** (1986), 135–143.

[Wells] R. Wells, *Compact real submanifolds of a complex manifold with nondegenerate holomorphic tangent bundles*, Math. Ann. **179** (1969), 123–129.

[Wh_{43}] H. Whitney, *The general type of singularity of a set of $2n-1$ smooth functions of n variables*, Duke Math. J. **10** (1943), 161–172.

[Wh_{44}] H. Whitney, *The self-intersections of a smooth n-manifold in $2n$-space*, Annals of Mathematics (2) **45** (1944), 220–246.

[Wh_{58}] H. Whitney, *Singularities of mappings of Euclidean spaces*, in Symp. Int. de Topología Algebraica, UNAM and UNESCO, Mexico City, 1958.

Received Received date / Revised version date

Department of Mathematical Sciences
Indiana University-Purdue University Fort Wayne
Fort Wayne, IN 46805-1499
E-mail address: CoffmanA@ipfw.edu
http://www.ipfw.edu/math/Coffman/