Invariants for Pairs of Almost Complex Structures

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1 Introduction

As an example of a phenomenon to which geometric residue theorems can be applied, Harvey and Lawson ([HL$_1$], [HL$_2$]) considered singular differential forms comparing two almost complex structures on a real vector bundle. The set of vectors in a fiber having the same image under either complex structure operator forms a complex vector subspace; points where this subspace jumps in dimension form the "coincidence locus" of the complex structures.

The formulas describing the cohomology class of the coincidence current are naturally generalized in two ways. First, instead of two structures on one real bundle, the structure on one vector bundle is compared with the structure on another in which the first is mapped injectively. Geometrically, this corresponds to comparing the almost complex structure of a manifold with that on complex tangents inherited from an immersion in an ambient manifold. Second, if the vectors over a coincidence locus form a bundle over a smooth, closed base space, then its chern numbers are given by a universal polynomial formula. Then the relationship between the complex coincidence locus of a map and the CR-singular set of its graph is described, with real-analytic examples. The last sections apply the theory to anticommuting complex structures, and to the coincidence of several complex structures.

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2 The Linear Algebra of Coincidence

Consider two complex vector spaces, as real vector spaces together with complex structure operators, $V^r = (V_{\mathbb{R}}^{2r}, J^V)$, $F^n = (F_{\mathbb{R}}^{2n}, J^F)$, and an injective $\mathbb{R}$-linear map $\alpha : V_{\mathbb{R}} \to F_{\mathbb{R}}$. α is said to be $\mathbb{C}$-linear with respect to J^V and J^F if

$$\alpha J^V \vec{v} = J^F \alpha \vec{v} \tag{1}$$

holds for all vectors $\vec{v} \in V_{\mathbb{R}}$. Since not all $\mathbb{R}$-linear maps are $\mathbb{C}$-linear, an immediate query about α might regard the nature of the set of vectors satisfying equation (1). Those vectors such that $\alpha J^V\vec{v} = J^F\alpha\vec{v}$ form a real vector space $K \subseteq V_{\mathbb{R}}$, and K is a complex vector space with respect to J^V:

$$\vec{v} \in K \Rightarrow \alpha J^V(J^V\vec{v}) = -\alpha\vec{v} = J^F(J^F\alpha\vec{v}) = J^F\alpha(J^V\vec{v}) \Rightarrow J^V\vec{v} \in K.$$

The image αK is a complex subspace of F:

$$\alpha\vec{v} \in \alpha K \Rightarrow J^F(\alpha\vec{v}) = \alpha J^V\vec{v} \in \alpha K.$$

The subspace K can have complex dimension j, $0 \leq j \leq r$. The map α restricted to K is a $\mathbb{C}$-linear isomorphism with respect to $J^V|_K$ and $J^F|_{\alpha K}$.

Example 2.1 If α is the identity map ($V_{\mathbb{R}} = F_{\mathbb{R}}$) and the complex structures J^V and J^F agree, then K, V, and F are $\mathbb{C}$-linearly isomorphic.

Example 2.2 If $(V_{\mathbb{R}}, J^V) = (F_{\mathbb{R}}, -J^F)$, then $K = \{\vec{0}\}$. This is the "complex conjugate," $V = \bar{F}$, and the complex structures J^F and $-J^F$ map every $\vec{v}$ to different images.

The vector space K can also be interpreted in terms of the kernel of a $\mathbb{C}$-linear map of complexifications, $V \otimes \mathbb{C} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, and $F \otimes \mathbb{C} = F_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. The map $\alpha : V_{\mathbb{R}} \hookrightarrow F_{\mathbb{R}}$ complexifies as $\alpha_{\mathbb{C}} : V \otimes \mathbb{C} \hookrightarrow F \otimes \mathbb{C}$, and the vector spaces break into eigenspaces as follows:

$$\begin{array}{rcl} V^{1,0} & = & \{\vec{v} \in V \otimes \mathbb{C} : J^V_{\mathbb{C}}\vec{v} = i\vec{v}\} \\ V^{0,1} & = & \{\vec{v} \in V \otimes \mathbb{C} : J^V_{\mathbb{C}}\vec{v} = -i\vec{v}\} \\ F^{1,0} & = & \{\vec{v} \in F \otimes \mathbb{C} : J^F_{\mathbb{C}}\vec{v} = i\vec{v}\} \\ F^{0,1} & = & \{\vec{v} \in F \otimes \mathbb{C} : J^F_{\mathbb{C}}\vec{v} = -i\vec{v}\} \end{array}$$

Define θ to be the inclusion of $V^{0,1}$ in $V \otimes \mathbb{C}$ and note that $\varphi = \frac{1}{2} - \frac{i}{2}J^F_{\mathbb{C}}$ is the projection of $F \otimes \mathbb{C}$ onto $F^{1,0}$.

Consider the composite map $\varphi\alpha_{\mathbb{C}}\theta : V^{0,1} \to F^{1,0}$. It is $\mathbb{C}$-linear with respect to i on $V^{0,1}$ and $J^F_{\mathbb{C}} = i$ on $F^{1,0}$. $\theta = \frac{1}{2} + \frac{i}{2}J^V_{\mathbb{C}}$ is the identity on $V^{0,1}$, and the composition can then be expressed as

$$\varphi\alpha_{\mathbb{C}}\theta = (\frac{1}{2} - \frac{i}{2}J^F_{\mathbb{C}})\alpha_{\mathbb{C}}(\frac{1}{2} + \frac{i}{2}J^V_{\mathbb{C}}) = \frac{1}{4}(\alpha_{\mathbb{C}} + i\alpha_{\mathbb{C}}J^V_{\mathbb{C}} - iJ^F_{\mathbb{C}}\alpha_{\mathbb{C}} + J^F_{\mathbb{C}}\alpha_{\mathbb{C}}J^V_{\mathbb{C}}). \quad (2)$$

The kernel of this map is the preimage by $\alpha_{\mathbb{C}}$ of the intersection of $\alpha_{\mathbb{C}}V^{0,1}$ with $F^{0,1}$, the kernel of φ. If $\alpha_{\mathbb{C}}\vec{v}$ is in this intersection, then

$$J^F_{\mathbb{C}}\alpha_{\mathbb{C}}\vec{v} = -i\alpha_{\mathbb{C}}\vec{v} = \alpha_{\mathbb{C}}(-i)\vec{v} = \alpha_{\mathbb{C}}J^V_{\mathbb{C}}\vec{v}.$$

Conversely, if $J^F_{\mathbb{C}}\alpha_{\mathbb{C}}\vec{v} = \alpha_{\mathbb{C}}J^V_{\mathbb{C}}\vec{v}$, then evaluating composition (2) shows that $\vec{v}$ is in the kernel. The subspace K also complexifies as $K^{1,0} \oplus K^{0,1}$. These remarks show that $K^{0,1} = \{\vec{v} \in V^{0,1} | J^F_{\mathbb{C}}\alpha_{\mathbb{C}}\theta\vec{v} = \alpha_{\mathbb{C}}J^V_{\mathbb{C}}\theta\vec{v}\}$ is the kernel of $\varphi\alpha_{\mathbb{C}}\theta$, and that it is isomorphic to the conjugate $\bar{K}$ of the coincidence subspace K.

REMARK: The map α need not be injective for this linear algebra to work—the domain V may have any dimension and α may be singular. Vectors $\vec{v}$ in the intersection of the subspaces K and ker α have the property $\alpha(J^V\vec{v}) = J^F\alpha(\vec{v}) = \vec{0}$. However, to simplify the geometric constructions that follow, only injective α will be considered. ∎

3 Grassmannian Constructions

As a universal version of the linear construction, consider the set of all complex r-subspaces of the complex vector space $F \otimes \mathbb{C}$ (subspaces are with respect to multiplication by i as the structure operator). Let V be the tautological complex r-bundle over this grassmannian $\mathbb{C}G(r, F \otimes \mathbb{C})$. Then V is a subbundle of the trivial complex $2n$-bundle $F \otimes \mathbb{C}$, and φ is a projection on the trivial n-bundle $F^{1,0}$. Those planes $V \in \mathbb{C}G(r, F \otimes \mathbb{C})$ that intersect $F^{0,1}$ in at least j complex dimensions form a subvariety, a degeneracy locus D_j, where the bundle map $\varphi|V$ has a kernel of complex dimension $\geq j$. These varieties are singular except for D_0 and $D_r \cong \mathbb{C}G(r, F^{0,1})$. As a partial desingularization of D_j, form the complex grassmannian bundle $\pi_j : \mathbb{C}G(j, V) \to \mathbb{C}G(r, F \otimes \mathbb{C})$. If U^j is the tautological bundle of j-planes in V (candidates for the kernel of φ), then the inclusion of U^j in V, followed by the map $\varphi : V \to F^{0,1}$, defines a section s_j of $\mathrm{Hom}(U^j, \pi_j^* F^{1,0})$. The zero locus of this section, corresponding to a drop by j in the rank of φ, has real codimension $2jn$ in the total space $\mathbb{C}G(j, V)$, and projects to the degeneracy locus D_j, which has real codimension $2jn - 2j(r-j) = 2j(n-r+j)$ in $\mathbb{C}G(r, F \otimes \mathbb{C})$.

This can be generalized to allow F to be a complex vector bundle over a smooth base space X; then $\mathbb{C}G(r, F \otimes \mathbb{C})$ is a grassmann bundle over X. A real m-bundle $T^m_{\mathbb{R}} \to X$, with $m = 2r$, and complex structure J^T, and an injective map $\alpha : T_{\mathbb{R}} \to F_{\mathbb{R}}$ determine a "conjugate Gauss map," $\gamma_{T,\alpha} : X \to \mathbb{C}G(r, F \otimes \mathbb{C})$, by $\gamma_{T,\alpha}(x) = (\alpha_x)_{\mathbb{C}} T^{0,1}_x \subseteq F_x \otimes \mathbb{C}$.

$$\begin{tikzcd}
\mathrm{Hom}(U^j, \pi_j^* F^{1,0}) \arrow[d, shift left] & V \arrow[d] & F \arrow[d] \\
\mathbb{C}G(j, V) \arrow[u, shift left, "s_j"] \arrow[r, "\pi_j"] & \mathbb{C}G(r, F \otimes \mathbb{C}) \arrow[r, shift left, "\mu"] & X \arrow[l, shift left, "\gamma_{T,\alpha}"]
\end{tikzcd}$$

Definition/Lemma 3.1 The "rank j coincidence locus" Q_j of the triple (F, T, α) is the set $\mu(\gamma_{T,\alpha}(X) \cap D_j) \subseteq X$. (F, T, α) is "generic" if $\gamma_{T,\alpha}(X)$ and $C_j := D_j \setminus D_{j+1}$ intersect transversely for all $0 \leq j \leq r$. In this case, $Q_j \setminus Q_{j+1}$ is a smooth (possibly empty) submanifold of X, of real codimension $2j(n-r+j)$.

Lemma 3.2 If N_j is the locus of "CR singularities" of the image $\alpha T_{\mathbb{R}}$ in F, where $\alpha T_x \cap J^F_x \alpha T_x$ has complex dimension $\geq j$, then $Q_j \subseteq N_j$.

PROOF: N_j is defined independently of any complex structure on T. $\alpha T_x \cap J^F \alpha T_x$ is the largest complex subspace of F_x contained in αT_x. For T_x to contain a j-subspace where the coincidence relation is satisfied, αT_x must at least contain a j-subspace of F_x. ■

Over the set $Q_j \setminus Q_{j+1}$, the set of coincidence subspaces K^j_x, where α_x is $\mathbb{C}$-linear, forms a bundle. In terms of the grassmannian construction, K^j is the conjugate of a pullback of U^j.

Example 3.3 If $T = F$ (abbreviating α is the identity, $J^T = J^F$), then $\gamma_{T,id}(X) \subseteq D_r$. Unless $r = 0$, this is not a generic situation.

Example 3.4 T and F can be isomorphic (by α, possibly the identity) as real bundles and have different complex structures so that (T, J^T) defines a Gauss map $\gamma_{T,\alpha}$ transverse to the submanifolds C_j. The real codimension of Q_j in X is $2j^2$.

Example 3.5 If T is the conjugate bundle of F, then, since no complex lines in T are complex in F, $\gamma_{T,id}(X) \cap D_1 = \emptyset$, and the triple (F, T, id) is generic.

Example 3.6 A "real structure" on the complex bundle $F = (F_{\mathbb{R}}, J)$ is a $\mathbb{R}$-linear map $C : F \to \bar{F}$ that is $\mathbb{C}$-linear, i.e., $CJ = -JC$, and such that $\bar{C}C$ is the identity on F, where $\bar{C} : \bar{F} \to F$ is the same real-linear map, but considered as $\mathbb{C}$-linear with respect to $-J$ and J. In particular, C is a $\mathbb{C}$-isomorphism of F and $\bar{F}$, so it is not generic with respect to coincidence. However, the triple (F, F, C) is generic, with $Q_1 = \emptyset$. A bundle F with a real structure must have odd chern classes all zero; this will also follow from the topological results of the next section. In [Wakakuwa], $(F_{\mathbb{R}}, J, C)$ is called an "almost complex-product structure of the first kind," where C is an "almost product" structure. For $t \in \mathbb{R}$, the operator $J_t = \sqrt{1+t^2}J + tC$ is a complex structure on $F_{\mathbb{R}}$, agreeing with J at $t = 0$, but coinciding on no vector with either J or $-J$ for $t \neq 0$. Similarly, $J'_t = -J_{-t} = -\sqrt{1+t^2}J + tC$ agrees with $-J$ at $t = 0$, but coincides on no vector with either J or $-J$ for $t \neq 0$.

As a geometric application, let T be the tangent bundle TX of a smooth m-manifold X, $m = 2r$, with complex structure J^T. If $f : X \to A$ is an immersion into another almost complex $2n$-manifold, then the tangent bundle (TA, J^A) pulls back to $(F, J^F) = (f^*TA, f^*J^A)$ over X. The map of T into F is the differential map $\alpha = f^*df$.

Example 3.7 [Audin-Lafontaine] f is a "(J^T, J^A)-holomorphic," or "pseudoholomorphic" immersion if $df \circ J^T = J^A \circ df$. Unless X is zero-dimensional, (F, T, f^*df) is not generic with respect to coincidence; $\gamma_{T,f^*df}(X) \subseteq D_r$.

Example 3.8 The codimension of the locus Q_1 for a generic df is $2(n - r + 1) = 2n - m + 2$. If $m \leq n$, then $Q_1 = \emptyset$.

Example 3.9 ([Eells-Lemaire]) If f is a harmonic map from a Riemann surface X to a Kähler manifold, then it is either holomorphic ($Q_1 = X$) or Q_1 is discrete (codimension ≥ 2).

4 Thom-Porteous Formulas and Examples

Theorem 4.1 If X is a compact, oriented, smooth manifold with real dimension $2j(n - r + j)$, and (F, T, α) is generic, then

$$\sum_{x \in Q_j} \mathrm{ind}(x) = \int_X \Delta^{(j)}_{n-r+j}(c(F - \bar{T})),$$

where $\mathrm{ind}(x)$ is the oriented intersection number of $\gamma_{T,\alpha}(X)$ and C_j at $\gamma_{T,\alpha}(x)$.

PROOF: The symbol $\Delta^{(a)}_b$, applied to a graded sum $c_0 + c_1 + c_2 + \ldots$, stands for the determinant of the $a \times a$ matrix with p, q entry c_{b-p+q}. In this case, the entries are the graded components of the (formal) quotient $c(F-\bar{T}) = (cF)/(c\bar{T})$.

Pushing forward the current $\mathrm{Zero}(s_j)$ and the chern form of the bundle $\mathrm{Hom}(U^j, \pi_j^* F)$ gives an equation of cohomology classes on $\mathbb{C}G(r, F \otimes \mathbb{C})$:

$$\pi_{j*}[\mathrm{Div}(s_j)] = \Delta^{(j)}_{n-r+j}(c(F^{1,0} - V)).$$

Such formulas are due to Giambelli, Thom, and [Porteous$_1$], and are considered, together with useful determinantal identities, in [Fulton] and [HL$_2$].

By the lemma on transversality of generic maps, $\gamma_{T,\alpha}(X)$ and C_j meet transversely at isolated points. The pushforward current $\pi_{j*}[\mathrm{Zero}(s_j)]$, when restricted to $\mathbb{C}G(r, F\otimes\mathbb{C}) \setminus D_{j+1}$, is equal to the restriction of the current $[D_j]$, since the projection π_j is a local diffeomorphism of $\mathrm{Zero}(s_j)$ over the set C_j. The chern class formula pulls back by $\gamma_{T,\alpha}$ to X by functoriality. ■

The quotient $c(F-\bar{T}) = cF/c\bar{T}$ of total chern classes is calculated using the given complex structures: $cF = c(F_{\mathbb{R}}, J^F)$, and $c\bar{T} = c(T_{\mathbb{R}}, -J^T)$. A real vector bundle may admit finitely or infinitely many complex structures with different chern classes, as examples of [Thomas], [Hiller], and [Nash] show. However, there are some relations restricting which chern classes can occur. A given complex structure induces an orientation on the real vector bundle; then the top chern class c_rT is equal to the euler class χT of the oriented bundle. The opposite complex structure $-J^T$ induces the same orientation as J^T if r is even, and reverses the orientation if r is odd. The pontrjagin class of the real bundle does not depend on its orientation. The following familiar relations hold, if $cT = 1 + c_1 + c_2 + \ldots + c_r$ and $p(T_{\mathbb{R}}) = 1 + p_1 + \ldots + p_r$.

$$\begin{aligned} c\bar{T} &= 1 - c_1 + c_2 - c_3 + \ldots \pm c_r \\ cTc\bar{T} &= 1 - p_1 + p_2 - \ldots \pm p_r \\ &= 1 + (2c_2 - c_1^2) + (2c_4 - 2c_1c_3 + c_2^2) + \ldots \end{aligned} \tag{3}$$

The first few terms of the quotient $cF/c\bar{T}$ are

$$\begin{aligned} \frac{cF}{c\bar{T}} = {} & 1 + (c_1F + c_1T) \\ & + (c_1^2T - c_2T + c_1Fc_1T + c_2F) \\ & + (c_1^3T - 2c_1Tc_2T + c_3T + c_1Fc_1^2T - c_1Fc_2T + c_2Fc_1T + c_3F) \\ & + (c_1^4T - 3c_1^2T + c_2^2T + 2c_1Tc_3T - c_4T + c_1Fc_1^3T - 2c_1Fc_1Tc_2T \\ & \; + c_1Fc_3T + c_2Fc_1^2T - c_2Fc_2T + c_3Fc_1T + c_4F) + \ldots \end{aligned}$$

Example 4.2 ([HL$_2$]) If $T_{\mathbb{R}}$ and $F_{\mathbb{R}}$ are equal as real (unoriented) bundles, with relatively generic complex structures, the coincidence currents satisfy the cohomological relation

$$[Q_j] = \Delta^{(j)}_j(c(F - \bar{T})).$$

T and F have symmetric roles in this scenario, and the equality $\Delta^{(j)}_j(c(F-\bar{T})) = \Delta^{(j)}_j(c(T-\bar{F}))$ also follows from a determinantal identity. In this example, diffeomorphisms $f : X \to X$ of an almost complex manifold (X, J^X) induce a complex structure f^*J^X; some of the diffeomorphisms are such that the triple $((TX, J^X), (TX, f^*J^X), f^*df)$ is generic. (This example was correctly analyzed, but stated with incorrect conclusion in [HL$_2$].)

Example 4.3 The case where $J^T = -J^F$ is generic and the numerator and denominator are equal in the quotient $cF/c\bar{T} = 1 + 0 + \ldots + 0$. For $j > 0$, the locus $Q_j = \emptyset$, and $\Delta_j^{(j)}(c(F - \bar{T})) = 0$ in the cohomology ring.

Example 4.4 Suppose $f : X \to A$ is a pseudoholomorphic immersion; then X admits a complex normal bundle ν such that $TX \oplus \nu = F = f^*TA$. The triple $(F, \overline{TX}, df)$ is generic and $Q_j = \emptyset$ for $j > 0$. Computing the coincidence cohomology class,

$$\Delta_{n-r+j}^{(j)}(c(F - TX)) = \Delta_{n-r+j}^{(j)}(c(\nu)).$$

Since ν has complex rank $n - r$, $\Delta_{n-r+j}^{(j)}(c(\nu))$ is a determinant of a matrix with zeroes on and above the diagonal. So, these formulas are not a new obstruction for pseudoholomorphic immersions.

Example 4.5 ([HL$_1$]) If J^1 and J^2 are two relatively generic complex structures on a vector bundle over the compact 2-manifold X, then Q_1 is a finite set, and the theorem gives the count

$$\sum_{x \in Q_1} \mathrm{ind}(x) = \int_X c_1(J^1) + c_1(J^2).$$

Let $X = (\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z}), i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix})$; then $f(x, y) = (x + y, y)$ is an orientation-preserving diffeomorphism of the torus, and $c_1TX = 0$. The differential in the x, y coordinates is $df = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. At no point on X does the equality $i \circ df = df \circ i$ hold.

Example 4.6 If $F = (TX, J^1)$ and $T = (TX, J^2)$ are two relatively generic complex structures on the compact 8-manifold X, then Q_2 is a finite set, and the theorem gives the formula (again, symmetric in F and T):

$$\begin{aligned} \sum_{x \in Q_2} \mathrm{ind}(x) &= \int_X \Delta_2^{(2)}(c(F - \bar{T})) \\ &= \int_X c_2^2T - c_3Tc_1F + c_2Tc_1^2F - c_1Tc_3T + c_1Tc_2Tc_1F \\ &\quad +c_2^2F - c_3Fc_1T + c_2Fc_1^2T - c_1Fc_3F + c_1Fc_2Fc_1T - 2c_2Tc_2F. \end{aligned}$$

Theorem 4.7 If X is compact, (F, T, α) is generic and $Q_{j+1} = \emptyset$, then Q_j is a submanifold of real codimension $2j(n - r + j)$ in X. The chern numbers of the bundle $K^j \to Q_j$ can be computed by applying the kernel bundle formula of [Pragacz] to the conjugate bundle $K^{0,1}$: If $\prod c_i(K^{0,1})^{\beta_i} = \sum m_{\mathbf{J}} s_{\mathbf{J}}(K^{0,1})$, where $m_{\mathbf{J}} \in \mathbb{Z}$ and $\sum i\beta_i = \frac{1}{2}\dim_{\mathbb{R}}(Q_j) = d$, then

$$\prod_i c_i(K^{0,1})^{\beta_i}[Q_j] = (-1)^d \sum_{\mathbf{J}} m_{\mathbf{J}} s_{j^{n-r+j}, \tilde{\mathbf{J}}}(F - \bar{T}).$$

$\tilde{\mathbf{J}}$ denotes the conjugate multiindex $(i_1, i_2, \ldots)$, $i_a = \mathrm{card}\{h : j_h \geq a\}$. The subscript $j^{n-r+j}, \tilde{\mathbf{J}}$ denotes the concatenation of $n-r+j$ j's and the multiindex $\tilde{\mathbf{J}}$. The symbol $s_{(i_1,\ldots,i_k)}$ is the determinant of a $k \times k$ matrix with p, q entry $s_{i_p - p+q}$. In this case, $1 + s_1 + \ldots$ denotes the segre class.

The determinants in this formula could immediately be rewritten in terms of chern classes using the determinantal identity $s_{\mathbf{I}}(E-F) = c_{\tilde{\mathbf{I}}}(E-F)$. The segre class is often more natural in enumerative constructions ([Fulton]); in this case, the first few terms of $s(F - \bar{T})$ are:

$$\begin{aligned} s &= 1 + s_1 + s_2 + s_3 + \ldots \\ &= 1 + c_1F + c_1T \\ &\quad + c_1^2F + c_1Fc_1T + c_2T - c_2F \\ &\quad + c_1^3F + c_1^2Fc_1T + c_1Fc_2T - 2c_1Fc_2F - c_1Tc_2F + c_3T + c_3F + \ldots \end{aligned}$$

Example 4.8 Suppose the same real four-plane bundle $F^4_{\mathbb{R}}$ over the compact 4-manifold X has two complex structures, $F^2 = (F_{\mathbb{R}}, J^F)$ and $T^2 = (F_{\mathbb{R}}, J^T)$. If the triple (F, T, id) is generic, the coincidence locus Q_1 is a real surface in X, and its cohomology class $[Q_1]$ is equal to the class $c_1F + c_1T$ by Theorem 4.1. The complex line bundle K^1 over Q_1 of vectors where $J^F = J^T$ has the following chern number:

$$\begin{aligned} \int_{Q_1} c_1K &= -\int_{Q_1} c_1K^{0,1} \\ &= \int_X s_{1,1}(F - \bar{T}) = \int_X \begin{vmatrix} s_1 & s_2 \\ 1 & s_1 \end{vmatrix} \\ &= \int_X c_2(F - \bar{T}) \\ &= \int_X c_1^2T - c_2T + c_1Fc_1T + c_2F \end{aligned}$$

For example, if $F = \bar{T}$, then $Q_1 = \emptyset$ and the cohomology classes $c_1F + c_1T$ and $c_1^2T - c_2T + c_1Fc_1T + c_2F$ are both zero. So far, this formula does not appear to be symmetric in F and T, however, using relation (3) gives:

$$\int_{Q_1} c_1K = \int_X p_1F_{\mathbb{R}} + c_2T + c_1Fc_1T + c_2F. \tag{4}$$

Example 4.9 Suppose the sphere S^6 admits a relatively generic pair of complex structures, $T = (TS^6, J^T)$ and $F = (TS^6, J^F)$. The chern classes are of the form $cT = 1 + c_3T$ and $cF = 1 + c_3F$, where c_3T and c_3F are equal to the euler class, up to a sign depending on the orientation induced by J^T and J^F. Q_1 is a real 4-submanifold of S^6, with $\int_{Q_1} c_1^2K = \int_X c_3T + c_3F$. This number is ± 4 if T and F have the same orientation, and 0 if they are oppositely oriented.

Example 4.10 Consider an immersion $f : X^{16} \to A^{18}$, where X and A are almost complex manifolds ($r = 8$, $n = 9$). If f is generic with respect to complex tangency, then $H = TX \cap (f^*J^ATX)$ has $j_0 = 7$ complex dimensions, except at isolated points where TX is a complex subspace of $k = 8$ dimensions in $F = f^*TA$. If f is generic with respect to complex coincidence, then the smooth locus of Q_1 is codimension 4 in X, and Q_2 is a 4-dimensional (codimension 12) submanifold of X. For X compact, the bundle K^2 over Q_2 has the same chern numbers as $K^{0,1}$:

$$\int_{Q_2} c_2 K^2 = \int_X \begin{vmatrix} s_2 & s_3 & s_4 & s_5 \\ s_1 & s_2 & s_3 & s_4 \\ 1 & s_1 & s_2 & s_3 \\ 0 & 1 & s_1 & s_2 \end{vmatrix},$$

$$\int_{Q_2} c_1^2 K^2 = \int_X \begin{vmatrix} s_2 & s_3 & s_4 & s_5 \\ s_1 & s_2 & s_3 & s_4 \\ 1 & s_1 & s_2 & s_3 \\ 0 & 1 & s_1 & s_2 \end{vmatrix} + \begin{vmatrix} s_2 & s_3 & s_4 & s_5 & s_6 \\ s_1 & s_2 & s_3 & s_4 & s_5 \\ 1 & s_1 & s_2 & s_3 & s_4 \\ 0 & 0 & 1 & s_1 & s_2 \\ 0 & 0 & 0 & 1 & s_1 \end{vmatrix}.$$

Example 4.11 Tsanov and [Pontecorvo] consider a pair of complex structures on a twistor space Z constructed as follows: Let $M = (M^4, g)$ be a compact oriented riemannian manifold. The twistor space Z is the total space of the fiber bundle $t : SO(TM, g)/U(2) \to M$, where local sections correspond to almost complex structures. The tangent space $T_z Z$ is split by the metric into a vertical part $T_z \mathbb{C}P^1$ and a horizontal part $T_{t(z)}M$— the horizontal part is given the complex structure operator defined by the point z. The direct sum of the two complex structure operators defines the tautological twistor almost complex structure, (Z, J) which is integrable if and only if g is anti-self-dual. If M is also a complex surface, a different description of Z as the projectivization of a holomorphic bundle gives a different almost complex structure (Z, I), which is integrable without requiring g to be anti-self-dual. The two almost complex structures I and J may have different chern classes. The cohomology ring of Z is a module over $H^*(M; \mathbb{R})$, generated by the cohomology class h over the fiber and subject to the relation $4h^2 = c_1^2$, where $t^*c(TM) = 1 + c_1 + c_2$.

$$\begin{aligned} c(TZ, I) &= (1+2h)(1+c_1+c_2) = 1 + 2h + c_1 + c_2 + 2hc_1 + 2hc_2 \\ c(TZ, J) &= (1+2h)(1+2h+c_2) = 1 + 4h + c_2 + c_1^2 + 2hc_2 \end{aligned}$$

The invariance relation (3) holds, and $c_3(I) = c_3(J)$, but the other chern numbers may disagree:

$$\begin{aligned} \int_Z c_1^3(I) &= \int_Z 8h^3 + 6hc_1^2 = \int_M 8c_1^2 \\ \int_Z c_1c_2(I) &= \int_Z 2hc_2 + 2hc_1^2 = \int_M 2(c_2 + c_1^2) \\ \int_Z c_1^3(J) &= \int_Z 64h^3 = \int_M 16c_1^2 \\ \int_Z c_1c_2(J) &= \int_Z 4h(c_2 + c_1^2) = \int_M 4(c_2 + c_1^2) \end{aligned}$$

So, if I and J have the same chern classes, the chern numbers c_1^2 and c_2 of TM must be zero. [Pontecorvo] uses this to conclude that if g is anti-self-dual, then I and J cannot be homotopic.

Perturbing I and J so that they are relatively generic, the locus Q_2 is expected to be empty, and the locus Q_1 is expected to be a codimension 2 submanifold in Z. Its current of integration represents the cohomology class $c_1(I) + c_1(J) = 6h + c_1$. The chern number of the line bundle K over Q_1 is given by the formula

$$\begin{aligned}
\int_{Q_1} c_1^2 K &= \int_Z \Delta_3^{(1)}(c(J)/c(-I)) = \int_Z s_{111} = \int_Z s_1^3 - 2s_1s_2 + s_3 \\
&= \int_Z c_3(J) + c_2(J)c_1(I) - c_1(J)c_2(I) + c_1(J)c_1^2(I) \\
&\quad + c_3(I) + c_1^3(I) - 2c_1(I)c_2(I) \qquad (5) \\
&= \int_Z 24h^3 + 12h^2c_1 + 8hc_1^2 - 2hc_2 + 2c_1^3 - c_1c_2 \\
&= \int_M 14c_1^2 - 2c_2
\end{aligned}$$

Again, equation (5) is not symmetric in I and J until relation (3) is used:

$$\int_{Q_1} c_1^2 K = \int_Z c_3(I) + c_3(J) + c_1(I)c_2(J) + c_2(I)c_1(J) + (c_1(I) + c_1(J))p_1TZ.$$

QUESTION: The geometric symmetry of the roles of I and J in the previous example and Example 4.8 does not seem to be exhibited until relation (3) is taken into account. Is there a purely combinatorial explanation of this phenomenon, for example, some identity among symmetric functions? ■

5 Coincidence as CR-Singularities of a Graph

The map $\alpha : T_{\mathbb{R}} \to F_{\mathbb{R}}$ defines its graph $\underline{\alpha}$ as a real-linear inclusion of the image $\underline{\alpha}T$ in $T_{\mathbb{R}} \oplus F_{\mathbb{R}}$ of the map $\vec{v} \mapsto (\vec{v}, \alpha(\vec{v}))$. $T_{\mathbb{R}} \oplus F_{\mathbb{R}}$ has the direct sum complex structure, $J_x^{\oplus}(\vec{v}, \vec{w}) = (J_x^T\vec{v}, J_x^F\vec{w})$. Denote $T \oplus F = (T_{\mathbb{R}} \oplus F_{\mathbb{R}}, J^{\oplus})$.

Lemma 5.1 $K_x \cong \underline{\alpha}T_x \cap J_x^{\oplus}\underline{\alpha}T_x$.

PROOF: The claim is that $\underline{\alpha}_x$ is a $\mathbb{C}$-linear isomorphism when restricted to K_x, and that its image is the maximal $J_x^{\oplus}$-complex subspace of $\underline{\alpha}T_x$.

$$\begin{aligned}
\vec{v} \in K \quad &\Longleftrightarrow \quad (J^T\vec{v}, J^F\alpha\vec{v}) = (J^T\vec{v}, \alpha J^T\vec{v}) \iff J^{\oplus}(\vec{v}, \alpha\vec{v}) \in \underline{\alpha}T \\
&\Longleftrightarrow \quad (\vec{v}, \alpha\vec{v}) \in \underline{\alpha}T \cap J^{\oplus}\underline{\alpha}T.
\end{aligned}$$

■

If, in addition to α being generic with respect to coincidence, $\underline{\alpha}T$ is a subbundle of real rank $m = 2r$ in the complex bundle $T \oplus F$ of complex rank $n + r$, which is generically included with respect to loci N_j of CR-singularities, then a cohomological version of formula (10.5) of [HL_2] applies:

$$\begin{aligned}[N_j] &= \Delta^{(j)}_{(n+r)-2r+j}(c(T\oplus F - T\otimes\mathbb{C}))\\ &= \Delta^{(j)}_{n-r+j}(\frac{cTcF}{cTc\bar{T}}) = \Delta^{(j)}_{n-r+j}(c(F-\bar{T})) = [Q_j].\end{aligned}$$

This shows that, for sufficiently general maps α, Theorem 4.1 can be derived as a corollary to cohomological formulas for CR singularities. (These formulas are investigated in more detail and generalized in [C].)

The relationship between complex coincidence and the CR structure of the graph seems to be well-known, but not formulated as explicitly as this in the literature. (cf [Freeman] and §4.2, [Chirka])

Example 5.2 The graph of a smooth map $f : X \to A$ (not necessarily an immersion) defines an embedding $\underline{f} : X \to X\times A$. The coincidence locus Q_j of df is the same as the locus N_j of complex tangents of the image of $d\underline{f} = \underline{df}$ in $X\times A$.

Example 5.3 ([Eells-Wood]) If f is a smooth map between connected, compact, oriented Riemannian surfaces, $f : (X, g_X) \to (A, g_A)$, the degree of f is an integer. Giving X and A complex structures compatible with the metrics, and using [Webster]'s formula for the image of the graph of f in $(X\times A, J^{\oplus})$,

$$\sum_{x\in N_1} \mathrm{ind}(x) = \int_X c_1(TX\oplus f^*TA) = \chi X + (\deg\, f)\chi A.$$

Reversing the orientation and complex structure on A changes the index sum to $\chi X - (\deg\, f)\chi A$, and similarly, reversing the orientation and complex structure on X gives index sum $-\chi X + (\deg\, f)\chi A$. The sign of the index differs from the [Eells-Wood] formulas, which use the complexified bundle map $T^{1,0}X \to T^{0,1}A$ instead of (2).

If f is a generic perturbation of a holomorphic map, with ramification $v(q) > 1$ at finitely many branch points $q \in X$, then the Riemann-Hurwitz theorem applies ([Griffiths-Harris]),

$$\chi X = (\deg\, f)\chi A - \sum(v(q) - 1),$$

so the number of points where f is $\mathbb{C}$-linear is $2\chi X + \sum(v(q) - 1)$.

It is well-known that in this two-dimensional case, f is conformal (angle-preserving) at points where df is $\mathbb{C}$-linear, and indirectly conformal (angle-reversing) at points where df is $\mathbb{C}$-antilinear. ([Ahlfors])

Example 5.4 If X is the almost complex manifold $(X_{\mathbb{R}}, J)$ and $\bar{X} = (X_{\mathbb{R}}, -J)$, then the graph of the identity diffeomorphism $f : X \to \bar{X}$ is the diagonal embedding $\underline{f} : X \to X\times\bar{X}$. The image is totally real in the "complexification" $X\times\bar{X}$ (cf [Eastwood]); again, for $j > 0$, this is consistent with the cohomological obstruction to total reality:

$$\Delta^{(j)}_j(c(\underline{f}^*TX\oplus T\bar{X} - TX\otimes\mathbb{C})) = 0.$$

Example 5.5 Something similar holds for the kernel bundle formulas: in the scenario of Example 4.8, equation (4) can be rewritten:

$$\int_{Q_1} c_1 K = p_1 F_{\mathbb{R}} + c_2(T \oplus F).$$

This is [Webster]'s formula for the complex tangent locus N_1 of a real 4-plane subbundle $F_{\mathbb{R}}$ of a complex 4-bundle, $T \oplus F$.

6 Cartographic Applications

Example 6.1 Considering $\mathbb{C}P^1$ as the sphere S^2 with the usual complex structure, the graph of the identity diffeomorphism embeds $\mathbb{C}P^1$ as the diagonal complex submanifold of $\mathbb{C}P^1 \times \mathbb{C}P^1$, or as a totally real submanifold of $\mathbb{C}P^1 \times \overline{\mathbb{C}P^1}$. Perturbing the identity map to a generic map gives a diffeomorphism $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ with four points of complex coincidence by Example 4.5. Equivalently, the graph of f is a sphere S^2 with four complex tangents in $\mathbb{C}P^1 \times \mathbb{C}P^1$.

The coordinate charts for $\mathbb{C}P^1$ are two complex lines. A smooth map between one of these charts and the unit sphere in $\mathbb{R}^3$ is given by stereographic projection:

$$\begin{aligned} z &= \frac{x_1 + ix_2}{1 - x_3} \\ (x_1, x_2, x_3) &= \left(\frac{z + \bar{z}}{z\bar{z} + 1}, \frac{z - \bar{z}}{i(z\bar{z} + 1)}, \frac{z\bar{z} - 1}{z\bar{z} + 1}\right) \end{aligned}$$

Figure 1.

In particular, the complex structure operator $J = i$ on the complex plane induces the complex structure on S^2, and if the diffeomorphism f is written in terms of the local coordinates x, y so that $z = x + iy \mapsto u(x, y) + iv(x, y)$, the coincidence relation $J \circ df = df \circ J$ is the pair of Cauchy-Riemann equations for complex functions:

$$\begin{aligned} df \circ J &= \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u_y & -u_x \\ v_y & -v_x \end{pmatrix} \\ J \circ df &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} -v_x & -v_y \\ u_x & u_y \end{pmatrix} \\ &\Longrightarrow \quad u_x = v_y, \ u_y = -v_x. \end{aligned}$$

This condition is equivalent to f being conformal (angle-preserving) [Ahlfors].

The best-known diffeomorphisms of the sphere are conformal at every point. The following composition gives a generic diffeomorphism of the sphere, one expected to satisfy the Cauchy-Riemann equations only on a finite set. First, stretch the unit sphere in $\mathbb{R}^3$ into the ellipsoid $x^2 + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$ by a linear transformation, with $1 < a < b$. (In Figure 2, $a = 2$, $b = 3$ are used.)

Figure 2.

Second, a radial projection maps the ellipsoid back onto the unit sphere.

Figure 3.

In terms of the coordinates $z = x + iy$, the composite f is given by the real analytic equations

$$\begin{aligned} u(x,y) &= \frac{2x}{\sqrt{b^2(x^2+y^2-1)^2+4x^2+4a^2y^2}\,(1-\frac{b(x^2+y^2-1)}{\sqrt{b^2(x^2+y^2-1)^2+4x^2+4a^2y^2}})}, \\ v(x,y) &= \frac{2ay}{\sqrt{b^2(x^2+y^2-1)^2+4x^2+4a^2y^2}\,(1-\frac{b(x^2+y^2-1)}{\sqrt{b^2(x^2+y^2-1)^2+4x^2+4a^2y^2}})}. \end{aligned}$$

A computer-assisted calculation finds exactly four points (x,y) in the plane where the Cauchy-Riemann relations hold:

$$(\pm\frac{\sqrt{b^2+a^2b^2-2a^2\pm 2a\sqrt{(b^2-1)(b^2-a^2)}}}{b\sqrt{a^2-1}}, 0).$$

Corresponding to the symmetry of f, the four-element subset Q_1 of S^2 in $\mathbb{R}^3$ forms a rectangle with sides parallel to the x_1- and x_3-axes.

With the conformal structure on the ellipsoid induced by the ambient euclidean metric, the stretching map from the sphere preserves angles at the four “umbilic points” of the ellipsoid. ([Boehm-Prautzsch], [Porteous$_2$])

Example 6.2 As another application of Example 5.3, consider the degree 0 map $f : S^2 \to \mathbb{R}^2$, given by projecting the unit sphere in $\mathbb{R}^3$ orthogonally onto a plane. The equations

$$\begin{aligned} f(z) &= (\frac{z+\bar{z}}{z\bar{z}+1}, \frac{z-\bar{z}}{i(z\bar{z}+1)}) \\ f(x+iy) = (u(x,y), v(x,y)) &= (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}) \end{aligned}$$

for f are smooth, and fold along a great circle. Evidently, f is directly conformal at one pole, $z = 0$, and indirectly conformal at the opposite pole $z = \infty$. Since a degree 0 map from S^2 is expected to have two points where it is directly conformal, this projection f is not generic and $([z = 0], 0)$ is a complex tangent with index 2 in the sphere graphed inside $\mathbb{C}P^1 \times \mathbb{C}$.

The shear $A(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ is holomorphic $\mathbb{C} \to \mathbb{C}$ only for $s = 0$. A simple perturbation of f is $A(s) \circ f$, and this composition is conformal when

$J \circ A(s) \circ df = A(s) \circ df \circ J$:

$$\begin{aligned}
J \circ A(s) \circ df &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \\
&= \frac{2}{(1+x^2+y^2)^2} \begin{pmatrix} 2xy & -x^2+y^2-1 \\ -x^2+y^2-2sxy+1 & sx^2-sy^2-2xy+s \end{pmatrix} \\
A(s) \circ df \circ J &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
&= \frac{2}{(1+x^2+y^2)^2} \begin{pmatrix} sx^2-sy^2-2xy+s & x^2-y^2+2sxy-1 \\ x^2-y^2+1 & 2xy \end{pmatrix}.
\end{aligned}$$

The two real solutions (x, y) of the Cauchy-Riemann equations are

$$\pm\left(\sqrt{\frac{-s(s-\sqrt{s^2+4})}{2(s^2+4)}}, \sqrt{\frac{-2s}{(s^2+4)(s-\sqrt{s^2+4})}}\right), \text{ if } s \geq 0,$$

and

$$\pm\left(\sqrt{\frac{-s(s+\sqrt{s^2+4})}{2(s^2+4)}}, \sqrt{\frac{-2s}{(s^2+4)(s+\sqrt{s^2+4})}}\right), \text{ if } s \leq 0.$$

Example 6.3 The unit sphere in $\mathbb{R}^3$ can also be projected radially onto the x, y-plane from the point $(0, 0, 2)$, again giving a degree zero map with one (index ± 2) point each of direct and indirect conformality.

Varying the point of projection along the line $\{(r, 0, 2)\}$ projects the sphere onto an ellipse. Not too surprisingly, the map is conformal at exactly those points projected onto the foci of the ellipse. The four points are the intersections of the sphere with the lines connecting the poles to the vertex of the cone.

Figure 4.

The minor semiaxis of the ellipse, parallel to the y-axis, has constant length $2/\sqrt{3}$. The major semiaxis has length $\frac{2}{3}\sqrt{r^2+3}$, and the ellipse meets the x-axis at coordinates $-r - \frac{4\sqrt{r^2+3}}{(r+\sqrt{r^2+3})^2+3} < 0$ and $-r + \frac{4\sqrt{r^2+3}}{(r-\sqrt{r^2+3})^2+3} > 0$. The foci are at $r/3$ and $-r$.

It is a well-known theorem that, in this case, the ellipses defined by the intersections of the planes $z = 1$ and $z = -1$ with the cone have two of their foci at the poles of the sphere, and that they are similar (and so conformal) to any parallel ellipse on the cone.

7 Anticommuting Complex Structures

Suppose I and J are two complex structures on a vector space or bundle F that anticommute: $IJ + JI = 0$. Then their product IJ is a new complex structure, as well as real-linear combinations of the form $aI + bJ + cIJ$ such that $a^2 +$

$b^2+c^2=1$. F, together with the anticommuting complex structures I, J, could be called “quaternionic;” there are examples of manifolds with anticommuting almost complex structures: [Joyce]. I and J never coincide and so are relatively generic with respect to coincidence.

The geometry of real subspaces of a quaternion vector space is more complicated than the complex vector space situation. ([Dlab-Ringel]) However, the complex subspaces of a quaternion vector space are studied with familiar constructions.

Let F be a complex vector space $(F_{\mathbb{R}}, I)$ of even complex dimension $n=2s$, and suppose F admits another complex structure J so that I and J anticommute. Let $\mathbb{C}G(r,F)$ be the grassmannian of I-complex subspaces of complex dimension r in F. Given a plane $V\in\mathbb{C}G(r,F)$, the intersection $V\cap JV$ is a quaternionic vector space: $V\cap JV$ is invariant under I and J, and their restrictions form a pair of anticommuting complex structures on $V\cap JV$.

Denoting by V the tautological complex r-bundle over $\mathbb{C}G(r,F)$, the real bundle $V_{\mathbb{R}}$ can be considered as a real $2r$-subbundle of the trivial complex vector bundle $(F_{\mathbb{R}}, J)$. If this inclusion were generic, the CR-singular set $N_r\subseteq\mathbb{C}G(r,F)$, where $V_{\mathbb{R}}$ is a J-complex subspace, would have real codimension $2r(n-r)$, and so be isolated. However, for r even, N_r is not isolated; those planes which are simultaneously I-complex and J-complex in (F,I,J) form the quaternionic grassmannian $\mathbb{H}G(r/2,F)$, of real dimension $r(n-r)$. If further, $2r\le n$, since the quaternionic subspace $V\cap JV$ must be at least 4-dimensional, the first-order CR-singular set $N_1\setminus N_2$ is empty, although generically of codimension $2(n-2r+1)$.

Example 7.1 The I-complex 2-planes in $F=\mathbb{H}^2$ form a complex manifold of complex dimension 4. Any plane containing a J-complex line must actually be J-complex, and so a quaternionic line in $\mathbb{H}^2$. These planes form the quaternionic projective space $\mathbb{H}P^1$, a real 4-sphere. This construction appears in twistor geometry ([Eastwood], [Ward-Wells]), where $\mathbb{C}G(2,4)$ is the compactified complexified Minkowski space, $\mathbb{C}P^3$ is the projective twistor space, and a projection π is induced by the $\mathbb{C}$-isomorphism $\mathbb{C}^4\to\mathbb{H}^2$:

$$\begin{tikzcd}\mathbb{C}P^3 \arrow[r, "\pi"] & \mathbb{H}P^1 \arrow[r, hook] & \mathbb{C}G(2,\mathbb{H}^2)\end{tikzcd}$$

so that π has fiber $\mathbb{C}P^1=\mathbb{C}G(1,\mathbb{H}^1)$ and the inclusion is totally real. This is another example of Webster’s formula, where $0=\int_{S^4}c_2(T\mathbb{C}G(2,4)|_{S^4})+p_1TS^4$. The restriction of the complex bundle to S^4 is essentially the complexification of TS^4, so $c_2=p_1=0$.

Example 7.2 If F is a bundle with anticommuting complex structures I and J, it is known (cf [Vaisman]) that the complex vector bundle (F,I) has all odd chern classes zero. The easy proof is that J is a $\mathbb{C}$-linear isomorphism of $F=(F_{\mathbb{R}},I)$ and $\bar{F}=(F_{\mathbb{R}},-I)$. The first coincidence current $[Q_1]$ is zero, so applying Theorem 4.7 to the chern form $c_1^qK^1$ simplifies to the expression

$$0=\Delta^{(1)}_{q+1}\left(\frac{c(F_{\mathbb{R}},I)}{c(F_{\mathbb{R}},-J)}\right),$$

for $q \geq 0$, so $c(F_{\mathbb{R}}, I) = c(F_{\mathbb{R}}, -J)$. The same argument holds for the complex structure IJ, which does not coincide with I or J:

$$c(F_{\mathbb{R}}, I) = c(F_{\mathbb{R}}, -J) = c(F_{\mathbb{R}}, IJ) = c(F_{\mathbb{R}}, -I).$$

More generally, this shows that the odd chern classes of $(F_{\mathbb{R}}, I)$ are zero when I and any other two complex structures on $F_{\mathbb{R}}$ are mutually non-coincident.

8 Several Complex Structures

The graph of a $\mathbb{R}$-linear map α can be generalized to the graph of finitely many maps $\alpha_i : T^{m=2r} \to (F_i, J_i) = F_i^{n_i}$, $i = 1 \dots p$. Then the subspace of vectors in T where all the maps α_i are $\mathbb{C}$-linear corresponds to a complex subspace in the graph $\underline{\alpha}$ of $\alpha_1 \oplus \dots \oplus \alpha_p$ in $T \oplus F_1 \oplus \dots \oplus F_p$:

$$\begin{aligned} \alpha_i J^T \vec{v} = J_i \alpha_i \vec{v} \ \forall i \quad \Longleftrightarrow \quad & J^{\oplus} \underline{\alpha} \vec{v} \\ & = (J^T \vec{v}, J_1 \alpha_1 \vec{v}, \dots, J_p \alpha_p \vec{v}) \\ & = (J^T \vec{v}, \alpha_1 J^T \vec{v}, \dots, \alpha_p J^T \vec{v}) \\ & = \underline{\alpha} J^T \vec{v} \in \underline{\alpha} T. \end{aligned}$$

$Q_j(\alpha_1, \dots, \alpha_p)$, the locus where all the maps α_i are $\mathbb{C}$-linear on the same j-subspace, generically has codimension $2j(n_1 + \dots + n_p - r + j)$, and is contained in, but not equal to, $Q_j(\alpha_1) \cap \dots \cap Q_j(\alpha_p)$. Using the complex tangent formula gives

$$\begin{aligned} [Q_j(\alpha_1, \dots, \alpha_p)] &= \Delta^{(j)}_{r+n_1+\dots+n_p-2r+j}(c(T \oplus F_1 \oplus \dots \oplus F_p) - c(T \otimes \mathbb{C})) \\ &= \Delta^{(j)}_{n_1+\dots+n_p-r+j}(cF_1 \cdot \dots \cdot cF_p - c\bar{T}). \end{aligned}$$

Example 8.1 A manifold M of real dimension $m = 2r > 0$ with three relatively generic almost complex structures has no tangent vectors where all three agree. The expected codimension of $Q_1(id_{TM}, id_{TM})$ is $2(r + r - r + 1) = m + 2$.

Example 8.2 A complex line bundle T, with generic maps to two other complex line bundles, $\alpha_i : T \to (F_i, J_i)$, will have a coincidence locus Q_1 of codimension $2(1 + 1 - 1 + 1) = 4$. The cohomology formula is

$$\begin{aligned} [Q_1] &= \Delta^{(1)}_2(cF_1 \cdot cF_2 - c\bar{T}) \\ &= c_1^2 T + (c_1 F_1 + c_1 F_2) c_1 T + c_1 F_1 c_1 F_2. \end{aligned}$$

Example 8.3 Consider the 4-manifold $\mathbb{C}P^2$ of complex lines in $\mathbb{C}^3$, with respect to the usual complex structure i. Inside $\mathbb{C}^3$, let F_1 and F_2 be a pair of real planes, which have their own complex structure but are totally real with respect to i. Also let p_1 and p_2 be projections of $\mathbb{C}^3$ onto these planes:

$$F_1 = (\{(x,0,y,0,0,0)\}, J_1 = \begin{pmatrix} \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix}),\ p_1 = \begin{pmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ & & & & & 0 \end{pmatrix},$$

$$F_2 = (\{(0,0,0,x,0,y)\}, J_2 = \begin{pmatrix} 0 & 0 \\ 0 & \begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{matrix} \end{pmatrix}),\ p_2 = \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 1 \end{pmatrix}.$$

The inclusion of the tautological line bundle T in the trivial bundle $\mathbb{C}^3 \to \mathbb{C}P^2$, followed by the projections p_1 and p_2, gives two $\mathbb{R}$-linear maps to the trivial line bundles, $T \to F_1$ and $T \to F_2$. The coincidence locus Q_1 in $\mathbb{C}P^2$ is the set of complex lines in $\mathbb{C}^3$ on which both the projections p_1 and p_2 are $\mathbb{C}$-linear. The equations $p_1 i\vec{v} = J_1 p_1 \vec{v}$ and $p_2 i\vec{v} = J_2 p_2 \vec{v}$ have solution set $\{(x,y,y,-x,-x,-y)\}$, which is the complex line $\mathbb{C} \cdot (1,-i,-1)$. So, Q_1 is a point, consistent with the formula

$$\sum_{x \in Q_1} \mathrm{ind}(x) = \int_{\mathbb{C}P^2} c_1^2 T = 1.$$

Of course, this is equivalent to the coincidence geometry of $p_1 \oplus p_2 : T \to F_1 \oplus F_2$.

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