Notes on Axiomatic Geometry

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These Notes are compiled from classroom handouts for Math 560, Foundations of Geometry, at Indiana University - Purdue University Fort Wayne. They supplement the textbook: both the third edition [G3] and the fourth edition [G4].

1 Incidence geometry

Undefined Terms: "Point." "Line." "On."

By a model we mean a set $\mathcal{P}$ of points, and a set $\mathcal{L}$ of lines, and a relation "on" which, for each given point and given line, is either true or false. The "on" relation could be specified as a subset of the set of ordered pairs $\mathcal{P} \times \mathcal{L}$.

We will also need the terms "$=$" and "$\neq$" from Set Theory, and it will also be convenient to use the word "set" and the usual set theory symbols "$\{\ :\ \}$," "$\in$," "$\notin$," "$\cup$," and "$\cap$." Note "$\in$" and "on" are not necessarily the same, although they may be related concepts in some models.

As on [G3] p. 12, [G4] p. 12 (and [G3] p. 50, [G4] p. 69), sometimes it's convenient to use phrases which mean the same thing as "on," for example, instead of saying "the point P is on the line ℓ and the point P is on the line m," one might say "ℓ and m meet at P." Instead of saying "P is on ℓ and Q is on ℓ," one might say "ℓ goes through P and Q." In Section 3 we will use the symbol $P\mathrm{I}\ell$.

Incidence Axioms

Here are four properties that a model of points and lines could have. The phrase "at least" in these sentences is just for emphasis. They would mean the same thing with "at least" deleted.

- **Axiom $I-1$. There exist at least two points.**
- **Axiom $I-2$. Any two points lie on exactly one line.**
  This is our version of Euclid's Postulate 1 ([E] Book 1). The word "exactly" means there are actually two statements, called "existence" and "uniqueness":
  - ◇ $I-2-i$. If P is a point, and Q is a point, and $P \neq Q$, then there exists a line ℓ such that P is on ℓ and Q is on ℓ.
  - ◇ $I-2-ii$. If P is a point, and Q is a point, and $P \neq Q$, and ℓ is a line, and m is a line, and P is on ℓ, and Q is on ℓ, and P is on m, and Q is on m, then $\ell = m$.
- **Axiom $I-3$. For every line, there exist at least two points on that line.**
- **Axiom $I-4$. For every line, there exists at least one point not on that line.**

Here are some Theorems about points and lines which can be proved using only the incidence axioms $I-1, \ldots, I-4$, and some Definitions which use the undefined terms.

Theorem 1.1. *For every point P, there is at least one line ℓ such that P is not on ℓ.*

Remark. This is Proposition 2.4 of [G3] and [G4], and part of Exercise 2.6 of [G3] and [G4]. ∎

Theorem 1.2. *For every point P, there exist a line ℓ and a line m such that $\ell \neq m$, and P is on ℓ, and P is on m.*

Remark. This is Proposition 2.5 of [G3] and [G4], and part of Exercise 2.6 of [G3] and [G4]. ∎

Exercise 1.3. The five incidence axioms are independent. Show this, for each axiom, by constructing a model which does not satisfy that axiom, but does satisfy the remaining four. (This is a modification of Exercise 2.7 of [G3] and [G4].)

Definition 1.4. A set of points S is collinear means: there exists a line ℓ such that every point $P \in S$ is on ℓ. Abbreviation: sometimes, instead of saying "the set $\{A, B, C\}$ is collinear," we'll just say "the points A, B, C are collinear."

Theorem 1.5. *Given points P, Q, R, if the set $\{P, Q, R\}$ is non-collinear, then P, Q, and R are distinct.*

Proof. The hypothesis means that there does not exist any line ℓ such that P, Q, and R are all on ℓ. We want to show that $P \neq Q$, $P \neq R$, and $Q \neq R$. Suppose, first, that all three points are equal: $P = Q = R$. Then, by $I - 1$, there exists some other point S not equal to P, and by $I - 2 - i$, there exists some line ℓ such that P is on ℓ and S is on ℓ. Then, P, Q and R are all on ℓ because they're equal to P, which contradicts the hypothesis that $\{P, Q, R\}$ is a non-collinear set. The second possibility is that two of the points, say P and Q, are equal, and $R \neq P$. Then by $I - 2 - i$, there exists some line ℓ such that P is on ℓ, and R is on ℓ, but again, Q is then on ℓ, contradicting the non-collinear hypothesis. Since, similarly, assuming $Q = R$, or $P = R$, leads to the same contradiction, we can conclude that $P = Q$, $Q = R$, $P = R$, and $P = Q = R$ are all false, so the three points must be distinct. ∎

Theorem 1.6. *Given points A, B, C, and a line ℓ, if $A \neq B$, and A is on ℓ and B is on ℓ and C is not on ℓ, then $\{A, B, C\}$ is not collinear.*

Proof. If there were a line m such that A, B, and C are all on m, then A and B would be distinct points on both ℓ and m, and so $\ell = m$ by $I - 2 - ii$. So C is on $\ell = m$, but this contradicts the hypothesis that C is not on ℓ. ∎

Theorem 1.7. *There exists a set of points $\{P, Q, R\}$ which is non-collinear.*

Proof. There exist two points, $P \neq Q$, by $I - 1$. There exists a line ℓ so that P and Q are on ℓ, by $I - 2 - i$. There exists a point R so that R is not on ℓ, by $I - 4$. Theorem 1.6 applies to $P = A$, $Q = B$, $R = C$, and the line ℓ, so $\{P, Q, R\}$ is non-collinear. ∎

The next statement follows immediately from the previous Theorem, so we call it a "corollary" instead of a "theorem." Corollary 1.8 is taken as an axiom by [G3] p. 51 and [G4] p. 69, and it can then be used to prove Proposition 2.3 of [G3] and [G4]. Our approach, with the $I - 1$, $\dots$, $I - 4$ axiom system (above), assumes Proposition 2.3 of [G3] and [G4] as an axiom ($I - 4$), and uses it (in Theorem 1.7) to prove this result:

Corollary 1.8. *There exist three distinct, non-collinear points.*

Proof. Theorem 1.7 gives the existence of a non-collinear set of points and then Theorem 1.5 states that such a set has three distinct elements, which is all that is needed to prove the Corollary. However, the Proof of Theorem 1.7 is not just about existence, it gives a specific construction of the required set of three points, and can also be used to directly explain (without Theorem 1.5) why the points are distinct: for P, Q, R, and ℓ as in the Proof of Theorem 1.7, the Proof already stated $P \neq Q$. $R \neq P$ because P is on the line ℓ and R is not on ℓ; similarly, $R \neq Q$. ∎

Theorem 1.9. *Given a point A, a line ℓ, and a set of points S, if A is not on ℓ, and $A \in S$, and S is a collinear set, then there is at most one point X such that $X \in S$ and X is on ℓ.*

Proof. From the definition of collinear, let m be a line so that every element of S is a point on m. Suppose, toward a contradiction, that there are at least two distinct points, X, Y so that $X \in S$, $Y \in S$, X is on ℓ and Y is on ℓ. Then, X and Y are distinct points on ℓ and m, so $\ell = m$ by $I - 2 - ii$. However, $A \in S$ implies A is on m, which contradicts the other hypothesis that A is not on ℓ. $\blacksquare$

Definition 1.10. A set of lines T is concurrent means: there exists a point P such that P is on every line $\ell \in T$.

Theorem 1.11. *There exist three distinct, non-concurrent lines.*

Remark. This is Proposition 2.2 of [G3] and [G4], and part of Exercise 2.6 of [G3] and Exercise 2.5 of [G4]. $\blacksquare$

Definition 1.12. A line ℓ and a line m are parallel means: there does not exist a point P such that P is on ℓ and P is on m. Abbreviation: $\ell \| m$.

Theorem 1.13. *Given lines ℓ and m, if $\ell \neq m$, and $\ell \not\| m$, then there exists exactly one point P which is on both ℓ and m.*

Remark. This is Proposition 2.1 of [G3] and [G4], and part of [G3] Exercise 2.6. Hint: this uses only the definition of $\|$, and $I - 2 - ii$. $\blacksquare$

Theorem 1.14. *Given lines ℓ and m, if there is exactly one point P which is on both ℓ and m, then $\ell \neq m$, and $\ell \not\| m$.*

Hint. This uses only the definition of $\|$, and $I - 3$. $\blacksquare$

Theorem 1.15. *Given lines ℓ and m, if $\ell \| m$, then $\ell \neq m$.*

Proof. Suppose, toward a contradiction, that $\ell = m$. Then, by $I - 3$, there exists a point P which is on ℓ, and if $\ell = m$, then P is also a point on m. This contradicts the definition of $\ell \| m$. $\blacksquare$

Here are some properties which a model may or may not have. We could call these properties "axioms" because they describe relationships between the undefined terms, and can't be proved using just the previously given axioms.

The Euclidean Parallel Property For every line ℓ, and every point P not on ℓ, there exists a unique line m such that P is on m and $\ell \| m$.

Note that this statement ([G3] p. 19, [G4] p. 21) is not exactly the same statement as Euclid's Postulate 5 ([E] Book 1, and [G3] p. 20, [G4] p. 22, figure 1.11 and above), which involves angles and other as yet undefined concepts.

The Elliptic Parallel Property ([G3] p. 53, [G4] p. 73) For every line ℓ, there are no lines parallel to ℓ.

It could also be stated as "any two lines have at least one point in common," and then Theorem 1.13 says that any two distinct lines have exactly one point in common.

The Hyperbolic Parallel Property ([G3] p. 55, [G4] p. 75) For every line ℓ, and every point P not on ℓ, there exist at least two distinct lines through P which are parallel to ℓ.

For each of these three properties, there is a model with that property. There are also models with none of the above properties. The existence of at least one parallel to every line, through any point not on it, is related to Euclid's Proposition 31 ([E] Book 1).

Theorem 1.16. *At most one of the above three properties can be satisfied.*

Proof. We'll need Theorem 1.7 and the notation from its Proof, which used axioms $I-1$, $I-2$, and $I-4$. There are two points $P \neq Q$ on a line ℓ, and a point R not on ℓ. If every line through R is not parallel to ℓ, then the Euclidean and Hyperbolic properties fail to hold. If there is exactly one line through R parallel to ℓ, then the Elliptic and Hyperbolic properties fail to hold. If there is more than one line through R parallel to ℓ, then the Elliptic and Euclidean properties fail to hold. These are the only three possibilities for ℓ and R, and in each case, two of the three failed to hold. ∎

The idea of the Proof is that it is enough to check just one line and one point, in order to rule out some parallel property. However, checking just one line is certainly not enough to establish that one of the properties does hold.

Notation 1.17. Let $\sim$ denote the following relation among lines: $\ell \sim m$ means that ℓ and m are parallel or equal.

This use of the $\sim$ symbol only applies to this Section — we will use $\sim$ in later Sections for some different equivalence relations.

Theorem 1.18. *The relation $\sim$ is reflexive and symmetric. If the Euclidean, or Elliptic, property holds, then $\sim$ is also transitive, so it is an equivalence relation.*

Proof. The reflexive property is obvious: $\ell \sim \ell$ because $\ell = \ell$. To prove the symmetric property, there are two cases from the definition of $\ell \sim m$. First, if $\ell = m$, then $m = \ell$, so $m \sim \ell$. Second, if $\ell \| m$, then by definition of $\|$, there is no point P such that P is on ℓ and P is on m. It is just a rule of logic that we can switch the order of statements in an "and" sentence, so there is no point P such that P is on m and P is on ℓ, which implies $m \| \ell$, and $m \sim \ell$.

Checking transitivity (if $\ell \sim m$ and $m \sim n$, then $\ell \sim n$) requires checking five simple cases.

- Case 1: $\ell = m$ and $m = n$. We can conclude that $\ell = n$ by the transitivity of equality. This case is the only possibility allowed by the Elliptic property (since there are no parallel lines, $\ell \sim m$ is the same as $\ell = m$). This case could also happen in the Euclidean case, but so could the remaining four.
- Case 2: $\ell = m$ and $m \| n$. It follows that $\ell \| n$, so $\ell \sim n$.
- Case 3: $\ell \| m$ and $m = n$. It follows that $\ell \| n$, so $\ell \sim n$.
- Case 4: $\ell \| m$ and $m \| n$, and there is no point P such that P is on ℓ and P is on n. This is the definition of $\ell \| n$, so $\ell \sim n$.
- Case 5: $\ell \| m$ and $m \| n$, and there is a point P such that P is on ℓ and P is on n. P cannot be on m since $\ell \| m$. So, there are two lines, ℓ and n, which are parallel to m and which go through a point not on m. The uniqueness part of the Euclidean property says that $\ell = n$, and so $\ell \sim n$.

∎

The transitivity property of Theorem 1.18 appears in [G3] p. 59 and [G4] p. 82, and is discussed further in Chapter 4, [G3] starting on p. 129, and [G4] starting on p. 175. It is not too different from Euclid's Proposition 30 ([E] Book 1), which used [E] Postulate 5. It is interesting that in the above Proof, none of the incidence axioms was required, and the Euclidean property was used only in Case 5.

Corollary 1.19. *Given that the Euclidean, or Elliptic, property holds, and given lines ℓ, m, and n, if $\ell \| m$ and $\ell \not\sim n$, then $m \not\sim n$.*

Proof. By definition of $\sim$, $\ell \sim m$. Suppose, toward a contradiction, that $m \sim n$. Then, by the transitivity of $\sim$, $\ell \sim n$, which contradicts the hypothesis. We didn't need any incidence axioms for this, either. ∎

Corollary 1.19 is not necessarily true for models where the Hyperbolic property holds, or where none of the three parallel properties holds.

Theorem 1.20. *For lines ℓ, m, the following are equivalent (which means if either statement is true, then so is the other):*
$\ell \not\sim m \iff$ there exists exactly one point which is on both ℓ and m.

Proof. This just uses the $\sim$ notation to restate Theorems 1.13 and 1.14, which required only incidence axioms $I-2-ii$ and $I-3$. ∎

To summarize this Theorem and the previous Corollary, we've shown that the Incidence Axioms, together with the Euclidean parallel property, imply that if a line n meets one of a pair of parallel lines in exactly one point, then it meets the other in exactly one point. Such a line n is called a <u>transversal</u> ([G3] p. 20, and [G4] p. 22).

Definition 1.21. For points P, Q with $P \neq Q$, the symbol $\overleftrightarrow{PQ}$ refers to the *set* of points, $\{A$ such that P, Q, A are collinear$\}$.

This notation appears on [G3] p. 14 and [G4] p. 15, but it's used in a different way. In particular, I mean that $\overleftrightarrow{PQ}$ is just a set, it's not necessarily a line. The book ([G3] p. 13, and [G4] p. 14) uses "$\{\ell\}$" to mean the set of points on the line ℓ, but that notation won't appear in these Notes.

Theorem 1.22. *Given points A and C, if $A \neq C$, then:*

1. *$\overleftrightarrow{AC} = \overleftrightarrow{CA}$,*
2. *the points A and C are elements of the set $\overleftrightarrow{AC}$,*
3. *if there is a line ℓ so that A and C are on ℓ, then every element of $\overleftrightarrow{AC}$ is a point on ℓ, and,*
4. *any subset of $\overleftrightarrow{AC}$ is a collinear set.*

Proof. By definition, $D \in \overleftrightarrow{AC}$ means $\{A, C, D\}$ is a collinear set, which means there is some line m so that A is on m, and C is on m, and D is on m. This immediately implies every element of $\{C, A, D\}$ is on m, so $\{C, A, D\}$ is a collinear set (the ordering doesn't matter for elements of a set, or for the definition of collinear), and this, by definition, means $D \in \overleftrightarrow{CA}$. This shows $\overleftrightarrow{AC} \subseteq \overleftrightarrow{CA}$, and the other subset relation is just as easy to show, so $\overleftrightarrow{AC} = \overleftrightarrow{CA}$.

By Axiom $I-2-i$, the sets $\{A, A, C\} = \{A, C\} = \{A, C, C\}$ are collinear (there exists some line through A and C), so $A \in \overleftrightarrow{AC}$ and $C \in \overleftrightarrow{AC}$.

If $D \in \overleftrightarrow{AC}$, then, by definition, A, C, and D are on some line k; if, also, A and C are on ℓ, then A and C are distinct points on k and ℓ, so $k = \ell$ by $I-2-ii$, which shows D is on ℓ.

By $I-2-i$, there is some line n so that A and C are on n. Suppose $S \subseteq \overleftrightarrow{AC}$, which by definition of subset means that if $D \in S$, then $D \in \overleftrightarrow{AC}$. Then, by definition, A, C, and D are on some line k, and again by $I-2-ii$, A and C are distinct points on k and n, so $k = n$ and D is on n. This shows every element of S is a point on n, the definition of "S is collinear." ∎

Theorem 1.23. *Given points A, B, C, if A, B, C are distinct and collinear, then* $\overleftrightarrow{AC} = \overleftrightarrow{CA} = \overleftrightarrow{BC} = \overleftrightarrow{CB} = \overleftrightarrow{AB} = \overleftrightarrow{BA}$.

Proof. For distinct points A, B, C, the previous Theorem immediately gives $\overleftrightarrow{AC} = \overleftrightarrow{CA}$, $\overleftrightarrow{BC} = \overleftrightarrow{CB}$, and $\overleftrightarrow{AB} = \overleftrightarrow{BA}$.

If, in addition to being distinct, A, B, and C are collinear, then by Definition 1.4, there is some line m so A, B, and C are all on m. To show, for example, $\overleftrightarrow{AC} \subseteq \overleftrightarrow{AB}$, suppose $D \in \overleftrightarrow{AC}$, which just means $\{D, A, C\}$ is a collinear set, so the points lie on some line ℓ. However, ℓ and m have distinct points A, C in common, so $\ell = m$ by $I-2-ii$, and this shows that D is on m. Since A, B, and D are on m, they are collinear, which is the definition of $D \in \overleftrightarrow{AB}$, proving $\overleftrightarrow{AC} \subseteq \overleftrightarrow{AB}$. A similar argument will show $\overleftrightarrow{AB} \subseteq \overleftrightarrow{AC}$, so the sets are equal. Repeating the argument gives $\overleftrightarrow{AC} = \overleftrightarrow{BC}$, so all the equalities hold. ∎

Theorem 1.24. *Given points A, B, C, if A, B, and C are not collinear, then* $\overleftrightarrow{AB} \cap \overleftrightarrow{BC} = \{B\}$.

Proof. By Theorem 1.5, the points are distinct, so the sets are well-defined. The relation $\{B\} \subseteq \overleftrightarrow{AB} \cap \overleftrightarrow{BC}$ follows immediately from Theorem 1.22. To show the other subset relation, suppose $X \in \overleftrightarrow{AB} \cap \overleftrightarrow{BC}$. Then, by definition of intersection, $X \in \overleftrightarrow{AB}$, and by definition of $\overleftrightarrow{AB}$, there is some line ℓ so that A, B, and X are on ℓ. Similarly, there is some line m so that B, C, and X are on m. If $X \neq B$, then X and B are distinct points on ℓ and m, so $\ell = m$ by $I-2-ii$, contradicting the hypothesis that A, B, C are not collinear. ∎

Theorem 1.25. *Given points A, B, C, D, if $C \neq D$, and $\{A, B, C\}$ is non-collinear, then $\overleftrightarrow{AB} \cap \overleftrightarrow{CD}$ is a set with at most one element.*

Proof. $A \neq B$ by Theorem 1.5, so $\overleftrightarrow{AB}$ is well-defined. Suppose P and Q are points in the intersection: $\{P, Q\} \subseteq \overleftrightarrow{AB} \cap \overleftrightarrow{CD}$. Then, by Theorem 1.22, $\{P, Q, A, B\} \subseteq \overleftrightarrow{AB}$ is collinear, lying on some line ℓ. Similarly, P, Q, C, and D lie on some line m. If $P \neq Q$, then ℓ and m have distinct points in common, so $\ell = m$ by $I-2-ii$, but this contradicts the assumption that there is no line such that A, B, and C are on that line. The conclusion is that there cannot be distinct points in the intersection: the intersection could be empty, or if it contains a point P, then every point in the intersection must equal P. ∎

2 The coordinate plane as a model for incidence geometry

(or, "More than you wanted to know about $Ax + By + C = 0$")

In "plane coordinate geometry," one considers a set of ordered pairs (x, y) as points, and equations $Ax + By + C = 0$ as lines. Using real, complex, rational, or integer points may lead to different geometric properties. The book ([G3] p. 58, [G4] p. 82) calls a model that satisfies $I - 1$ through $I - 4$ and the Euclidean parallel property an affine plane. Here are some details, relating coordinate geometry to affine planes. The real affine plane is mentioned in the book ([G3] pp. 61, 67, [G4] pp. 84); it is a special case of F^2, the affine plane over F, where F is a field ([G4] p. 86).

Let $\mathbb{K}$ be an integral domain. (Students who have not yet seen this concept in algebra class may pretend $\mathbb{K}$ is one of these number systems: $\mathbb{Z}$ (the integers), $\mathbb{Q}$ (the rationals), $\mathbb{R}$ (the reals), or $\mathbb{C}$ (the complex numbers)). Just to recall the properties of arithmetic that we'll need, assume $\mathbb{K}$ has two "operations," $+$ and $\cdot$, at least two different elements 0 and 1 so that $x+0 = x \cdot 1 = x$, and $+$ and $\cdot$ are commutative, associative, and distributive, and that for every x, there's an opposite $-x$ so that $x + (-x) = 0$. We will not assume that we can always divide by x: for example in the set of integers, the only time you can divide any number by x and still get an integer is when $x = 1$ or -1. We'll also assume the following property of multiplication:

$$\text{if } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0. \qquad (**)$$

Define a "point" to be an ordered pair (x, y), where both x and y are in $\mathbb{K}$. Define a "coefficient vector" to be an ordered triple (A, B, C) of elements of $\mathbb{K}$ so that A and B are not both 0. Define "scalar multiplication" of a scalar $k \in \mathbb{K}$ and a vector (A, B, C) so that $k \cdot (A, B, C) = (k \cdot A, k \cdot B, k \cdot C)$. (We're using $\cdot$ for both scalar multiplication, and just the usual multiplication of numbers.)

Notation 2.1. Define a relation $\sim$ among coefficient vectors so that $(A, B, C) \sim (D, E, F)$ means, "there are constants $k, j \in \mathbb{K}$, $k \neq 0$ and $j \neq 0$, so that $k \cdot (A, B, C) = j \cdot (D, E, F)$."

Theorem 2.2. *$\sim$ is an equivalence relation.*

Proof. This is a Theorem of algebra and set theory; we aren't doing any geometry yet. The reflexive property is easy: $(A, B, C) \sim (A, B, C)$ (just pick $j = k = 1$). The symmetric property says that if $(A, B, C) \sim (D, E, F)$, then $(D, E, F) \sim (A, B, C)$ (this is also easy). The transitive property is the interesting one: if $(A, B, C) \sim (D, E, F)$ and $(D, E, F) \sim (G, H, I)$, then $(A, B, C) \sim (G, H, I)$. Using the definition, $(A, B, C) \sim (D, E, F)$ means we can assume there exist constants k_1, j_1 so that $k_1 \cdot (A, B, C) = j_1 \cdot (D, E, F)$, and $(D, E, F) \sim (G, H, I)$ means there exist constants k_2, j_2 so that $k_2 \cdot (D, E, F) = j_2 \cdot (G, H, I)$. Multiply the first equation by k_2 to get $k_2 \cdot (k_1 \cdot (A, B, C)) = k_2 \cdot (j_1 \cdot (D, E, F))$. Using the commutativity and associativity of multiplication, this gives $(k_2 \cdot k_1) \cdot (A, B, C) = j_1 \cdot (k_2 \cdot (D, E, F))$, and by the second assumed equation, $j_1 \cdot (k_2 \cdot (D, E, F)) = j_1 \cdot (j_2 \cdot (G, H, I)) = (j_1 \cdot j_2)(G, H, I)$. So, $(k_2 \cdot k_1) \cdot (A, B, C) = (j_1 \cdot j_2) \cdot (G, H, I)$, and $k_2 \cdot k_1$ and $j_1 \cdot j_2$ are both non-0 (by the $**$ property!), which is all we need to show $(A, B, C) \sim (G, H, I)$. ∎

We will try to define "line" as an "equivalence class" of coordinate vectors. Recall that the equivalence class of something is the set of all things equivalent to it.

Notation 2.3. The equivalence class of the coefficient vector (A, B, C) is a set, which will be denoted $[A : B : C]$. This means that if (A, B, C) is a coefficient vector, then it is an element of the set $[A : B : C]$, and any coefficient vector (D, E, F) such that $(A, B, C) \sim (D, E, F)$ is also in the set $[A : B : C]$. We'll call $[A : B : C]$ the "line with coefficient vector (A, B, C)," and it's equal to the line $[D : E : F]$.

Example 2.4. $(2, 4, 6) \sim (3, 6, 9)$. Why? Let $k = 3$, $j = 2$. Then, $k \cdot (2, 4, 6) = (6, 12, 18)$, and $j \cdot (3, 6, 9) = (6, 12, 18)$, and this proves they are equivalent. So, they're both in the same equivalence class, $[2 : 4 : 6] = [3 : 6 : 9]$.

Now, define a point (x, y) is "on" the line $[A : B : C]$ to mean:

$$A \cdot x + B \cdot y + C = 0.$$

Example 2.4 can be interpreted geometrically by saying that $2x+4y+6 = 0$ and $3x+6y+9 = 0$ are equations for the same line, because the equations have equivalent coefficients.

Since there is more than one coefficient vector describing any line, it is very necessary to check that "on a line" doesn't depend on which coefficient vector we use to represent that line. That is, checking whether a point is on a line or not shouldn't depend on which equation of the line we're using!

Theorem 2.5. *If (x, y) satisfies $A \cdot x + B \cdot y + C = 0$, and $(A, B, C) \sim (D, E, F)$, then (x, y) satisfies $D \cdot x + E \cdot y + F = 0$.*

Proof. By definition of $\sim$, there are non-zero constants k and j so that $k \cdot A = j \cdot D$, $k \cdot B = j \cdot E$, and $k \cdot C = j \cdot F$. By hypothesis, $A \cdot x + B \cdot y + C = 0$, so multiply both sides of this equation by k (and use the distributive and associative laws) to get $(k \cdot A) \cdot x + (k \cdot B) \cdot y + (k \cdot C) = k \cdot 0 = 0$. Then, replace the coefficients and expand the expression to get $(j \cdot D) \cdot x + (j \cdot E) \cdot y + (j \cdot F) = j \cdot (D \cdot x + E \cdot y + F) = 0$. Now we use the $**$ property, together with the fact that $j \neq 0$ from the definition of $\sim$, to conclude $D \cdot x + E \cdot y + F = 0$. ∎

Now we can start checking the incidence axioms. Logically, we are assuming that the numbers themselves ("elements of $\mathbb{K}$") are our undefined terms, and the laws of arithmetic are our "axioms," which give relationships among the undefined terms. Then, all the Theorems on this handout are facts about numbers and algebra, and applying our definition of points and lines to these Theorems will show that our model has certain incidence properties.

For example, the next Theorem is exactly the statement of incidence axiom $I-1$! Remember, we are working with a specific model: this doesn't mean we're proving $I - 1$ will always hold in any model, and it doesn't mean we're proving it using the other incidence axioms. The next Theorem and Proof only mean that we're checking that it holds for this model, using the rules for algebra and not any incidence axioms. We are also going to prove a Theorem which re-states incidence axiom $I - 2$: Any two points lie on exactly one line. Once we've done that, then any of the Theorems of incidence geometry which follow from $I - 1$ and $I - 2$ will also be Theorems about the coordinate plane, which we won't need to re-prove using only algebra!

Generally speaking, this was one of the main reasons for stating the incidence axioms in terms of undefined quantities. It allows us to apply the incidence theorems to any model, as long as we can verify the axioms hold in that model.

Theorem 2.6. *There exist at least two points.*

Proof. Even better, there exist at least four points, for example $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$. They are distinct because, earlier, we explicitly assumed $0 \neq 1$. ∎

To verify $I-2$ requires checking both the $I-2-i$ and $I-2-ii$ parts: first, the existence says that there is at least one line $[A:B:C]$ through two given points, and second, the uniqueness: that if two points are on a line $[A:B:C]$, and $[D:E:F]$ also goes through those two points, then $[A:B:C]=[D:E:F]$.

Theorem 2.7. *Given distinct points, (x_1,y_1) and (x_2,y_2), there exists a coefficient vector (A,B,C) so that both points are on the line $[A:B:C]$.*

Proof. Let $A = y_1 - y_2$, $B = x_2 - x_1$, and $C = x_1 \cdot y_2 - x_2 \cdot y_1$. Notice that A could be 0, and B could be 0, but they can't both be zero, because then the two points would not be distinct. (Exercise: where did I get these formulas? Recall from precalculus the definition of slope, and the point-slope equation of a line.) Now, to check that (x_1,y_1) is on the line: $A\cdot x_1 + B\cdot y_1 + C = (y_1-y_2)\cdot x_1 + (x_2-x_1)\cdot y_1 + (x_1\cdot y_2 - x_2\cdot y_1) = 0$. Also, (x_2,y_2) is on the line: $A\cdot x_2 + B\cdot y_2 + C = (y_1-y_2)\cdot x_2 + (x_2-x_1)\cdot y_2 + (x_1\cdot y_2 - x_2\cdot y_1) = 0$. ∎

The following Theorem shows that if the two points from the previous Theorem, (x_1,y_1) and (x_2,y_2), are also on a line with a coefficient vector (D,E,F), then this vector is equivalent to the vector (A,B,C) we just calculated. They're in the same equivalence class, so they define the same line, and we have the uniqueness for $I-2-ii$.

Theorem 2.8. *Suppose (D,E,F) is a coefficient vector, and (x_1,y_1) and (x_2,y_2) are points on the line $[D:E:F]$. Then $(D,E,F) \sim (y_1-y_2, x_2-x_1, x_1\cdot y_2 - x_2\cdot y_1)$.*

Proof. By definition of "on," $D\cdot x_1 + E\cdot y_1 + F = 0$, and $D\cdot x_2 + E\cdot y_2 + F = 0$. The statement of the Theorem is that there exist non-zero constants k, j so that $k\cdot(D,E,F) = j\cdot(y_1-y_2, x_2-x_1, x_1\cdot y_2 - x_2\cdot y_1)$. Rather than just telling you what k and j are, let's try some calculations to find some k and j that will work. Multiply $D\cdot x_1 + E\cdot y_1 + F = 0$ by y_2 and $D\cdot x_2 + E\cdot y_2 + F = 0$ by y_1. (why? because the only thing we can do to coefficient vectors is scalar multiplication, and the only scalars sitting around are the coordinates of our given points.) Now, subtract (and, to simplify things, drop the "$\cdot$" multiplication symbol):

$$\begin{aligned} Dx_1y_2 + Ey_1y_2 + Fy_2 &= 0 \\ -(Dx_2y_1 + Ey_2y_1 + Fy_1) &= 0 \\ = D(x_1y_2 - x_2y_1) + F(y_2 - y_1) &= 0. \end{aligned}$$

Similarly, multiply the first by x_2 and the second by x_1:

$$\begin{aligned} Dx_1x_2 + Ey_1x_2 + Fx_2 &= 0 \\ -(Dx_2x_1 + Ey_2x_1 + Fx_1) &= 0 \\ = E(-x_1y_2 + x_2y_1) + F(x_2 - x_1) &= 0. \end{aligned}$$

These equations hold for any points and any coefficients, but the rest of the Proof of the Theorem will require some cases.

Case 1. $F \neq 0$ and $x_1y_2 - x_2y_1 \neq 0$. Let $k = x_1y_2 - x_2y_1$, and $j = F$, and use the relations we just derived.

$$\begin{aligned} k \cdot (D, E, F) &= (D(x_1y_2 - x_2y_1), E(x_1y_2 - x_2y_1), F(x_1y_2 - x_2y_1)) \\ &= (F(y_1 - y_2), F(x_2 - x_1), F(x_1y_2 - x_2y_1)) \\ &= j \cdot (y_1 - y_2, x_2 - x_1, x_1y_2 - x_2y_1). \end{aligned}$$

This proves $(D, E, F) \sim (y_1 - y_2, x_2 - x_1, x_1y_2 - x_2y_1)$, which is what we wanted.

Case 2. $x_1y_2 - x_2y_1 = 0$. Then the above equations imply $F \cdot (y_2 - y_1) = F \cdot (x_2 - x_1) = 0$. One of the quantities $(y_2 - y_1)$ or $(x_2 - x_1)$ must be non-zero, so $F = 0$ by the $**$ property, which is the next case.

Case 3. $F = 0$. The above equations imply $D \cdot (x_1y_2 - x_2y_1) = 0$ and $E \cdot (x_1y_2 - x_2y_1) = 0$, and since one of D and E must be non-zero, $x_1y_2 - x_2y_1 = 0$ by the $**$ property. $F = 0$ also means that $Dx_1 + Ey_1 = 0$ and $Dx_2 + Ey_2 = 0$, from the definition of "on," and we want to show that $(D, E, 0) \sim (y_1 - y_2, x_2 - x_1, 0)$.

Case 3.a. For $D \neq 0$ and $y_1 - y_2 \neq 0$, use scalars $k = y_1 - y_2$ and $j = D$ to get $k \cdot (D, E, 0) = (D(y_1 - y_2), E(y_1 - y_2), 0)$, and $j \cdot (y_1 - y_2, x_2 - x_1, 0) = (D(y_1 - y_2), D(x_2 - x_1), 0)$. Then, use the "on" relation to plug in $Dx_2 = -Ey_2$ and $Dx_1 = -Ey_1$, which will show $k \cdot (D, E, 0) = j \cdot (y_1 - y_2, x_2 - x_1, 0)$, so the vectors are equivalent.

Case 3.b. If $y_1 - y_2 = 0$, then subtracting the equations $Dx_1 + Ey_1 = 0$ and $Dx_2 + Ey_2 = 0$ gives $D(x_1 - x_2) = 0$, and $x_1 - x_2$ is non-zero because the points are distinct, so $D = 0$ by $**$, which is the next case.

Case 3.c. If $D = 0$, then $E \neq 0$, so $0x_1 + Ey_1 + 0 = 0$ and $0x_2 + Ey_2 + 0 = 0$, and by $**$, $y_1 = y_2 = 0$. Since the points are distinct, $x_1 \neq x_2$. Use $k = x_2 - x_1 \neq 0$ and $j = E$ to prove equivalence: $k \cdot (D, E, F) = (0, E(x_2 - x_1), 0) = j \cdot (0, x_2 - x_1, 0)$. ∎

Geometrically, Cases 2 and 3, where $F = 0$, are lines through the origin. Cases 3.b and 3.c, where $D = F = 0$, are the line $Ey = 0$, the x-axis.

This model also has incidence property $I - 4$:

Theorem 2.9. *If (A, B, C) is any coefficient vector, then there's some point not on the line $[A : B : C]$.*

Proof. If the line has no points on it (we haven't said anything about $I - 3$ yet, so this could happen), then $(0, 0)$ is a point not on the line. Otherwise, there is some point (x, y) on the line, so $Ax + By + C = 0$. I claim that either $(x + 1, y)$, or $(x, y + 1)$ is not on the line. If $(x + 1, y)$ is not on the line, then we're done; if it is on the line, then $A(x + 1) + By + C = 0$, so $Ax + A + By + C = 0$, and we already had $Ax + By + C = 0$, so $A = 0$. Geometrically, $A = 0$ means the line is horizontal, and we can use $0x + By + C = 0$ to show that $(x, y + 1)$ is not on the line: $Ax + B(y + 1) + C = 0x + By + B + C = B$, and this is non-zero because A and B can't both be zero. ∎

What about $I - 3$, that for every line, there are at least two points on it? Here, there's a problem with one of our number systems. Try the line $2x + 2y + 1 = 0$. All the coefficients are integers, but are there any integer solutions (x, y)? No; for any integers x and y, $2x + 2y$ will always be even, and -1 is odd, so $2x + 2y = -1$ has no solutions. This means that our original definition of "line" as just "an equivalence class of coefficient vectors" doesn't satisfy $I-3$ when $\mathbb{K}=\mathbb{Z}$.

If you like $I-3$, and you want to do incidence geometry with the integer grid, then we could try to fix the definition of line to mean "An equivalence class of coefficient vectors (A,B,C) so that the solution set $\{(x,y): A\cdot x+B\cdot y+C=0\}$ is non-empty." We are just throwing out all the "pointless" lines! This doesn't affect any of our Theorems, and the other three axioms still work. In fact, we only have to define a line to contain at least one point, since the existence of another point is automatic:

Theorem 2.10. *Suppose (A,B,C) is a coefficient vector, and (x,y) is on the line $[A:B:C]$. Then $(x+B,y-A)$ is a different point on the line.*

Proof. Plugging in, $A(x+B)+B(y-A)+C=Ax+AB+By-BA+C=Ax+By+C=0$. $(x+B,y-A)$ is different from (x,y) since A and B can't both be 0. ∎

One can conclude that any model with all the points (x,y), and all the non-empty lines, satisfies all the incidence axioms. The problem with the pointless lines will never come up when our coefficient set $\mathbb{K}$ happens to be a "field," that is, a number system where division by any two numbers in the set, $\frac{p}{q}$ with $q\neq 0$ gives a number still in the set. $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields, but not $\mathbb{Z}$.

Theorem 2.11. *If (A,B,C) is any coefficient vector, and $\mathbb{K}$ is a field (non-zero division is allowed), then there is at least one point on the line $[A:B:C]$. (And, by the previous Theorem, there are at least two points.)*

Proof. Case 1: $B\neq 0$. Then the equation $Ax+By+C=0$ can be solved for y by dividing by B: $y=\frac{-A}{B}x+\frac{-C}{B}$. The "$y$-intercept" is the point on the line when you plug in $x=0$: $(0,\frac{-C}{B})$.

Case 2: $A\neq 0$. Solve for x: $x=\frac{-B}{A}y+\frac{-C}{A}$. The "$x$-intercept" is the point $(\frac{-C}{A},0)$.

These are the only two cases, since A and B can't both be 0. ∎

The next Theorem works for any coefficient set $\mathbb{K}$.

Theorem 2.12. *Given a coefficient vector (A,B,C), and a point (x_0,y_0) not on the line $[A:B:C]$, there is a line ℓ such that (x_0,y_0) is on ℓ and $\ell\|[A:B:C]$.*

Proof. Let $\ell=[A:B:-Ax_0-By_0]$. It's easy to check (x_0,y_0) is on this line. Suppose there is some point (x,y) which is on both ℓ and $[A:B:C]$. Then $Ax+By+C=0$, and $Ax+By-Ax_0-By_0=0$. Solving the first equation gives $Ax+By=-C$, and plugging this into the second gives $-C-Ax_0-By_0=0$, which implies $Ax_0+By_0+C=0$, contradicting the hypothesis that (x_0,y_0) is not on the line $[A:B:C]$. ∎

In fact, by Corollary 1.8, incidence axioms $I-1,2,4$ imply there does exist such a line $[A:B:C]$, and a point (x_0,y_0) not on it, so Theorem 2.12 implies that the Elliptic parallel property does not hold.

The following Theorem requires dividing again, so it won't apply, for example, to the integer grid.

Theorem 2.13. *Suppose $\mathbb{K}$ is a field, and (A, B, C) and (D, E, F) are coefficient vectors such that $[A : B : C]$ and $[D : E : F]$ are parallel. Then $A \cdot E - B \cdot D = 0$.*

Proof. Suppose, toward a contradiction, that $A \cdot E - B \cdot D \neq 0$, so then we can divide by this number. It's easy to check that the point

$$(x, y) = \left(\frac{BF - CE}{AE - BD}, \frac{CD - AF}{AE - BD} \right)$$

is a point on both lines, contradicting the parallel hypothesis. ∎

In precalculus terms, $AE - BD = 0$ means the lines have the same slope! For example, if B and E are non-zero, then $AE = BD \implies A/B = D/E$. Geometrically, we can use this Theorem to show that the coordinate planes with coefficients in a field have the Euclidean parallel property. We've already shown there exists a parallel through any point not on a given line, so the next Theorem will prove uniqueness.

Theorem 2.14. *Suppose $\mathbb{K}$ is a field, and there is a coefficient vector (A, B, C), and a point (x_0, y_0) not on the line $[A : B : C]$. If ℓ and m are two lines parallel to $[A : B : C]$, with (x_0, y_0) on both lines, then $\ell = m$.*

Proof. We already know from Theorem 2.12 that $[A : B : -Ax_0 - By_0]$ is parallel to $[A : B : C]$ going through (x_0, y_0). Let ℓ have a coefficient vector (D, E, F), so by Theorem 2.13, $AE - BD = 0$. I want to show that $(D, E, F) \sim (A, B, -Ax_0 - By_0)$. If I can show this, then a similar argument will show that the coefficient vector for m is also equivalent to $(A, B, -Ax_0 - By_0)$, and by the transitivity from Theorem 2.2, ℓ and m will have equivalent coefficient vectors, so they'll be the same line.

Case 1. $B \neq 0$ and $E \neq 0$. Let $k = B$ and $j = E$, so that $k \cdot (D, E, F) = (BD, BE, BF)$, and using $AE - BD = 0$, plug in $BD = AE$ to get $k \cdot (D, E, F) = (AE, BE, BF)$. Then, $j \cdot (A, B, -Ax_0 - By_0) = (EA, EB, -EAx_0 - EBy_0)$, and using the fact that (x_0, y_0) is on ℓ, $Dx_0 + Ey_0 + F = 0$, so $0 = BDx_0 + BEy_0 + BF = AEx_0 + BEy_0 + BF$, which proves $BF = -AEx_0 - BEy_0$. This shows (D, E, F) and $(A, B, -Ax_0 - By_0)$ are equivalent.

Case 2. $E = 0$. Then $AE - BD = 0$ implies $BD = 0$. D can't be zero, so B must be 0 by $**$, which is the next case.

Case 3. $B = 0$. An argument similar to Case 2 implies E must also be 0. We want to show $(D, 0, F) \sim (A, 0, -Ax_0)$. We know $Dx_0 + 0y_0 + F = 0$, so $F = -Dx_0$. Let $k = A$ (which is non-zero because B is zero) and $j = D$ (similarly non-zero), so $k \cdot (D, 0, F) = (AD, 0, A(-Dx_0))$, and $j \cdot (A, 0, -Ax_0) = (DA, 0, D(-Ax_0))$. These are the same, which proves the equivalence we wanted. ∎

3 Incidence-preserving maps

Let $\mathcal{M}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathrm{I}_1)$ be a model of geometry: we will call $\mathcal{P}_1$ a set of points, and $\mathcal{L}_1$ a set of lines, and there is an incidence relation $\mathrm{I}_1 \subseteq \mathcal{P}_1 \times \mathcal{L}_1$ where for $P \in \mathcal{P}_1$, $\ell \in \mathcal{L}_1$, $(P, \ell) \in \mathrm{I}_1$ will be denoted $P\mathrm{I}_1\ell$. As in Section 1, $P\mathrm{I}_1\ell$ be said in different ways: "the point P is on (or incident with) the line ℓ," etc. At this point we are not assuming $\mathcal{M}_1$ satisfies any of the incidence axioms. Similarly, let $\mathcal{M}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathrm{I}_2)$ be another model.

Definition 3.1. Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, a function $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$, and a function $\beta : \mathcal{L}_1 \to \mathcal{L}_2$, the pair (α, β) is an isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ means: α is invertible, β is invertible, and for all $P \in \mathcal{P}_1$, $\ell \in \mathcal{L}_1$,

$$P\mathrm{I}_1\ell \implies \alpha(P)\mathrm{I}_2\beta(\ell), \tag{3.1}$$
$$\alpha(P)\mathrm{I}_2\beta(\ell) \implies P\mathrm{I}_1\ell. \tag{3.2}$$

The above Definition, written as an "if and only if" statement, appears in [G3] p. 56 and [G4] p. 79 — see also Example 1, [G3] p. 310 and [G4] p. 398. Here we will consider property (3.1) and its converse (3.2) separately; this approach is not taken by [G3] or [G4]. Definition 3.2 states a notion of an incidence-preserving map from a domain model $\mathcal{M}_1$ to a target model $\mathcal{M}_2$, where neither α nor β are necessarily invertible, but they do preserve the "on" relation in the sense of (3.1):

Definition 3.2. Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, any function $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$, and any function $\beta : \mathcal{L}_1 \to \mathcal{L}_2$, the pair (α, β) is an on-preserving map from $\mathcal{M}_1$ to $\mathcal{M}_2$ means: For all $P \in \mathcal{P}_1$, $\ell \in \mathcal{L}_1$,

$$P\mathrm{I}_1\ell \implies \alpha(P)\mathrm{I}_2\beta(\ell).$$

The implication stated by (3.2) is equivalent to its logical contrapositive: if P is not on ℓ, then the image point $\alpha(P)$ is not on the image line $\beta(\ell)$. This motivates the following terminology:

Definition 3.3. Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, any function $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$, and any function $\beta : \mathcal{L}_1 \to \mathcal{L}_2$, the pair (α, β) is an off-preserving map from $\mathcal{M}_1$ to $\mathcal{M}_2$ means: For all $P \in \mathcal{P}_1$, $\ell \in \mathcal{L}_1$,

$$\alpha(P)\mathrm{I}_2\beta(\ell) \implies P\mathrm{I}_1\ell.$$

Lemma 3.4. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, and invertible functions $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$, $\beta : \mathcal{L}_1 \to \mathcal{L}_2$, the following are equivalent:*

1. *(α, β) is an isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$;*
2. *(α, β) is both on-preserving and off-preserving;*
3. *(α, β) and $(\alpha^{-1}, \beta^{-1})$ are both on-preserving;*
4. *$(\alpha^{-1}, \beta^{-1})$ is an isomorphism from $\mathcal{M}_2$ to $\mathcal{M}_1$.*

Proof. The equivalence of the first two statements is just Definition 3.1, where for invertible α and β, an isomorphism is defined by being both on-preserving (property (3.1)) and off-preserving (property (3.2)).

We will next show that the third statement implies the second. To show that the off-preserving property holds, given $P \in \mathcal{P}_1$, $\ell \in \mathcal{L}_1$, use the assumption that $(\alpha^{-1}, \beta^{-1})$ is an on-preserving map: $\alpha(P)\mathrm{I}_2\beta(\ell) \implies \alpha^{-1}(\alpha(P))\mathrm{I}_1\beta^{-1}(\beta(\ell)) \implies P\mathrm{I}_1\ell$.

Conversely, to show the second statement implies the third, assume that (α, β) is off-preserving. To show that $(\alpha^{-1}, \beta^{-1})$ satisfies Definition 3.2, given $Q \in \mathcal{P}_2$, $m \in \mathcal{L}_2$, let $P = \alpha^{-1}(Q)$, $\ell = \beta^{-1}(m)$. Then, by the off-preserving property, $Q\mathrm{I}_2m \implies \alpha(P)\mathrm{I}_2\beta(\ell) \implies P\mathrm{I}_1\ell \implies \alpha^{-1}(Q)\mathrm{I}_1\beta^{-1}(m)$; this shows $(\alpha^{-1}, \beta^{-1})$ is an on-preserving map.

Finally, the third statement is equivalent to $(\alpha^{-1}, \beta^{-1})$ and $((\alpha^{-1})^{-1}, (\beta^{-1})^{-1})$ both being on-preserving maps (because $(\alpha^{-1})^{-1} = \alpha$ and $(\beta^{-1})^{-1} = \beta$). The above argument that the first and third statements are equivalent applies to $(\alpha^{-1}, \beta^{-1})$, so the third statement is equivalent to the fourth statement. ∎

The above Lemma didn't use any incidence axioms; the Proof is just a re-labeling argument to show how Definitions 3.1, 3.2, and 3.3 are logically related when α and β are both invertible. The remaining Lemmas and Theorems will assume $\mathcal{M}_1$ and $\mathcal{M}_2$ satisfy some of the incidence axioms.

Lemma 3.5. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose that $\mathcal{M}_1$ satisfies incidence axiom $I-2-i$, and $\mathcal{M}_2$ satisfies incidence axioms $I-2-ii$ and $I-3$. If (α, β) is an on-preserving map from $\mathcal{M}_1$ to $\mathcal{M}_2$, and α is onto, then β is also onto.*

Proof. Given any line $m \in \mathcal{L}_2$, we want to show there exists $\ell \in \mathcal{L}_1$ such that $\beta(\ell) = m$. By the $I-3$ property of $\mathcal{M}_2$, there exist $P, Q \in \mathcal{P}_2$ with $P \neq Q$, $P\mathrm{I}_2m$, and $Q\mathrm{I}_2m$. Because α is onto, there exist $R, S \in \mathcal{P}_1$ with $\alpha(R) = P$ and $\alpha(S) = Q$. R and S are distinct (otherwise $\alpha(R) = \alpha(S) = P = Q$), so by the $I-2-i$ property of $\mathcal{M}_1$, there exists a line $\ell \in \mathcal{L}_1$ such that $R\mathrm{I}_1\ell$ and $S\mathrm{I}_1\ell$. By the on-preserving property, $\alpha(R)\mathrm{I}_2\beta(\ell)$ and $\alpha(S)\mathrm{I}_2\beta(\ell)$, so $P\mathrm{I}_2\beta(\ell)$ and $Q\mathrm{I}_2\beta(\ell)$. By the $I-2-ii$ property of $\mathcal{M}_2$, $\beta(\ell) = m$. ∎

Lemma 3.6. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose that $\mathcal{M}_1$ satisfies incidence axioms $I-2-i$, $I-2-ii$, and $I-4$, and $\mathcal{M}_2$ satisfies incidence axiom $I-2-ii$. If (α, β) is an on-preserving map from $\mathcal{M}_1$ to $\mathcal{M}_2$, and β is one-to-one, then α is either one-to-one or constant.*

Proof. Suppose, toward a contradiction, that α is neither one-to-one nor constant. Not having the one-to-one property means that there are points $P, R \in \mathcal{P}_1$ such that $P \neq R$ and $\alpha(P) = \alpha(R) \in \mathcal{P}_2$. By the $I-2-i$ property of $\mathcal{M}_1$, there exists a line $\ell \in \mathcal{L}_1$ such that $P\mathrm{I}_1\ell$ and $R\mathrm{I}_1\ell$.

Claim: If Q is any point in $\mathcal{P}_1$ such that $\alpha(Q) \neq \alpha(P)$, then $Q\mathrm{I}_1\ell$.

To prove the claim, first note that $\alpha(Q) \neq \alpha(P)$ implies $Q \neq P$ and $Q \neq R$. By the $I-2-i$ property of $\mathcal{M}_1$, the following lines exist:

There exists a line $j \in \mathcal{L}_1$ such that $P\mathrm{I}_1j$ and $Q\mathrm{I}_1j$.

There exists a line $k \in \mathcal{L}_1$ such that $R\mathrm{I}_1k$ and $Q\mathrm{I}_1k$.

By the on-preserving property, $\alpha(P)\mathrm{I}_2\beta(j)$, $\alpha(Q)\mathrm{I}_2\beta(j)$, $\alpha(Q)\mathrm{I}_2\beta(k)$, and $\alpha(R)\mathrm{I}_2\beta(k)$. From $\alpha(P) = \alpha(R)$, the last incidence implies $\alpha(P)\mathrm{I}_2\beta(k)$.

The distinct points $\alpha(Q) \neq \alpha(P)$ are on the line $\beta(j)$ and on the line $\beta(k)$, so $\beta(j) = \beta(k)$ by the $I-2-ii$ property of $\mathcal{M}_2$. Then $j = k$ because β is one-to-one. Now $R\mathrm{I}_1j$, so P and R are distinct points on both ℓ and j, and $\ell = j$ by the $I-2-ii$ property of $\mathcal{M}_1$. To summarize, the distinct points P, R, Q are all on the same line $\ell = j = k$, and this proves the Claim.

Now, α not being a constant map means there exists some point $S \in \mathcal{P}_1$ such that $\alpha(S) \neq \alpha(P)$. The Claim applies (with $Q = S$) to show that $S\mathrm{I}_1\ell$.

By the $I-4$ property of $\mathcal{M}_1$, there is a point $T \in \mathcal{P}_1$ so that T is not on ℓ. If $\alpha(T) \neq \alpha(P)$, then the Claim applies (with $Q = T$) to show $T\mathrm{I}_1\ell$, a contradiction. So the only remaining case is that $\alpha(T) = \alpha(P)$.

Because $S\mathrm{I}_1\ell$, $T \neq S$. By the $I-2-i$ property of $\mathcal{M}_1$, there exists a line $m \in \mathcal{L}_1$ such that $T\mathrm{I}_1 m$ and $S\mathrm{I}_1 m$. By the on-preserving property, $\alpha(T)\mathrm{I}_2\beta(m)$ and $\alpha(S)\mathrm{I}_2\beta(m)$. Recalling that $\alpha(S)$ and $\alpha(P) = \alpha(T)$ are on $\beta(\ell)$ and that $\alpha(S) \neq \alpha(P)$, the distinct points $\alpha(S)$ and $\alpha(T)$ are both on the line $\beta(m)$ and on the line $\beta(\ell)$, so $\beta(m) = \beta(\ell)$, by the $I-2-ii$ property of $\mathcal{M}_2$. Then $\ell = m$ because β is one-to-one, but this contradicts T not being on ℓ.

The conclusion is that all cases lead to a contradiction, so α must be either one-to-one or constant. ∎

Lemma 3.7. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose that $\mathcal{M}_1$ satisfies incidence axioms $I-2-i$ and $I-3$, and $\mathcal{M}_2$ satisfies incidence axiom $I-2-ii$. If (α, β) is an on-preserving map from $\mathcal{M}_1$ to $\mathcal{M}_2$, and α is one-to-one and β is one-to-one, then (α, β) is off-preserving.*

Proof. Suppose toward a contradiction that there is some $P \in \mathcal{P}_1$ and $\ell \in \mathcal{L}_1$ with $\alpha(P)\mathrm{I}_2\beta(\ell)$ but not $P\mathrm{I}_1\ell$. By property $I-3$ of $\mathcal{M}_1$, there is some point $Q \in \mathcal{P}_1$ with $Q\mathrm{I}_1\ell$. It follows that $Q \neq P$, so by property $I-2-i$ of $\mathcal{M}_1$, there is some line $m \in \mathcal{L}_1$ with $P\mathrm{I}_1 m$ and $Q\mathrm{I}_1 m$. It follows from P not being on ℓ that $\ell \neq m$.

By the on-preserving hypothesis, $\alpha(Q)\mathrm{I}_2\beta(\ell)$ and $\alpha(Q)\mathrm{I}_2\beta(m)$ and $\alpha(P)\mathrm{I}_2\beta(m)$. Because α is one-to-one, $\alpha(P) \neq \alpha(Q)$, and these distinct points are on both lines $\beta(\ell)$ and $\beta(m)$, so by the $I-2-ii$ property of $\mathcal{M}_2$, $\beta(\ell) = \beta(m)$. This contradicts the earlier observation that $\ell \neq m$ and the hypothesis that β is one-to-one. ∎

Theorem 3.8. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose that $\mathcal{M}_1$ satisfies incidence axioms $I-2-i$, $I-2-ii$, $I-3$, and $I-4$, and $\mathcal{M}_2$ satisfies incidence axioms $I-1$, $I-2-ii$, and $I-3$. For $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ and $\beta : \mathcal{L}_1 \to \mathcal{L}_2$, the following are equivalent:*

1. *α is onto, β is one-to-one, and (α, β) is an on-preserving map;*
2. *(α, β) is an isomorphism.*

Proof. As in Lemma 3.4, every isomorphism is an on-preserving map, so we only need to show the converse and assume α is onto, β is one-to-one, and (α, β) is an on-preserving map.

The onto property of α implies the onto property of β by Lemma 3.5, so the one-to-one function β is invertible. Also because β is one-to-one, Lemma 3.6 applies: α is either one-to-one or constant. The $I-1$ property for $\mathcal{M}_2$ is that $\mathcal{P}_2$ has at least two elements, and the onto property of α means that α cannot be constant, so α is one-to-one, and invertible. Lemma 3.7 gives the off-preserving property, so (α, β) is an isomorphism. ∎

Lemma 3.9. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose that $\mathcal{M}_1$ satisfies incidence axiom $I-2-ii$, and $\mathcal{M}_2$ satisfies incidence axiom $I-3$. If (α, β) is an off-preserving map from $\mathcal{M}_1$ to $\mathcal{M}_2$, and α is onto, then β is one-to-one.*

Proof. For $\ell, m \in \mathcal{L}_1$ with $\beta(\ell) = \beta(m)$, by the $I-3$ property of $\mathcal{M}_2$, there exist $P, Q \in \mathcal{P}_2$ with $P \neq Q$. $P\mathrm{I}_2\beta(\ell)$, and $P\mathrm{I}_2\beta(m)$. Because α is onto, there exist $R, S \in \mathcal{P}_1$ with $\alpha(R) = P$ and $\alpha(S) = Q$, so $R \neq S$. By the off-preserving property, $\alpha(R)\mathrm{I}_2\beta(\ell) \implies R\mathrm{I}_1\ell$, $\alpha(R)\mathrm{I}_2\beta(m) \implies R\mathrm{I}_1 m$, $\alpha(S)\mathrm{I}_2\beta(\ell) \implies S\mathrm{I}_1\ell$, and $\alpha(S)\mathrm{I}_2\beta(m) \implies S\mathrm{I}_1 m$. By the $I-2-ii$ property of $\mathcal{M}_1$, $\ell = m$, which shows β is one-to-one. ∎

Lemma 3.10. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose that $\mathcal{M}_1$ satisfies incidence axiom $I-2-ii$, and $\mathcal{M}_2$ satisfies incidence axioms $I-2-i$ and $I-3$. If $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ is onto and $\beta : \mathcal{L}_1 \to \mathcal{L}_2$ is onto, and (α, β) is off-preserving, then (α, β) is on-preserving.*

Proof. We want to show that for any $P \in \mathcal{P}_1$ and $\ell \in \mathcal{L}_1$, if $P\mathrm{I}_1\ell$, then $\alpha(P)\mathrm{I}_2\beta(\ell)$, so assume $P\mathrm{I}_1\ell$. By the $I-3$ property of $\mathcal{M}_2$, there is some point $Q \in \mathcal{P}_2$ so that $Q\mathrm{I}_2\beta(\ell)$, and by the onto property of α, there is some $R \in \mathcal{P}_1$ with $\alpha(R) = Q$. By hypothesis, $Q\mathrm{I}_2\beta(\ell) \implies \alpha(R)\mathrm{I}_2\beta(\ell) \implies R\mathrm{I}_1\ell$. If $\alpha(R) = \alpha(P)$, then we're done: $\alpha(P) = Q \implies \alpha(P)\mathrm{I}_2\beta(\ell)$. Otherwise, for $\alpha(R) \neq \alpha(P)$, by the $I-2-i$ property of $\mathcal{M}_2$, there is some line $m \in \mathcal{L}_2$ so that $\alpha(R)\mathrm{I}_2 m$ and $\alpha(P)\mathrm{I}_2 m$. From the onto property of β, there is some line $n \in \mathcal{L}_1$ with $\beta(n) = m$. By the off-preserving hypothesis, $\alpha(R)\mathrm{I}_2\beta(n) \implies R\mathrm{I}_1 n$ and $\alpha(P)\mathrm{I}_2\beta(n) \implies P\mathrm{I}_1 n$. From $\alpha(R) \neq \alpha(P)$, $P \neq R$, so by the $I-2-ii$ property of $\mathcal{M}_1$, $\ell = n$. It follows that $\beta(\ell) = \beta(n) = m$, and from $\alpha(P)\mathrm{I}_2 m$ that $\alpha(P)\mathrm{I}_2\beta(\ell)$. ∎

Theorem 3.11. *Given models $\mathcal{M}_1$ and $\mathcal{M}_2$, suppose that $\mathcal{M}_1$ satisfies incidence axiom $I-2-ii$, and $\mathcal{M}_2$ satisfies incidence axioms $I-2-i$, and $I-3$. For $\alpha : \mathcal{P}_1 \to \mathcal{P}_2$ and $\beta : \mathcal{L}_1 \to \mathcal{L}_2$, the following are equivalent:*

1. *α is invertible, β is onto, and (α, β) is an off-preserving map;*
2. *(α, β) is an isomorphism.*

Proof. As in Lemma 3.4, every isomorphism is an off-preserving map, so we only need to show the converse and assume α is invertible, β is onto, and (α, β) is an off-preserving map.

The onto property of α implies the one-to-one property of β by Lemma 3.9, so the onto function β is invertible. Also because β is onto, Lemma 3.10 gives the on-preserving property, so (α, β) is an isomorphism. ∎

4 Order geometry

Undefined Terms: "Point." "Line." "On." "Between."

We will continue to use the terminology from Incidence geometry, and we will assume all of the I axioms, but we won't need all of them for every proof. We will not assume any of the "parallel properties" unless otherwise mentioned.

Just as "on" is something that a "point" can do to a "line," the term "between" is something that a "point" B can do to an ordered pair of points, (A, C). We'll say "B is between A and C." It is convenient to abbreviate this as $A * B * C$.

Betweenness Axioms. Here are some properties that a model of points and lines (and "on" and "$*$") could have. The phrase "at least" in these sentences is just for emphasis. They would mean the same thing with "at least" deleted. The symbol $P \implies Q$ abbreviates the sentence "if P, then Q."

- **Axiom $B-1-i$. $A * B * C \implies A \neq C$.**
- **Axiom $B-1-ii$. $A * B * C \implies \{A, B, C\}$ is a collinear set.**
- **Axiom $B-1-iii$. $A * B * C \implies C * B * A$.**
- **Axiom $B-2$. If B and D are distinct points, then there is at least one point A so that $A * B * D$.**
- **Axiom $B-3-i$. If A, B, C are distinct, collinear points, then at least one of the following is true: $A * B * C$, or $A * C * B$, or $B * A * C$.**
- **Axiom $B-3-ii$. If B is between A and C, then C is not between A and B.** abbreviation: $A * B * C \implies (A * C * B$ is false).

Theorem 4.1. *$A * B * C \implies A, B, C$ are distinct.*

Proof. By $B-1-i$, $A \neq C$. If $B = C$, then $A * B * C \implies A * C * B$. However, $A * C * B$ is false by $B-3-ii$, so $B \neq C$. Using $B-1-iii$, $A * B * C \implies C * B * A$. If $B = A$, then $C * B * A \implies C * A * B$, but again, $B-3-ii$ says that $C * B * A \implies (C * A * B$ is false), so $B \neq A$. ∎

Theorem 4.2. *If $A * B * C$, then the following statements are all false: $A * C * B$, $B * C * A$, $C * A * B$, $B * A * C$.*

Proof. $A * C * B$ is false, by $B-3-ii$. $B-1-iii$ says $B * C * A \implies A * C * B$, so $B * C * A$ must also be false. Also by $B-1-iii$, $C * B * A$ is true, and then $B-3-ii$ says that $C * A * B$ is false, and since $B * A * C \implies C * A * B$ ($B-1-iii$ again), $B * A * C$ is also false. ∎

Theorem 4.3. *Given points P, Q, if $P \neq Q$, then there exists a line ℓ such that P is on ℓ and Q is on ℓ.*

Proof. Since $P \neq Q$, $B-2$ says there's some point A so that $A * P * Q$. By $B-1-ii$, $\{A, P, Q\}$ is a collinear set, which means there's a line ℓ so that A, P and Q are all on ℓ. ∎

The previous Theorem is exactly the statement of Axiom $I-2-i$! In fact, if we assume the betweenness axioms, then $I-2-i$ follows as a consequence, so we could call it a Theorem. If we have a model where we have checked the betweenness properties, then that model must also have the $I-2-i$ property. Another way to think about it is that if we have a model which does not satisfy $I-2-i$, then there is no way to define a "betweenness" relation which satisfies the $B-1$ and $B-2$ axioms.

Theorems 4.1, 4.2, 4.3 didn't use any I axioms. However, Theorem 4.3, and axioms $B-1-ii$ and $B-3-i$, do require the undefined terms "line" and "on" (since they refer to "collinear," Definition 1.4).

Lemma 4.4. *Given points A, B, P, and a line ℓ, if A is not on ℓ, and P is on ℓ, and $A*P*B$, then B is not on ℓ.*

Proof. By $B-1-ii$, $\{A,P,B\}$ is a collinear set. By Theorem 4.1, $A*P*B \implies P \neq B$. By Theorem 1.9, P is the only element of $\{A,P,B\}$ on ℓ, so B is not on ℓ. ∎

Lemma 4.5. *If $A*B*C$ and $A*C*D$, then A, B, C, D are distinct.*

Remark. This is Betweenness Exercise 3.1.a of [G3] and [G4]. ∎

Lemma 4.6. *If $A*B*C$ and $A*C*D$, then $\{A,B,C,D\}$ is a collinear set.*

Remark. This is Betweenness Exercise 3.1.b of [G3] and [G4]. ∎

Exercise 4.7. Are the B axioms independent? Try, as in Exercise 1.3, constructing models which satisfy all the incidence axioms, and where "between" means something other than the "Euclidean between," satisfying all but one of the six B properties.

With the new undefined term "between," we can state definitions of "segment," "ray," "opposite ray," "angle," and "triangle." Some informal definitions appear in the book's Chapter 1 (p. $14-17$ of [G3], p. $16-19$ of [G4]) and the Chapter 1 Exercises.

Definition 4.8. For distinct points A and B, the <u>segment</u> AB is the set $\{A\}\cup\{B\}\cup\{X : A*X*B\}$.

Theorem 4.9. *Given points A, B, if $A \neq B$, then $AB = BA$.*

Proof. The segment BA is defined as the set $\{B\}\cup\{A\}\cup\{X : B*X*A\}$. To prove equality of sets, we need to show $BA \subseteq AB$ and $AB \subseteq BA$. If P is an element of BA, then there are exactly three possibilities: $P = B \in AB$, or $P = A \in AB$, or $B*P*A$. In this last case, Axiom $B-1-iii$ gives $A*P*B$, so $P \in AB$. Since in each case, $P \in AB$, we can conclude $BA \subseteq AB$. The proof for the other subset relation is similar. ∎

Definition 4.10. For distinct points A and B, the <u>ray</u> $\overrightarrow{AB}$ is the set $\{A\}\cup\{B\}\cup\{X : A*X*B\}\cup\{X : A*B*X\}$.

The order of the two points is important (see Theorem 4.13). The point A in the above definition is called the <u>endpoint</u> of the ray. Note that "$A \neq B$" is part of the definition of segment AB and ray $\overrightarrow{AB}$. (What would happen if we allowed $A = B$? Then, $AB = \overrightarrow{AB} = \{A\}$, which is not terribly bad, just inconvenient.)

Theorem 4.11. $C * A * B \implies C \notin \overrightarrow{AB}$.

Proof. By Theorems 4.1 and 4.2, $C * A * B$ implies $C \neq A$, $C \neq B$, and also $A * C * B$ and $A * B * C$ are false, so $C \notin \overrightarrow{AB}$. ∎

It also follows immediately that if $C * A * B$, then $C \notin AB$.

Theorem 4.12. *Given points A, B, C, if $A \neq B$, and $\{A, B, C\}$ is a collinear set, and $C \notin \overrightarrow{AB}$, then $C * A * B$.*

Proof. This is a converse to the previous Theorem. Since A and B are elements of $\overrightarrow{AB}$, and C is not, A, B, and C are distinct. They are also collinear by hypothesis, so by axiom $B-3-i$, one of these must hold: $A*B*C$, $A*C*B$, or $B*A*C$. Since the first two possibilities imply $C \in \overrightarrow{AB}$, $B * A * C$ must be true, and $C * A * B$ follows from $B - 1 - iii$. ∎

Theorem 4.13. *Given points A, B, if $A \neq B$, then $\overrightarrow{AB} \neq \overrightarrow{BA}$.*

Proof. To show two sets are not equal, you need to find an element of one that's not an element of the other one. Axiom $B-2$ says that $A \neq B$ implies there exists a point C so that $C*A*B$, and so $B * A * C$ by $B - 1 - iii$. $C \in \overrightarrow{BA}$ by definition of ray $\overrightarrow{BA}$, and $C \notin \overrightarrow{AB}$ by Theorem 4.11. ∎

Theorem 4.14. *Given points A, B, if $A \neq B$, then $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$.*

Proof. This is Proposition 3.1.(i) of [G3] and [G4]. $A \neq B$ is required by the definitions of ray and segment, from which follow $AB \subseteq \overrightarrow{AB}$, and $BA \subseteq \overrightarrow{BA}$. By Theorem 4.9, $AB \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}$.

To show the other subset relation, suppose $C \in \overrightarrow{AB}$ and $C \in \overrightarrow{BA}$. If $C = A$ or $C = B$, then $C \in AB$. Otherwise, A, B, C are distinct, and collinear by $B - 1 - ii$ and the definition of ray. So, by $B - 3 - i$, there are three cases. If $A * B * C$, then $C * B * A$ by $B - 1 - iii$, so $C \notin \overrightarrow{BA}$ by Theorem 4.11. Similarly, if $B * A * C$, then $C * A * B \implies C \notin \overrightarrow{AB}$. The only remaining case is $A * C * B$, so $C \in AB$. ∎

The next Theorem refers to the set $\overleftrightarrow{AB}$ from Definition 1.21.

Theorem 4.15. *Given points A and B such that $A \neq B$, let*

$$U = \{A\} \cup \{B\} \cup \{X : A * B * X\} \cup \{X : A * X * B\} \cup \{X : X * A * B\}.$$

Then, $\overleftrightarrow{AB} = U$, and this expression for U is a disjoint union.

Proof. First, show $U \subseteq \overleftrightarrow{AB}$. By Theorem 1.22 (which used Axiom $I - 2 - i$, or now, Theorem 4.3), $\{A\} \cup \{B\} \subseteq \overleftrightarrow{AB}$. The other elements X of U satisfy $A*B*X$, or $A*X*B$, or $X*A*B$, and by $B - 1 - ii$, $\{A, B, X\} = \{A, X, B\} = \{X, A, B\}$ are collinear, so $X \in \overleftrightarrow{AB}$, and this gives $U \subseteq \overleftrightarrow{AB}$. Second, show the other subset. If $X \in \overleftrightarrow{AB}$, then by definition, $\{A, B, X\}$ is collinear. If the three points A, B, X are distinct, then by $B-3-i$, at least one of $A*X*B$, or $A*B*X$, or $X * B * A$ is true, so $X \in U$. If the points are not distinct, then $X = A$ or $X = B$ ($A = B$ is already ruled out by hypothesis), so $X \in U$. This shows $\overleftrightarrow{AB} \subseteq U$, and the sets are equal. The disjointness follows from Theorems 4.1 and 4.2. ∎

Corollary 4.16. *Given points A, B, if $A \neq B$, then $AB \subsetneq \overrightarrow{AB} \subsetneq \overleftrightarrow{AB}$. The unions which appeared in the definitions of segment and ray are also disjoint unions. Each of the sets AB and $\overrightarrow{AB}$ is a collinear set, and if there is a line ℓ so that A and B are on ℓ, then every point of AB and $\overrightarrow{AB}$ is on ℓ.*

Proof. The $\subsetneq$ symbol means "...is a subset of, but not equal to...". The relation $AB \subseteq \overrightarrow{AB}$ follows immediately from the definitions of ray and segment. The relation $\overrightarrow{AB} \subseteq U$ follows from the definition of ray and the formula for U. The inequality of sets follows from the fact that $\{X : A * B * X\}$ and $\{X : X * A * B\}$ are both non-empty, by $B-2$ and $B-1-iii$. The claims of the last sentence follow from Theorem 1.22. ∎

Theorem 4.15 is also related to Proposition 3.1.(ii) of [G3] and [G4], and Betweenness Exercise 3.2.a. of [G3] and [G4].

Definition 4.17. Two rays R_1 and R_2 are <u>opposite</u> means: they have the same endpoint, and $R_1 \cup R_2$ is a collinear set, and $R_1 \neq R_2$.

This is a re-statement of the definition from [G3] p. 16, [G4] p. 18. It is not exactly the definition stated on [G3] p. 75, [G4] p. 104, but as the book points out, the statements are related.

Theorem 4.18. $A * B * C \implies \overrightarrow{BA}$ *and* $\overrightarrow{BC}$ *are opposite rays.*

Proof. A, B and C are distinct by Theorem 4.1, so the rays are well-defined. The above Definition holds, since the rays have the same endpoint B, and they are distinct (for example, $A \in \overrightarrow{BA}$, and $A \notin \overrightarrow{BC}$ by Theorem 4.11), and it remains only to be checked that their union is collinear. By Corollary 4.16, $\overrightarrow{BA} \subseteq \overleftrightarrow{AB}$ and $\overrightarrow{BC} \subseteq \overleftrightarrow{BC}$. By $B-1-ii$, A, B, and C are collinear, so by Theorem 1.23, $\overleftrightarrow{AB} = \overleftrightarrow{BC}$. The set $\overrightarrow{BA} \cup \overrightarrow{BC} \subseteq \overleftrightarrow{AB}$ is collinear by Theorem 1.22. ∎

Corollary 4.19. *For any ray, there exists at least one opposite ray.*

Remark. This is part of Betweenness Exercise 3.5 of [G3] and [G4] . ∎

Definition 4.20. An <u>angle with vertex A</u> is a pair of rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, which are not equal, and not opposite.

The ordering of the pair of rays does not matter.

Notation 4.21. The angle formed by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ can be abbreviated $\angle BAC = \angle CAB$. (the common endpoint, A, goes in the middle of the three letters.) An even shorter (but ambiguous) abbreviation is $\angle A$.

Theorem 4.22. *Given points A, B, C, the following are equivalent:*

1. *$\{A, B, C\}$ is not a collinear set.*
2. *$\overrightarrow{AB}$ and $\overrightarrow{AC}$ form an angle.*
3. *$\overrightarrow{BC}$ and $\overrightarrow{BA}$ form an angle.*
4. *$\overrightarrow{CA}$ and $\overrightarrow{CB}$ form an angle.*

Proof. We'll show *1.* $\iff$ *2.*, the implications *1.* $\iff$ *3.* and *1.* $\iff$ *4.* being similar, and enough to establish the equivalence.

If $\{A, B, C\}$ is not a collinear set, then the points are distinct (Theorem 1.5), so $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are well-defined rays, with the same endpoint. If $\overrightarrow{AB} = \overrightarrow{AC}$, then $B \in \overrightarrow{AC} \subseteq \overleftrightarrow{AC}$ (by Corollary 4.16), which means A, B, and C are collinear, a contradiction. If $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays, then $\{A, B, C\}$ is a subset of the collinear set $\overrightarrow{AB} \cup \overrightarrow{AC}$, so A, B, C are collinear, again contradicting the hypothesis. So, the rays are distinct and not opposite, and they form an angle.

Conversely, let $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form an angle, so $A \neq B$ and $A \neq C$ by definition of ray, and by definition of angle, $\overrightarrow{AB} \neq \overrightarrow{AC}$, so $B \neq C$. Suppose, toward a contradiction, that $\{A, B, C\}$ is collinear. Then, by Corollary 4.16, $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$, and $\overrightarrow{AC} \subseteq \overleftrightarrow{AC} = \overleftrightarrow{AB}$ by Theorem 1.23. So, $\overrightarrow{AB} \cup \overrightarrow{AC} \subseteq \overleftrightarrow{AB}$, and $\overrightarrow{AB} \cup \overrightarrow{AC}$ is collinear, by Theorem 1.22. By the definition of "opposite ray," $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are either opposite or equal, both of which contradict the definition of angle. ∎

Notation 4.23. A set of three points $\{A, B, C\}$ which is not collinear can be called a triangle. The set will be labeled $\triangle ABC$, and the order of the points doesn't matter: $\triangle ABC = \triangle ACB = \triangle BAC = \triangle BCA = \triangle CAB = \triangle CBA$. Each of the points is a vertex. The three vertices are distinct by Theorem 1.5. Each of the line segments AB, BC, CA is called an edge of the triangle (or a side, but this is not the same as the "side" from Notation 5.1) . For each vertex, there is an angle formed by rays which have that vertex as endpoint, by Theorem 4.22. For example, at vertex A, the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form the angle $\angle A = \angle BAC$. (Sometimes, this is called an interior angle.) The edge BC is said to "subtend" angle $\angle BAC$ (or to be "opposite" angle $\angle A$, or to be "opposite" vertex A).

The edges of a triangle meet each other only at the vertices:

Lemma 4.24. *Given points A, B, C, if A, B, and C are not collinear, then $AB \cap BC = \{B\}$, and $BC \cap CA = \{C\}$, and $CA \cap AB = \{A\}$.*

Proof. As previously noted, the points are distinct, and the segments are well-defined. The relation $\{B\} \subseteq AB \cap BC$ follows immediately from Definition 4.8. To show the other subset relation, use Corollary 4.16 and Theorem 1.24: $AB \cap BC \subseteq \overleftrightarrow{AB} \cap \overleftrightarrow{BC} = \{B\}$. The other intersections are proved similarly. ∎

The following Definitions (4.25, 4.26) and Theorems (4.27, 4.28) are an elaboration on the Definition from [G3] p. 76 and [G4] p. 110.

Definition 4.25. For points A, B, and a line ℓ, the statement "A and B are on the same side of ℓ" means:

- $A = B$, and A is not on ℓ,
  OR
- $A \neq B$, and there is no point P such that P is an element of segment AB, and P is on ℓ.

Definition 4.26. For points A, B, and a line ℓ, the statement "A and B are on opposite sides of ℓ" means:

- A is not on ℓ, or B is not on ℓ,
  AND
- There is a point P so that $A * P * B$ and P is on ℓ.

It follows from Lemma 4.4 that if A and B are on opposite sides of ℓ, then both A and B are not on ℓ (so it would have been superfluous for the above Definition to state that both A and B are not on ℓ).

Theorem 4.27. *Given points A, B, and a line ℓ, if A and B are on opposite sides of ℓ, then A and B are not on the same side of ℓ.*

Proof. First, we mention what it means for "A and B are on the same side of ℓ" to be a false statement:

- $A = B$, and A is on ℓ
  OR
- $A \neq B$, and there is a point P such that $P \in AB$ and P is on ℓ.

Now, supposing A and B are on opposite sides of ℓ, there is a point P so that $A * P * B$ and P is on ℓ. By Axiom $B-1-i$, $A * P * B \implies A \neq B$, and by definition of segment, $P \in AB$. The conclusion follows immediately. ∎

The converse of Theorem 4.27 is false, since one of the points could be on the line, and then we would have a "neither same nor opposite" situation. However, if we exclude this case, then there is the following:

Theorem 4.28. *Given points A, B, and a line ℓ, if A is not on ℓ, and B is not on ℓ, and A and B are not on the same side of ℓ, then A and B are on opposite sides of ℓ.*

Proof. We need to check both parts of Definition 4.26. The first part is given as a hypothesis. The next step is to suppose, toward a contradiction, that $A = B$. Since A is not on ℓ by hypothesis, $A = B$ would put A and B on the same side, contradicting our other hypothesis. So, $A \neq B$, and the only way A and B could fail to satisfy the definition of "on the same side" is that there is some point $P \in AB$ so that P is on ℓ. P cannot be A or B since A and B are not on ℓ, so by definition of segment, $A * P * B$, which shows that the second part of the "opposite sides" definition holds. ∎

The following fact is going to be used several times, both in these Notes, and in the HW solution set. It almost looks like Definition 4.26, but is actually a theorem which will follow from that Definition.

Lemma 4.29 ("Lemma of the Line"). *Given points A, B, C, and a line ℓ, if A and B are on opposite sides of ℓ, and C is on ℓ, and A, B, and C are collinear, then $A * C * B$.*

Proof. By the second part of the definition of opposite sides, there's a point P on ℓ so that $A * P * B$, and by $B - 1 - i$, $A \neq B$. By $B - 1 - ii$, A, P, and B all lie on some line m, and by hypothesis, A, C, and B lie on some line n. The line n has two distinct points (A and B) in common with m, so $m = n$, by $I - 2 - ii$. So, P and C are both on m and on ℓ. If $P \neq C$, then $m = \ell$ by $I - 2 - ii$, but then A and B are on ℓ, contradicting the first part of Definition 4.26. Conclude: $P = C$, so $A * C * B$. ∎

Lemma 4.30. *Given points A, B, C, and a line ℓ, if $A \neq B$, and A is on ℓ, and C is not on ℓ, and $B \in \overrightarrow{AC}$, then B and C are on the same side of ℓ.*

Remark. This is Betweenness Exercise 3.9 of [G3] and [G4]. ∎

Theorem 4.31. *Given points B, C, D, and lines ℓ, m, if B is on ℓ, and C is on m, and B and D are on the same side of m, and C and D are on the same side of ℓ, and $\{B, C, D\}$ is collinear, then $B * D * C$.*

Proof. Step 1 is to show B, C, D are distinct. If $B = C$, then B and D are on the same side of ℓ, contradicting the hypothesis that B is on ℓ. By the definition of B and D on the same side of m, D is not on m, so $C \neq D$. By the definition of C and D on the same side of ℓ, D is not on ℓ, so $B \neq D$. Step 2 will show the conclusion follows from $B - 3 - i$, which says that since B, C, D are distinct and collinear, there are three cases. If $D * B * C$, then D and C are on opposite sides of ℓ. If $B * C * D$, then D and B are on opposite sides of m. However, by Theorem 4.27, "opposite sides" contradicts the "same side" hypotheses, so these two cases are ruled out. The only remaining case from $B - 3 - i$ is $B * D * C$. ∎

Notation 4.32. Let $\mathrm{C}(\ell)$ denote the "complement" of the line ℓ, defined by

$$\mathrm{C}(\ell) = \{P : P \text{ is a point not on } \ell\}.$$

$\mathrm{C}(\ell)$ is just the set of points not on ℓ (in Euclidean geometry, you should visualize the two open half planes), and of course, every line has its own complement, for example, $\mathrm{C}(m)$ is the set of points not on m. This isn't standard notation, and isn't in [G3] or [G4], but I couldn't think of anything better. By Axiom $I - 4$, there is at least one point not on ℓ, so $\mathrm{C}(\ell) \neq \emptyset$.

Theorem 4.33. *"Points A and B are on the same side of ℓ" is a reflexive and symmetric relation for elements $A, B \in \mathrm{C}(\ell)$.*

Proof. If $A \in \mathrm{C}(\ell)$, then A is not on ℓ, and the definition of "on the same side" says that A and A are on the same side of ℓ, which proves the reflexive property. If A and B are points in $\mathrm{C}(\ell)$ which are on the same side of ℓ, then either $A = B$ (in which case $B = A$ and B is not on ℓ, so B and A are on the same side of ℓ), or $A \neq B$, and there is no point $P \in AB$ with P on ℓ. By Theorem 4.9, $BA = AB$, so $B \neq A$, and there is no point $P \in BA$ with P on ℓ, and in this case also, B and A are on the same side of ℓ. ∎

Note that for points on the line (not in $\mathrm{C}(\ell)$), the "reflexive" property fails, so this is one reason to restrict our attention to $\mathrm{C}(\ell)$.

5 Order geometry with plane separation

The following "axiom" is called the "Plane Separation" property.

Axiom $B-4$. For every line ℓ, the "on the same side of ℓ" relation on the set $\mathrm{C}(\ell)$ is transitive.

So, for every line ℓ, "on the same side of ℓ" is an equivalence relation.

Notation 5.1. It follows from set theory that any equivalence relation on a set "partitions" that set into *disjoint*, *non-empty* equivalence classes. So, for each ℓ, the set $\mathrm{C}(\ell)$ is partitioned into disjoint equivalence classes, which we will call half planes. More specifically, for $A \in \mathrm{C}(\ell)$, the equivalence class of A is denoted

$$H(\ell, A) = \{X \in \mathrm{C}(\ell) : X \text{ and } A \text{ are on the same side of } \ell\},$$

and we will say it is the "half plane bounded by ℓ which contains A." The book ([G3] p. 77, [G4] p. 111) calls this set H_A (our notation includes the name of the line just to be more specific), and it could also be called the side of ℓ containing A (but this is not the same as the "side" from Notation 4.23).

Notice, for example, that the models $\mathbb{R}^3$ (points and lines in space) and $\mathbb{Z}^2$ (the integer grid with non-empty lines) satisfy all of the I axioms and the $B-1,2,3$ axioms, but do not satisfy $B-4$.

Theorem 5.2. *For every line ℓ, there are at least two distinct half planes bounded by ℓ.*

Proof. As mentioned earlier, by $I-4$, there is a point A not on ℓ, so there is at least one half plane $H(\ell, A)$. By $I-3$, there is some point O on ℓ (so $O \neq A$), and by $B-2$, there is some point B so that $B * O * A$. By $B-1-i$, $B \neq A$, and $O \in BA$ is a point on ℓ, so B and A are not on the same side. By Lemma 4.4, B is not on ℓ, so $H(\ell, B)$ is an equivalence class in $\mathrm{C}(\ell)$ distinct from $H(\ell, A)$. ∎

Recall that $H(\ell, B)$ and $H(\ell, A)$ aren't just different sets, they're actually disjoint! We just showed $B \notin H(\ell, A)$, which, using the transitivity axiom $B-4$, is enough to show that any point on the same side of ℓ as B is not on the same side of ℓ as A. This Theorem is part of Proposition 3.2 of [G3] and [G4], and the above Proof follows just lines (1)–(4) in the book's proof.

Theorem 5.3. *Given points A, B, C, and a line ℓ, if A and B are on opposite sides of ℓ, and B and C are on the same side of ℓ, then A and C are on opposite sides of ℓ.*

Remark. This is the Corollary to Axiom $B-4$ from [G3] p. 77 and [G4] p. 111, and Betweenness Exercise 3.1.c of [G3] and [G4]. ∎

Theorem 5.4 (Pasch 1)**.** *Given points A, B, C, D, and a line ℓ, if A, B, and C are distinct, and $D \in AB$, and D is on ℓ, then there is some point E so that $E \in AC \cup CB$ and E is on ℓ.*

Proof. If $D = A$ or $D = B$, let $E = D$, so E is on ℓ and $E \in AC \cup CB$, and we're done. Otherwise, $D \in AB \implies A * D * B$. If A is on ℓ, let $E = A$, and again we're done. If A is not on ℓ, then A and B are on opposite sides of ℓ (Definition 4.26) and B is not on ℓ (Lemma 4.4). If C is on ℓ, let $E = C$ (done!). Otherwise, neither B nor C is on ℓ, and there are exactly two cases.

i. If B and C are on the same side of ℓ, then by Theorem 5.3, A and C are on opposite sides of ℓ, meaning there is some E so that E is on ℓ and $A * E * C$, so $E \in AC \subseteq AC \cup CB$.

ii. If B and C are not on the same side of ℓ, then by Theorem 4.28, B and C are on opposite sides of ℓ, so there is some E so that E is on ℓ and $B * E * C$, so $E \in BC \subseteq AC \cup CB$. ∎

Under certain conditions, the point E is unique.

Theorem 5.5 (Pasch 2)**.** *Given $\triangle ABC$, a point D, and a line ℓ, if $A * D * B$, and D is on ℓ, and A and B are not both on ℓ, then there is exactly one point E so that $E \in AC \cup CB$ and E is on ℓ. If C is not on ℓ, then either $E \in AC$, or $E \in CB$, but not both.*

Proof. There exists at least one such point $E \in AC \cup BC$, by Theorem 5.4. By Definition 4.26, A and B are on opposite sides of ℓ, and by Lemma 4.4, both A and B are not on ℓ. Let c denote a line through A, B, and D ($B - 1 - ii$), so C is not on c (A, B, C not collinear by definition of $\triangle$).

Case 1. C is on ℓ. If $E \in AC$ and $E \neq C$, then $E \in \overrightarrow{CA}$ and E and A are on the same side of ℓ by Lemma 4.30, but this contradicts E is on ℓ. Similarly, if $E \in BC$ and $E \neq C$, then $E \in \overrightarrow{CB}$ and E and B are on the same side of ℓ, another contradiction. The only possibility is $E = C$, which proves uniqueness.

Case 2. C is not on ℓ. Then, $E \neq C$, and by Lemma 4.24, $AC \cap CB = \{C\}$, so $E \notin AC \cap CB$, which proves the last part of the Theorem.

Since E is on ℓ, A, B, C, E are distinct. E is on at least one of the segments CB or AC, so to prove uniqueness, there are two cases.

i. The first case is $E \in CB$. By Corollary 4.16, CB is a collinear set, all its points lying on some line a, with A not on a. By Theorem 1.9, since $C \in CB$ is not on ℓ, E is the only element of CB which is on ℓ.

Since $B \neq D$ (D is on ℓ, B is not), $D \in AB$ by definition of segment, and $CB \cap AB = \{B\}$ by Lemma 4.24 and Theorem 4.9, $D \notin CB$, so $D \neq E$.

Suppose, toward a contradiction, that there is some point $F \in AC$ such that F is on ℓ. It was already remarked that $E \notin CB \cap AC$, so $E \notin AC$, and $E \neq F$. Since $A \neq D$ (A is not on ℓ), $D \in AB$ by definition of segment, and $AC \cap AB = \{A\}$ by Lemma 4.24 and Theorem 4.9, $D \notin AC$, so $D \neq F$. This shows D, E, and F are distinct and collinear (all on ℓ).

Since $F \in AC$ and F is on ℓ and A and C are not on ℓ, $A \neq F$ and $C \neq F$, so $A * F * C$ by definition of segment, and there is some line b such that A, F, and C are on b, by $B - 1 - ii$. Since $\{A, B, C\}$ is not collinear, B is not on b. $A * D * B \implies D \in \overrightarrow{AB} \implies D$ and B are on the same side of b (Lemma 4.30), so D is not on b. Similarly, $E \in CB \implies E$ is not on b.

Applying $B - 3 - i$ to the distinct, collinear points D, E, F gives three cases.

a. If $D * F * E$, then Pasch 1 (Theorem 5.4) applies to the distinct points B, D, E, the point $F \in DE$, and the line b so that F is on b. There is some point G so that $G \in DB \cup BE$, and G

is on b. Since B, D, and E are not on b, either $D * G * B$, or $B * G * E$. If $D * G * B$, then G is on c (Corollary 4.16 applied to $G \in DB$). Also, $B * G * D$ and $B * D * A$ (by hypothesis and $B - 1 - iii$) implies $A \neq G$ (Lemma 4.5), so A and G are distinct points on b and c, so $b = c$ by $I - 2 - ii$, but this contradicts C not on c. Similarly, if $B * G * E$, then $B * E * C$ implies G and C are distinct points on b and a, contradicting A not on a.

The next two cases proceed similarly, and all three show that the supposed existence of F leads to a contradiction.

b. If $D * E * F$, then Pasch 1 applies to the distinct points A, D, F, the point $E \in DF$, and the line a so that B, C, and E are on a. There is some point H so that $H \in DA \cup AF$ and H is on a. Since A, D, and F are not on a (by a Lemma 4.30 argument as above), either $D * H * A$ or $A * H * F$. If $D * H * A$, then $A * D * B$ implies H and B are distinct points on a and c, contradicting A not on a. If $A * H * F$ and $A * F * C$, then C and H are distinct points on a and b, again contradicting A not on a.

c. If $E * D * F$, then Pasch 1 applies to the distinct points C, E, F, the point $D \in EF$, and the line c so that A, B, and D are on c. There is some point I so that $I \in EC \cup CF$ and I is on c. Since C, E, and F are not on c, either $E * I * C$ or $C * I * F$. If $E * I * C$, then $C * E * B$ implies I and B are distinct points on a and c, contradicting A not on a. If $C * I * F$ and $A * F * C$, then A and I are distinct points on b and c, contradicting C not on c.

ii. The second case is $E \in AC$, and an argument analogous to case i. shows there can be no point $F \in AC \cup CB$ such that F is on ℓ and $F \neq E$. ∎

The conclusions of Pasch 2 also hold when A, B, C are collinear and distinct — this is Theorem 5.17. Theorems 5.4, 5.5, and 5.17 together are called "Pasch's Theorem," a version of which is stated in the book, [G3] p. 80 and [G4] p. 114. The above Proof of Pasch 2 (using Pasch 1) is based on an argument appearing in [Golos], Theorem 5.22, p. 82.

Theorem 5.6. *Given points B, D, if $B \neq D$, then there exists a point C such that $B * C * D$.*

Remark. This is stated as the second part of Axiom $B - 2$ by the book, [G3] p. 74 and [G4] p. 108. It is also Betweenness Exercise 3.6 of [G3] and [G4]. ∎

Theorem 5.7. *If $A * B * C$ and $A * C * D$, then $B * C * D$.*

Proof. (This is part of Proposition 3.3 of [G3] and [G4].) By Lemma 4.5 and Lemma 4.6, A, B, C, D are distinct and collinear, lying on some line ℓ. By $I - 4$, there is a point E not on ℓ (so $E \neq C$). By $I - 2 - i$, there is a line m so that C and E are on m. A is not on m (if it were, then A and C would be distinct points on ℓ and m, and $\ell = m$, by $I - 2 - ii$, contradicts E not on ℓ). Similarly, B is not on m. By hypothesis, $A * C * D$, and this implies A and D are on opposite sides of m.

Now, we need to show A and B are on the same side of m. Supposing A and B are not on the same side of m, Theorem 4.28 says A and B are on opposite sides of m. Since $\{A, B, C\}$ is collinear and C is on m, $A * C * B$ by the Lemma of the Line (Lemma 4.29). However, the hypothesis $A * B * C$ and Axiom $B - 3 - ii$ imply $A * C * B$ is false, a contradiction.

Since A and B are on the same side of m, and A and D are on opposite sides of m, Theorem 5.3 (which used $B-4$) says that B and D are on opposite sides of m. Since $\{B, C, D\}$ is collinear, and C is on m, $B * C * D$ by the Lemma of the Line. ∎

Corollary 5.8. *If $A * B * C$ and $A * C * D$, then $A * B * D$.*

Remark. This is the rest of Proposition 3.3 of [G3] and [G4], and Betweenness Exercise 3.2.b of [G3] and [G4]. ∎

Corollary 5.9. *If $B * C * D$ and $A * B * D$, then $A * B * C$ and $A * C * D$.*

Remark. This is the converse to Proposition 3.3 of [G3] and [G4], and Betweenness Exercise 3.2.c of [G3] and [G4]. ∎

Theorem 5.10. *If $A * B * C$ and $B * C * D$, then $A * B * D$.*

Remark. This is part of the Corollary on p. 79 of [G3] and p. 113 of [G4], and part of Betweenness Exercise 3.2.d of [G3] and [G4]. ∎

Corollary 5.11. *If $A * B * C$ and $B * C * D$, then $A * C * D$.*

Remark. This is the rest of Corollary on p. 79 of [G3] and p. 113 of [G4], and the rest of Betweenness Exercise 3.2.d of [G3] and [G4]. ∎

To summarize the above results (Theorem 5.7, Corollaries 5.8, 5.9, Theorem 5.10, Corollary 5.11, which are Proposition 3.3 and its Corollary from [G3] and [G4]), you should think that two "$*$" statements about four points imply the other two $*$ statements that you expect from your intuition about $\mathbb{R}^2$. There are analogues of these results for $\mathbb{R}^3$ also, but they would require some other axiom system since $B - 4$ does not hold.

You should also be careful about the following inconclusive situation: $A*B*C$ and $A*B*D$. None of the above results applies to this case, and you can draw examples where C is between B and D, or D is between B and C, or even $C = D$.

Theorem 5.12. *$A * B * C \implies AB \subseteq AC$.*

Remark. This is part of Betweenness Exercise 3.3.a of [G3] and [G4]. ∎

Corollary 5.13. *$A * B * C \implies CB \subseteq CA$*

Remark. This is the rest of Betweenness Exercise 3.3.a of [G3] and [G4]. ∎

Theorem 5.14. *$A * B * C \implies AC \subseteq AB \cup BC$.*

Remark. This is Betweenness Exercise 3.3.b of [G3] and [G4]. ∎

Corollary 5.15. *$A * B * C \implies AC = AB \cup BC$.*

Proof. (This is part of Proposition 3.5 of [G3] and [G4].) Theorem 5.14 established the $\subseteq$ relation. $AB \subseteq AC$ by Theorem 5.12, and $BC \subseteq AC$ by Corollary 5.13 and Theorem 4.9 (which says $CB = BC$ and $CA = AC$). These two subsets together imply $AB \cup BC \subseteq AC$, so the sets are equal. ∎

Theorem 5.16. *$A * B * C \implies \{B\} = AB \cap BC$.*

Remark. This is the rest of Proposition 3.5 of [G3] and [G4], and Betweenness Exercise 3.3.c of [G3] and [G4]. ∎

Theorem 5.17 (Pasch 3)**.** *Given points A, B, C, D, and a line ℓ, if A, B, C are collinear, and $A \neq C$, and $B \neq C$, and $A * D * B$, and D is on ℓ, and A and B are not both on ℓ, then there is exactly one point E so that $E \in AC \cup CB$ and E is on ℓ. If C is not on ℓ, then either $E \in AC$, or $E \in CB$, but not both.*

Proof. By $B-1-i$, $A \neq B$, so A, B, C are distinct. For the uniqueness part of the Theorem, $\overleftrightarrow{AC} = \overleftrightarrow{CB}$ by Theorem 1.23. By Corollary 4.16, $AC \subseteq \overleftrightarrow{AC}$ and $CB \subseteq \overleftrightarrow{CB} = \overleftrightarrow{AC}$, so $AC \cup CB \subseteq \overleftrightarrow{AC}$, and this implies $AC \cup CB$ is a collinear set (Theorem 1.22). Since A and B are not both on ℓ, there can be, by Theorem 1.9, at most one point E such that $E \in AC \cup CB$ and E is on ℓ.

For the existence, just let $E = D$, so E is on ℓ by hypothesis. To show $E = D \in AC \cup CB$, use the three cases from $B-3-i$, applied to the distinct, collinear points A, B, C.

Case 1. If $A * B * C$, then $A * D * B \implies D * B * C$ (Theorem 5.7), so $D \notin BC$ (Theorem 4.11). It also follows that $A * D * C$ (Corollary 5.8), so $D \in AC$.

Case 2. If $A * C * B$, then $A * D * B \implies D \in AB \subseteq AC \cup CB$ by Theorem 5.14.

Case 3. If $B * A * C$, then $A * D * B \implies B * D * A$ ($B-1-iii$), and $D * A * C$ (Theorem 5.7), so $D \notin AC$ (Theorem 4.11). It also follows that $B * D * C$ (Corollary 5.8), so $D \in BC$.

Since $BC = CB$ (Theorem 4.9), Cases 1., 2., 3. showed $D \in AC \cup CB$. Cases 1. and 3. also showed that D is not in both AC and CB.

For the last part of the Theorem, all that remains is to consider Case 2., and assume C is not on ℓ. Then since D is on ℓ, $D \neq C$, so $D \notin AC \cap CB = \{C\}$, using $A * C * B$ and Theorem 5.16. ∎

Lemma 5.18. *Given points A, B, C, P, if $A * B * C$, and B is not between P and C, then A is not between P and C.*

Remark. This is Betweenness Exercise 3.4.a of [G3] and [G4]. Hint: there are two cases, depending on whether P is collinear with the other points. ∎

Theorem 5.19. $A * B * C \implies \overrightarrow{BC} \subsetneq \overrightarrow{AC}$.

Remark. This is part of Betweenness Exercise 3.4.b of [G3] and [G4]. ∎

Corollary 5.20. $A * B * C \implies \overrightarrow{BA} \subsetneq \overrightarrow{CA}$.

Remark. This is the rest of Betweenness Exercise 3.4.b of [G3] and [G4]. ∎

Theorem 5.21. $A * B * C \implies \overrightarrow{AC} = AB \cup \overrightarrow{BC}$.

Proof. $AB \subseteq AC \subseteq \overrightarrow{AC}$, by Theorem 5.12 and Corollary 4.16, and $\overrightarrow{BC} \subseteq \overrightarrow{AC}$ by Theorem 5.19. Together, these imply $AB \cup \overrightarrow{BC} \subseteq \overrightarrow{AC}$. To show the other subset relation, suppose $X \in \overrightarrow{AC}$. Then, there are four cases, starting with $X = A \in AB$, and $X = C \in \overrightarrow{BC}$. If $A * X * C$, then $X \in AC = AB \cup BC \subseteq AB \cup \overrightarrow{BC}$, by Corollary 5.15 and Corollary 4.16. The remaining case is $A * C * X$, which together with $A * B * C$ implies, by Theorem 5.7, $B * C * X$, so $X \in \overrightarrow{BC}$. In each case, X is an element of $AB \cup \overrightarrow{BC}$. ∎

Theorem 5.22. $A * B * C \implies \overrightarrow{BA} \cap \overrightarrow{BC} = \{B\}$.

Remark. This is part of Proposition 3.6 of [G3] and [G4], and Betweenness Exercise 3.4.c of [G3] and [G4]. ∎

Theorem 5.23. $A * B * C \implies \overrightarrow{AB} = \overrightarrow{AC}$.

Remark. This is the rest of Proposition 3.6 of [G3] and [G4], and part of Betweenness Exercise 3.5 of [G3] and [G4]. Hint: show equality of sets in the usual way, by checking both subset relations. Use Theorem 4.12. ∎

Theorem 5.24. *For any distinct points A, B, there is exactly one ray R with endpoint A such that $B \in R$.*

Proof. One such ray is $R = \overrightarrow{AB}$, which proves existence. To prove uniqueness, we need to check that if R' is some ray with endpoint A, and $B \in R'$, then $R' = R$. Let $R' = \overrightarrow{AD}$, and assume $B \in R'$, so that by definition of ray (and the hypothesis $A \neq B$), one of the following must hold: $B = D$ (in which case $\overrightarrow{AD} = \overrightarrow{AB}$), or $A * B * D$, or $A * D * B$. In both of these last two cases, $\overrightarrow{AD} = \overrightarrow{AB}$ by Theorem 5.23. So, in every case, $R' = R$. ∎

Theorem 5.25. *For any ray, there exists exactly one opposite ray.*

Remark. This is the rest of Betweenness Exercise 3.5 of [G3] and [G4]. ∎

Theorem 5.26 ("Line Separation"). $A * B * C \implies \overleftrightarrow{AC} = \overrightarrow{BA} \cup \overrightarrow{BC}$.

Proof. By Theorem 4.1, the points A, B, and C are distinct, so the rays and line are well-defined, and A, B, and C are collinear, so Theorem 1.23 applies.

Corollary 4.16 says that $\overrightarrow{BA} \subseteq \overleftrightarrow{BA} = \overleftrightarrow{AC}$, and $\overrightarrow{BC} \subseteq \overleftrightarrow{BC} = \overleftrightarrow{AC}$, so $\overrightarrow{BA} \cup \overrightarrow{BC} \subseteq \overleftrightarrow{AC}$.

To show the other subset relation, suppose $D \in \overleftrightarrow{AC}$. One possibility is that $D \in \overrightarrow{BC}$, and if this holds, then $D \in \overrightarrow{BA} \cup \overrightarrow{BC}$. The only other possibility is that $D \notin \overrightarrow{BC}$, so $D \neq C$, $D \neq B$, and we want to show $D \in \overrightarrow{BA}$. First, note $D \notin \overrightarrow{BC} \implies \overrightarrow{BD} \neq \overrightarrow{BC}$. $D \in \overleftrightarrow{AC} = \overleftrightarrow{BC}$, so $\{B, C, D\}$ is a collinear set of three distinct points, and Theorem 1.23 applies again to give $\overleftrightarrow{BC} = \overleftrightarrow{BD}$. By Corollary 4.16, $\overrightarrow{BD} \subseteq \overleftrightarrow{BD} = \overleftrightarrow{BC}$, so $\overrightarrow{BD} \cup \overrightarrow{BC} \subseteq \overleftrightarrow{BC}$, which (by Theorem 1.22) means $\overrightarrow{BD} \cup \overrightarrow{BC}$ is a collinear set, and the rays are opposite (Definition 4.17). By uniqueness of the opposite ray (Theorem 5.25, and the fact that $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are opposite rays by Theorem 4.18), $D \in \overrightarrow{BD} = \overrightarrow{BA}$. This shows every element $D \in \overleftrightarrow{AC}$ is in either $\overrightarrow{BC}$ or $\overrightarrow{BA}$. ∎

Note that Theorem 5.22 stated that the two rays in the above union overlap only in one point (the common endpoint B). The above Theorem is Proposition 3.4 of [G3] and [G4]; the book proves it directly from the B axioms and Proposition 3.3 of [G3] and [G4], without using the uniqueness of the opposite ray.

Theorem 5.27. *Given points A, B, C, and a line ℓ, if A and B are on opposite sides of ℓ, and B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.*

Proof. By Definition 4.26 and Lemma 4.4, A, B, and C are not on ℓ, and there exist points X on ℓ and Y on ℓ such that $A * X * B$ and $B * Y * C$. By Theorem 4.1, $A \neq B$, $A \neq X$, $B \neq X$, $B \neq C$, $B \neq Y$, and $C \neq Y$. There are three cases.

Case 1. $A = C$. Then, by Definition 4.25, A and C are on the same side of ℓ, which is what we wanted.

Case 2. $\{A, B, C\}$ is collinear and $A \neq C$. The three points lie on some line m, and X is on m and Y is on m by $B - 1 - ii$ and $I - 2 - ii$. If $X \neq Y$, then $\ell = m$ by $I - 2 - ii$, contradicting A not on ℓ, so $X = Y$ and we have $A * X * B$ and $B * X * C$. By $B - 3 - i$, there are three cases for the distinct, collinear points A, B, C.

If $A * B * C$, then $A * X * B$ implies, by Theorem 5.7, $X * B * C$, which contradicts $B * X * C$, by Theorem 4.2. If $B * A * C$, then Theorem 5.7, applied to $B * X * A$ ($B - 1 - iii$), gives $X * A * C$, so by definition of ray, $C \in \overrightarrow{XA}$, and A and C are on the same side of ℓ by Lemma 4.30. Similarly for the third case, if $A * C * B$, then $B * C * A$ and $B * X * C$ imply $X * C * A$, so $C \in \overrightarrow{XA}$ and A and C are on the same side of ℓ.

Case 3. $\{A, B, C\}$ is not collinear. By $I - 2 - i$, A and B lie on some line c, and C is not on c. Similarly, B and C lie on a line a and A is not on a, and A and C lie on a line b, with B not on b. By Corollary 4.16 applied to $X \in AB$ and $Y \in BC$, X is on c and Y is on a. If $X = Y$, then B and X are distinct points on c and a, and $a = c$ by $I - 2 - ii$, contradicting C not on c; so, $X \neq Y$.

Lemma 4.30 applies twice: $A * X * B \implies X \in \overrightarrow{AB}$, and X and B are on the same side of b. $B * Y * C \implies Y \in \overrightarrow{CB}$, and Y and B are on the same side of b. $B - 4$ says that X and Y are on the same side of b.

Suppose, toward a contradiction, that A and C are not on the same side of ℓ, so by Theorem 4.28, there is some point Z on ℓ so that $A * Z * C$. A, Z, and C are distinct by Theorem 4.1, and Z is on b (Corollary 4.16 applied to $Z \in AC$).

$Z \in \overrightarrow{AC}$ and $A \neq Z$ imply Z and C are on the same side of c. $Y \in \overrightarrow{BC}$ and $B \neq Y$ imply Y and C are on the same side of c. $B - 4$ says that Z and Y are on the same side of c.

$Z \in \overrightarrow{CA}$ and $C \neq Z$ imply Z and A are on the same side of a. $X \in \overrightarrow{BA}$ and $B \neq X$ imply X and A are on the same side of a. $B - 4$ says that Z and X are on the same side of a.

If $X = Z$, then A and X are distinct points on c and b, and $b = c$ by $I - 2 - ii$, contradicting B not on b; so, $X \neq Z$. If $Y = Z$, then C and Y are distinct points on a and b, and $a = b$ by $I - 2 - ii$, contradicting B not on b; so, $Y \neq Z$. We can conclude that X, Y, and Z are distinct, and collinear (all on ℓ), so by $B - 3 - i$ there are three cases.

i. If $X * Y * Z$, then Z not on a implies X and Z are on opposite sides of a, contradicting the above claim that X and Z are on the same side of a (Theorem 4.27).

ii. Similarly, if $X * Z * Y$, then X not on b implies X and Y are on opposite sides of b, another contradiction.

iii. Finally, if $Y * X * Z$, then Z not on c implies Z and Y are on opposite sides of c, a contradiction.

The conclusion is that A and C must be on the same side of ℓ. ∎

Theorem 5.27 is assumed as the second "plane separation" axiom, $B - 4 - ii$, by the book ([G3] p. 77 and [G4] p. 111, and Figure 3.8), but the above Proof shows that assuming $B - 4 - ii$, in addition to all the other betweenness and incidence axioms, is superfluous. The book also uses $B-4-ii$ to give a proof of Pasch 2 ([G3] p. 80 and [G4] p. 114) which is shorter than the Proof of Theorem 5.5 in these Notes.

Exercise 5.28. Use Pasch's Theorem to give a shorter proof of Theorem 5.27.

Corollary 5.29. *For every line ℓ, there are exactly two half planes bounded by ℓ.*

Proof. (This is the rest of Proposition 3.2 of [G3] and [G4].) By the construction from the Proof of Theorem 5.2, there are at least two half planes, $H(\ell, A)$ and $H(\ell, B)$ for points A and B on opposite sides of ℓ. Any half plane bounded by ℓ is, by definition, of the form $H(\ell, C)$ for some $C \in \mathrm{C}(\ell)$, and there are two cases. If B and C are on the same side of ℓ, then $H(\ell, C) = H(\ell, B)$. If B and C are not on the same side of ℓ, then B and C are on opposite sides, by Theorem 4.28, so Theorem 5.27 applies immediately, saying A and C are on the same side of ℓ, so $H(\ell, C) = H(\ell, A)$. The conclusion is that every half plane is either $H(\ell, A)$ or $H(\ell, B)$. ∎

6 Interiors and convexity

Definition 6.1. Given an angle $\angle BAC$, there exists a line ℓ so that A and B are on ℓ ($A \neq B$ by definition of ray, so Axiom $I-2-i$ applies), and C is not on ℓ by Theorem 4.22. So, there is a set $H(\ell, C)$, the half plane containing C, bounded by ℓ. Similarly, there is a line m so that A and C are on m, B is not on m, and there is a set $H(m, B)$, the half plane containing B, bounded by m. Define the <u>interior of $\angle BAC$</u> to be the set $H(\ell, C) \cap H(m, B)$.

The above Definition appears in the book: [G3] p. 81 and [G4] p. 115.

Lemma 6.2. *Given $\angle BAC$, and a point D, if D is in the interior of $\angle BAC$, then the sets $\{A, B, D\}$ and $\{A, C, D\}$ are not collinear.*

Proof. From Definition 6.1, $A \neq B$ and A and B are on the line ℓ, and $D \in H(\ell, C)$ means D is not on ℓ, so Theorem 1.6 applies, and $\{A, B, D\}$ is not collinear. The $\{A, C, D\}$ case is similar. ∎

Theorem 6.3. *Given $\angle CAB$, and a point D, suppose B, C, and D are collinear. Then, the following are equivalent: D is in the interior of $\angle CAB \iff B * D * C$.*

Remark. This is Proposition 3.7 of [G3] and [G4], and Betweenness Exercise 3.10 of [G3] and [G4]. ∎

Theorem 6.4. *Given $\angle CAB$, and points D and X, suppose D is in the interior of $\angle CAB$ and $\{A, D, X\}$ is a collinear set. Then, the following are equivalent:*

1. *$X \neq A$ and $X \in \overrightarrow{AD}$,*
2. *X is in the interior of $\angle CAB$.*

Remark. This is Proposition 3.8, parts a. and b. of [G3] and [G4], and Betweenness Exercise 3.11.a,b of [G3] and [G4]. ∎

Theorem 6.5. *Given $\angle CAB$, and points D and E, if D is in the interior of $\angle CAB$, and $E * A * C$, then B is in the interior of $\angle DAE$.*

Remark. This is Proposition 3.8.c of [G3] and [G4], and the rest of Betweenness Exercise 3.11.c of [G3] and [G4]. ∎

Theorem 6.6 ("The Crossbar Theorem")**.** *Given $\angle CAB$ and a point D, if D is in the interior of $\angle CAB$, then the set $\overrightarrow{AD} \cap BC$ has exactly one element, Y. The point Y satisfies $Y \neq A$ and $B * Y * C$.*

Remark. This Theorem is stated in the book: [G3] p. 82 and [G4] p. 116. It is Betweenness Exercise 3.12 of [G3]. ∎

Definition 6.7. Given $\triangle ABC$, let a be a line through B and C ($I - 2 - i$ applies because A, B, C are distinct, as mentioned in Notation 4.23), so A is not on a (because A, B, C are non-collinear by definition of $\triangle$). Similarly, let b be a line through A and C, so B is not on b, and let c be a line through A and B, so C is not on c. Define the interior of $\triangle ABC$ to be the intersection of half planes:

$$H(a, A) \cap H(b, B) \cap H(c, C).$$

Note that the interior of the triangle is equal to the intersection of the interiors of the three angles, $\angle BAC$, $\angle ABC$, $\angle ACB$. This Definition is stated in the book: [G3] p. 82 and [G4] p. 117.

Lemma 6.8. *Given $\triangle ABC$, if $X \in AB \cup BC \cup CA$, then X is not an element of the interior of $\triangle ABC$. Let a, b, c be the lines from Definition 6.7; if $X \in AB$, then X is on c, if $X \in BC$, then X is on a, and if $X \in CA$, then X is on b.*

Proof. If $X \in AB$, then X is on c, by Corollary 4.16, so $X \notin H(c, C)$, and X is not in the interior. Similarly using Corollary 4.16 and the definition of half plane, if $X \in BC$, then X is on a and not in $H(a, A)$, and if $X \in CA$, then X is on b and not in $H(b, B)$. ∎

Theorem 6.9. *Given $\triangle ABC$, and a point D, if D is an element of the interior of $\triangle ABC$, then any ray R with endpoint D has the property that the set $R \cap (AB \cup BC \cup CA)$ contains exactly one point. Further, if the ray does not contain a vertex, it meets exactly one side.*

Remark. This is Proposition 3.9.b of [G3] and [G4], and part of Betweenness Exercise 3.13 of [G3] and Betweenness Exercise 3.12 of [G4]. ∎

Theorem 6.10. *Given $\triangle ABC$, and a line ℓ, there exists a point which is on ℓ, and not an element of the interior of $\triangle ABC$.*

Remark. This is Betweenness Exercise 3.14 of [G3] and Betweenness Exercise 3.13 of [G4]. ∎

Definition 6.11. Given a set of points, S, the set S is convex means: for all points $A \in S$, $B \in S$, the following implication holds: If $A * X * B$, then $X \in S$.

This Definition is stated in the book: [G3] p. 107 and [G4] p. 149.

Theorem 6.12. *The set $\emptyset$ is convex. The set of all points (in a given model of order geometry) is a convex set. Given a point P, the set $\{P\}$ is convex. Given points P and Q, if $P \neq Q$, then the set $\overleftrightarrow{PQ}$ is convex.*

Proof. The convexity of the empty set is trivial, for lack of a counterexample. Similarly, by $B-1-i$, there are no points X such that $P * X * P$, so in particular, there aren't any such points not in S, and $\{P\}$ is convex.

The set of all points, in a model of points where $*$ is defined, is also convex for trivial reasons: if A and B are any points (in the set of all points), and if X is a point such that $A*X*B$, then X is in the set of all points (the fact that it is between A and B is not needed).

Suppose $A, B \in \overleftrightarrow{PQ}$. Then, $\{A, B, P, Q\}$ is a collinear set by Theorem 1.22, all the points lying on some line ℓ. If $A * X * B$, then by $B-1-ii$, there is some line n so that A, X, and B are on n, and by $B-1-i$, $A \neq B$, so A and B are distinct points on ℓ and n, and $\ell = n$ by $I-2-ii$. This implies X is on ℓ, so X, P and Q are all on ℓ, which means $X \in \overleftrightarrow{PQ}$. ∎

Theorem 6.13. *A ray is a convex set.*

Proof. Suppose $A, B \in \overrightarrow{PQ}$. Then $\overrightarrow{PQ} \subseteq \overleftrightarrow{PQ}$ (Corollary 4.16), and the Proof of Theorem 6.12 showed that if $A*X*B$, then X, P, and Q are collinear and $A \neq B$. We want to show $X \in \overrightarrow{PQ}$. From the definition of ray, there are some cases to check.

Case 1: $A = P$. Then, $B \in \overrightarrow{PQ}$, $B \neq A \implies \overrightarrow{PQ} = \overrightarrow{AB}$ by Theorem 5.24, and $X \in AB \subseteq \overrightarrow{AB} = \overrightarrow{PQ}$.

Case 2: $B = P$. Then, $A \in \overrightarrow{PQ}$, $A \neq B \implies \overrightarrow{PQ} = \overrightarrow{BA}$ by Theorem 5.24, and $A*X*B \implies B*X*A$ $(B-1-iii)$, so $X \in BA \subseteq \overrightarrow{BA} = \overrightarrow{PQ}$.

Case 3: $A \neq P$, so $\overrightarrow{PQ} = \overrightarrow{PA}$ by Theorem 5.24. Since $B = P$ has been considered and $B = A$ has been ruled out, $B \in \overrightarrow{PA}$ has two cases from the definition of ray remaining. Suppose, toward a contradiction, that $X \notin \overrightarrow{PA}$. By Theorem 4.12, $X * P * A$.

i. The first case is $P * B * A$. $X * P * A \implies X * B * A$, by Corollary 5.9, contradicting $A * X * B$ by Theorem 4.2.

ii. The second case is $P * A * B$. $X * P * A \implies X * A * B$, by Corollary 5.11, which also contradicts $A * X * B$.

These contradictions show $X \notin \overrightarrow{PQ}$ is false, which is what we wanted. ∎

Theorem 6.14. *A half plane is a convex set.*

Remark. This is Betweenness Exercise 3.19.a of [G3] and Betweenness Exercise 3.18.a of [G4]. ∎

A "closed" half plane is also convex:

Theorem 6.15. *Given points A, B, C and a line c, if $A \neq B$ and A and B are on c and C is not on c, then the set $\overleftrightarrow{AB} \cup H(c, C)$ is convex.*

Proof. Given $P, Q \in \overleftrightarrow{AB} \cup H(c, C)$, suppose $P*X*Q$. We want to show that $X \in \overleftrightarrow{AB} \cup H(c, C)$, and there are three cases.

Case 1. $P \in \overleftrightarrow{AB}$ and $Q \in \overleftrightarrow{AB}$ imply $X \in \overleftrightarrow{AB}$ by Theorem 6.12.

Case 2. If $P \in \overleftrightarrow{AB}$ and $Q \in H(c, C)$, then P is on c by Theorem 1.22. By definition of $H(c, C)$, Q is not on c and Q and C are on the same side of c. $P * X * Q \implies X \in \overrightarrow{PQ}$, so X

and Q are on the same side of c by Lemma 4.30, and by $B-4$, X and C are on the same side of c, so $X \in H(c,C)$. The case where $Q \in \overleftrightarrow{AB}$ and $P \in H(c,C)$ is similar.

Case 3. $P \in H(c,C)$ and $Q \in H(c,C)$ imply $X \in H(c,C)$ by Theorem 6.14. ∎

Theorem 6.16. *Given any set $\mathcal{J}$, suppose that for each $i \in \mathcal{J}$ there is a set of points S_i. Let S be the intersection of all the sets S_i. If every S_i is convex, then S is also convex.*

Remark. This is a modification of Exercise 4.25.a of [G3].

The set $\mathcal{J}$ is called the "index set" (it could be a finite or infinite set) and the statement "S is the intersection of all the sets S_i" means S is a set, and that a point X is in S if and only if $X \in S_i$ for all $i \in \mathcal{J}$. The intersection is sometimes denoted

$$S = \bigcap_{i \in \mathcal{J}} S_i.$$

∎

Theorem 6.17. *Given points A, B, if $A \neq B$, then the segment AB is a convex set.*

Proof. This follows immediately from Theorems 4.14, 6.13, and 6.16. ∎

Theorem 6.18. *The interior of an angle is a convex set. The interior of a triangle is a convex set.*

Remark. This is Betweenness Exercise 3.19 parts b. and c. of [G3], and Betweenness Exercise 3.18 parts b. and c. of [G4]. ∎

Definition 6.19. Given a set of points S, a <u>convex hull of S</u> is a set $ch(S)$ with the following three properties:

- $S \subseteq ch(S)$,
- $ch(S)$ is a convex set,
- If T is a convex set and $S \subseteq T$, then $ch(S) \subseteq T$.

Theorem 6.20. *Given any set S, there exists a unique convex hull $ch(S)$.*

Proof. To prove existence, first use the fact that S is a subset of the set of all points, and the set of all points is convex by Theorem 6.12. So, there is at least one convex set containing S; let $ch(S)$ be the intersection of *all* convex sets which contain S. Such an intersection is convex by Theorem 6.16. It is contained in any convex set T which contains S, because it is the intersection of T with all the other convex sets containing S. If $X \in S$, then X is a point in every convex set containing S, so X is in the intersection of all such sets, which shows $S \subseteq ch(S)$.

To prove uniqueness, suppose $ch(S)$ and $ch_2(S)$ are two sets which both satisfy the three properties from Definition 6.19. Then, they are both convex sets containing S, by the first two parts of the definition, and $ch(S) \subseteq ch_2(S)$ by the third part of the definition of $ch(S)$, and $ch_2(S) \subseteq ch(S)$ by the third part of the definition of $ch_2(S)$. We can conclude that the two sets are equal. ∎

Theorem 6.21. *Given points A, B, if $A \neq B$, the convex hull of the set $\{A, B\}$ is the segment:*

$$ch(\{A, B\}) = AB.$$

Proof. A and B are elements of AB, and AB is convex by Theorem 6.17. To show the third part of the definition of convex hull, suppose T is a convex set, such that $\{A, B\} \subseteq T$. If $A * X * B$, then, since A and B are elements of the convex set T, X is also in T. This shows every element $X \in AB$ is also an element of T, so $AB \subseteq T$, and $AB = ch(\{A, B\})$. ∎

Theorem 6.22. *Given a non-collinear set of points $\{A, B, C\}$, let $\mathcal{I}$ denote the interior of $\triangle ABC$. Then, the convex hull of the set $\{A, B, C\}$ is:*

$$ch(\{A, B, C\}) = AB \cup BC \cup CA \cup \mathcal{I}.$$

Remark. This is Exercise 4.26 of [G3]. ∎

The following notes on quadrilaterals are related to the discussion in the book: [G3] p. 29, p. 127, p. 141, and [G4] p. 44, p. 187, p. 199.

Definition 6.23. An ordered quadruple of four points, (A, B, C, D) is a <u>quadrilateral</u> means: the sets $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, and $\{B, C, D\}$ are all non-collinear. The four points are called <u>vertices</u>, and they are distinct (Theorem 1.5). The four segments AB, BC, CD, DA are called <u>edges</u> (or "sides" as in Notation 4.23), and the two segments AC, BD are called <u>diagonals</u>. A pair of edges with a common endpoint is <u>adjacent</u>, and otherwise <u>opposite</u>.

Notation 6.24. A convenient abbreviation for a quadrilateral is $\square ABCD$, and the order of the four points matters. However, since $\square BCDA$, $\square CDAB$, $\square DABC$, $\square DCBA$, $\square CBAD$, $\square BADC$, and $\square ADCB$ have the same vertices and edges as $\square ABCD$, it is convenient to say they are the same quadrilateral. Note that there are 24 ways to order the four points, eight of which were already named, and the rest falling into two other sets of eight. For example, $\square ABCD$, $\square ABDC$, and $\square ACBD$ are all different.

Adjacent edges meet only at one point, by Lemma 4.24. An ordered quadruple of four points, without any non-collinearity conditions, could be called a <u>degenerate quadrangle</u>.

Definition 6.25. Given points A, B, C, D, such that $A \neq B$ and $C \neq D$, the segments AB, CD are <u>semiparallel</u> means: there does not exist a point P such that $P \in CD$ and $\{A, B, P\}$ is collinear, and there does not exist a point Q such that $Q \in AB$ and $\{C, D, Q\}$ is collinear.

Definition 6.26. $\square ABCD$ is a <u>simple</u> quadrilateral means: both pairs of opposite edges are disjoint. $\square ABCD$ is a <u>convex quadrilateral</u> means: it is a simple quadrilateral, and one of its two pairs of opposite edges is semiparallel.

Theorem 6.27. *Given $\square ABCD$, if $\square ABCD$ is simple, and one pair of opposite edges is semiparallel, then so is the other pair.*

Remark. This is Exercise 4.23 of [G3], p. 141; see also Exercise 4.28 of [G4], p. 199. ∎

Definition 6.28. Given a quadrilateral $\square ABCD$, let p be a line through A, B, let q be a line through B, C, let r be a line through C, D, and let s be a line through D, A. If AB and CD are semiparallel, then by Definition 6.25, C and D are on the same side of p and A and B are on the same side of r, and by Definition 6.26, $\square ABCD$ is a convex quadrilateral. By Theorem 6.27, BC and DA are also semiparallel, so B and C are on the same side of s, and A and D are on the same side of q. Define the <u>interior of the convex quadrilateral $\square ABCD$</u> to be the set

$$H(p, D) \cap H(q, A) \cap H(r, B) \cap H(s, C).$$

This Definition is stated on p. 127 of [G3]; a related notion is stated in Exercise 4.29 of [G4], p. 199.

Theorem 6.29. *The interior of a convex quadrilateral is a convex set, and the diagonals of a convex quadrilateral intersect at exactly one point. This point is in the interior of the quadrilateral.*

Remark. This is related to Exercises 4.24 and 4.25 of [G3], and Exercises 4.28 and 4.29 of [G4]. ∎

7 Review of complex numbers

The set $\mathbb{C}$ of complex numbers, $\mathbb{C} = \{x + yi : x \in \mathbb{R}, y \in \mathbb{R}\}$ forms a "field" in the sense of abstract algebra: there are elements $0 + 0i$ (abbreviated 0) and $1 + 0i$ (abbreviated 1), $+$ and $\cdot$ binary operations, additive inverses $(z + (-z) = 0)$ and division $z_1 \cdot (z_2^{-1}) = \frac{z_1}{z_2}$ is always allowed except for $z_2 = 0$. The operations $+$ and $\cdot$ are commutative, associative, and distributive, and positive integer powers are defined as usual: z^n means $z \cdot z \cdots z$, with n factors. The real numbers $\mathbb{R}$ are a subset of $\mathbb{C}$: if $x \in \mathbb{R}$, then $x = x + 0i$ is also a complex number. Numbers $x + iy$ are usually graphed as ordered pairs on the cartesian plane, so that the real number line is the x-axis. The y-axis contains numbers of the form $0 + iy$, and this is called the <u>imaginary axis</u>, to distinguish it from the <u>real axis</u>.

The addition, subtraction, and multiplication rules for complex numbers are just like the rules for polynomials, with the additional rule that $i^2 = -1$. It is convenient to group the real terms together, and to write the imaginary part as a real number times i:

$$\begin{aligned}
(a + bi) + (x + yi) &= (a + x) + (b + y)i \\
(a + bi) - (x + yi) &= (a - x) + (b - y)i \\
(a + bi) \cdot (x + yi) &= ax + xbi + ayi + (bi)(yi) = (ax - by) + (xb + ay)i
\end{aligned}$$

The quotient of complex numbers can be simplified by multiplying the numerator and denominator by the same quantity:

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di} \cdot \frac{c - di}{c - di} = \frac{ac - adi + cbi + (bi)(-di)}{c^2 - cdi + cdi - (di)(di)}$$

$$= \frac{(ac + bd) + (bc - ad)i}{c^2 - i^2 d^2} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} \cdot i.$$

The complex conjugate of a complex number $z = x + iy$ is the complex number $\bar{z} = x - iy$. Note that $z \cdot \bar{z} = (x+iy)(x-iy) = x^2 + y^2$ is always a real, non-negative number. The distance formula (Pythagorean theorem) states that the distance from $z = x + yi$ to the origin $0 + 0i$ is $\sqrt{x^2 + y^2}$. This distance is called the absolute value $|z|$, so $|z|^2 = x^2 + y^2 = z \cdot \bar{z}$. For $z_1 = a + bi$, and $z_2 = c + di$, the calculation of the quotient can be rewritten:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}.$$

The distance formula for two complex numbers z_1, z_2 is $|z_1 - z_2|$. (Expanding this into the $x_1 + y_1 i$, $x_2 + y_2 i$ form gives the Cartesian distance formula.) The equation of a circle with center $c \in \mathbb{C}$, and radius $r \geq 0$ is

$$\{z : |z - c| = r\}.$$

The equation of a line in $\mathbb{C}$ is

$$\{z : Dz + \bar{D}\bar{z} + C = 0\},$$

for any $C \in \mathbb{R}$, and $D \in \mathbb{C}$, $D \neq 0$. (In class, I showed how to arrive at this, starting with the Cartesian equation $Ax + By + C = 0$.)

The argument of a complex number z is the radian measure of the angle formed by the following two rays: the positive x-axis, and the ray whose endpoint has coordinate 0, which goes through the point with coordinate z. This is denoted $arg(z)$, and usually is considered to have values in $[0, 2\pi)$, with angles measured counter-clockwise, or values in $(-\pi, \pi]$, with negative angles measured clockwise. The complex number $z = 0$ does not have an angle, and the numbers on the negative x-axis have angle π. This is the same angle measure usually called "θ" in polar coordinates, and the "r" coordinate is $|z| = \sqrt{x^2 + y^2}$. The relationship between $\theta = arg(z)$ and the $x + yi$ expression is $\tan(\theta) = y/x$, for $x \neq 0$. When $x = 0$, the positive multiples of i have $arg = \pi/2$, and the negative multiples of i have $arg = 3\pi/2$ (counter-clockwise), or $arg = -\pi/2$ (clockwise).

It can be shown that for two complex numbers z, w, the length $|z \cdot w|$ is equal to the product of the lengths $|z| \cdot |w|$, and the argument of $z \cdot w$ is the *sum* of the arguments of z and w.

Exercise 7.1. Calculate the following complex numbers (meaning, state your answer in the form $a + bi$ for exact real numbers a and b):

- $\frac{1}{i} =$
- $i^3 =$
- $i^4 =$
- $i^{1998} =$
- $\frac{30-47i}{3+4i} =$
- $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i)^2 =$

Exercise 7.2. Let $z = x + yi$ and $w = a + bi$. Prove the following statements:

- $\overline{(z + w)} = \bar{z} + \bar{w}$.
- $\overline{(z \cdot w)} = \bar{z} \cdot \bar{w}$.
- $|z| = |\bar{z}|$.
- $|z \cdot w| = |z| \cdot |w|$.

Exercise 7.3. Prove the identity $\tan(arg(z \cdot w)) = \tan(arg(z) + arg(w))$.

Hint. Use the trigonometric function identities, for example on [G3] p. 401 and [G4] p. 490. $\blacksquare$

Exercise 7.4. If z is the complex number $x + iy$, then its <u>real part</u> is the real number x, abbreviated $\mathrm{Re}(z)$. The <u>imaginary part</u> is the real number y, abbreviated $\mathrm{Im}(z)$.

Show $\mathrm{Re}(z) = \dfrac{z + \bar{z}}{2}$, and $\mathrm{Im}(z) = \dfrac{z - \bar{z}}{2i}$. Also show $\mathrm{Im}(i \cdot z) = \mathrm{Re}(z)$, and $\mathrm{Re}(i \cdot z) = -\mathrm{Im}(z)$.

8 Linear fractional transformations

The following Theorem is part of Proposition 9.26 of [G3] and [G4].

Theorem 8.1. *Given $\{a, b, c, d\} \subseteq \mathbb{C}$, suppose $ad - bc \neq 0$, so $f(z) = \frac{az+b}{cz+d}$ is a linear fractional transformation. If f "fixes the real axis" (meaning $z \in \mathbb{R} \implies f(z) \in \mathbb{R} \cup \{\infty\}$), then there's some scalar $k \in \mathbb{C}$, $k \neq 0$, so that $\{ka, kb, kc, kd\} \subseteq \mathbb{R}$.*

This means that the function $f(z)$ can be written as a linear fractional transformation with all <u>real</u> coefficients:

$$f(z) = \frac{(ka)z + (kb)}{(kc)z + (kd)}.$$

Here's the proof, using only algebra. The book gives a proof which is more geometric; it writes f as a composite of simpler transformations. [C] generalizes the Theorem to higher dimensions, and gives a proof using "complex homogeneous coordinates."

Proof. Take any $z \in \mathbb{R}$, so $z = \bar{z}$, and by hypothesis, $f(z)$ is also real (or infinite), so it is equal to its conjugate, $\overline{f(z)}$:

$$\frac{az + b}{cz + d} = \overline{\left(\frac{az + b}{cz + d}\right)} = \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} = \frac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}}. \tag{8.1}$$

Case 1. $c = 0$. Then, $ad - bc \neq 0 \implies ad \neq 0 \implies a \neq 0$ and $d \neq 0$. Equation (8.1) becomes $\frac{az+b}{0z+d} = \frac{\bar{a}z+\bar{b}}{0z+\bar{d}}$, and cross-multiplying by d and $\bar{d}$ (both non-zero) gives:

$$(a\bar{d} - \bar{a}d)z + (b\bar{d} - \bar{b}d) = 0.$$

This equation holds for all $z \in \mathbb{R}$, including $z = 0$, so $b\bar{d} = \bar{b}d$ (which implies $b\bar{d}$ is real), and also for $z = 1$, so $(a\bar{d} - \bar{a}d) \cdot 1 + 0 = 0 \implies a\bar{d} = \bar{a}d$, and $a\bar{d}$ is also real. Let $k = \bar{d} \neq 0$, so that $\{a\bar{d}, b\bar{d}, 0\bar{d}, d\bar{d}\}$ are all real.

Case 2. $c \neq 0$. Then, Equation (8.1) holds for all real z, except possibly $z = -d/c$. Cross-multiplying gives

$$\begin{aligned} (az+b)(\bar{c}z+\bar{d}) - (\bar{a}z+\bar{b})(cz+d) &= 0 \\ \Longrightarrow\ (a\bar{c}-\bar{a}c)z^2 + (b\bar{c}+a\bar{d}-\bar{a}d-\bar{b}c)z + (b\bar{d}-\bar{b}d) &= 0, \end{aligned} \tag{8.2}$$

for all real z except possibly one. Since any quadratic polynomial which equals zero, on at least three different points, must be identically zero, all these coefficients are 0, so $a\bar{c} = \bar{a}c$, and $b\bar{d} = \bar{b}d$: $a\bar{c}$ and $b\bar{d}$ are real numbers.

The linear term from Equation (8.2) is also 0, and multiplying by cd will still give 0:

$$\begin{aligned} 0 &= cd(a\bar{d} - \bar{a}d + b\bar{c} - \bar{b}c) \\ \Longrightarrow 0 &= acd\bar{d} - \bar{a}cd^2 + bc\bar{c}d - \bar{b}c^2d \\ \Longrightarrow 0 &= acd\bar{d} - a\bar{c}d^2 + bc\bar{c}d - bc^2\bar{d} \\ \Longrightarrow 0 &= (ad-bc)(c\bar{d} - \bar{c}d). \end{aligned}$$

One step used the equalities $a\bar{c} = \bar{a}c$, and $b\bar{d} = \bar{b}d$. The last step is just factoring, and $ad - bc \neq 0$ by hypothesis, so we can conclude $c\bar{d} = \bar{c}d$, and $d\bar{c}$ is real.

A similar calculation results from multiplying by bc:

$$\begin{aligned} 0 &= bc(a\bar{d} - \bar{a}d + b\bar{c} - \bar{b}c) \\ \Longrightarrow 0 &= abc\bar{d} - \bar{a}bcd + b^2c\bar{c} - b\bar{b}c^2 \\ \Longrightarrow 0 &= a\bar{b}cd - abc\bar{d} + b^2c\bar{c} - b\bar{b}c^2 \\ \Longrightarrow 0 &= (ad-bc)(\bar{b}c - b\bar{c}). \end{aligned}$$

Again, $ad - bc$ can't be 0, so $\bar{b}c = b\bar{c}$, and $b\bar{c}$ is real. Let $k = \bar{c} \neq 0$, so $\{a\bar{c}, b\bar{c}, c\bar{c}, d\bar{c}\}$ are all real. ∎

The set of all L.F.T.s with real coefficients is called $PGL(2,\mathbb{R})$. The determinant $\delta = ad - bc$ is a non-zero real number. If all four coefficients are scaled by $|\delta|^{-1/2}$, then

$$f(z) = \frac{\frac{a}{\sqrt{|\delta|}}z + \frac{b}{\sqrt{|\delta|}}}{\frac{c}{\sqrt{|\delta|}}z + \frac{d}{\sqrt{|\delta|}}}$$

has determinant $\frac{ad-bc}{(\sqrt{|\delta|})^2} = \frac{ad-bc}{|ad-bc|}$, which is either 1, for $\delta > 0$, or -1, for $\delta < 0$.

The subset of $PGL(2,\mathbb{R})$ with determinant $\delta > 0$ (so that dividing by $\sqrt{\delta}$ gives a determinant $+1$) forms a subgroup, called $PSL(2,\mathbb{R})$.

For $f \in PGL(2,\mathbb{R})$,

$$f(i) = \frac{ai+b}{ci+d} = \frac{(b+ai)(d-ci)}{(d+ci)(d-ci)} = \frac{(bd+ac)+i(ad-bc)}{c^2+d^2}.$$

So, if $ad - bc > 0$, then the imaginary part of $f(i)$ is positive (so $f(i)$ is in the upper half plane), and if $ad - bc < 0$, then the imaginary part of $f(i)$ is negative (so f takes i to the lower half plane). More generally, it is not hard to check that if $z = x + iy$, then $\mathrm{Im}(f(z)) = \frac{(ad-bc)y}{|cz+d|^2}$. We can conclude that $PSL(2,\mathbb{R})$ is the set of L.F.T.s that preserve the real axis and the upper half plane (if $y > 0$, then $\mathrm{Im}(f(z)) > 0$), and the other elements in $PGL(2,\mathbb{R})$, with $\delta < 0$, preserve the real axis but switch the upper and lower half plane (if $y > 0$, then $\mathrm{Im}(f(z)) < 0$).

Exercise 8.2. Let γ be the circle with center O, and radius length r. In complex coordinates, let O have coordinate $c \in \mathbb{C}$, so the equation of γ is $\{z : |z - c| = r\}$. For a point $P \neq O$ with coordinate z, show that the inversion P' of P in γ has coordinate

$$F(z) = \frac{r^2}{\bar{z} - \bar{c}} + c,$$

by checking the following properties:

- $\overline{OP} \cdot \overline{OP'} = r^2$. (these are the products of the lengths, not the complex conjugates...)
- P' is on the ray $\overrightarrow{OP}$. In terms of coordinates, this means that $F(z) - c$ is a positive scalar multiple of $z - c$. (So, calculate $\frac{F(z)-c}{z-c}$, and show you get a positive number)
- Show that if $P \in \gamma$ (so $|z - c| = r$) then $P = P'$ (show $F(z) = z$).
- Check that the inverse of $F(z)$ is z. (This means: calculate $F(F(z))$, and simplify.)

Exercise 8.3. Suppose λ is the circle defined by $\{z : |z - d| = R\}$, and that $c \in \lambda$. Show that the image of λ under inversion in γ is a line. Hint: plug the formula $z = \frac{r^2}{\bar{w}-\bar{c}} + c$ into the equation $(z - d)\overline{(z - d)} = R^2$. Then $c \in \lambda \implies |c - d|^2 = R^2$ leads to a convenient cancellation. Simplify the resulting equation until it is in the $Dw + \bar{D}\bar{w} + C = 0$ form.

Exercise 8.4. <u>Stereographic projection</u> is a function f with input $z \in \mathbb{C}$, and output an element of the sphere $S = \{(X,Y,Z) : X^2 + Y^2 + Z^2 = 1\} \subseteq \mathbb{R}^3$. Here's the formula:

$$\begin{aligned} X &= \frac{z + \bar{z}}{1 + z\bar{z}} \\ Y &= \frac{(-i)(z - \bar{z})}{1 + z\bar{z}} \\ Z &= \frac{z\bar{z} - 1}{1 + z\bar{z}} \end{aligned}$$

- Show that the image of z is on the sphere. (simplify $X^2 + Y^2 + Z^2$)
- Denote the inverse of $f : \mathbb{C} \to S$ by $f^{-1} : S \to \mathbb{C}$. Show that $f^{-1}(X,Y,Z) = \frac{X+Yi}{1-Z}$, for $Z \neq 1$. (Check: $(f \circ f^{-1})(X,Y,Z) = (X,Y,Z)$, assuming $X^2 + Y^2 + Z^2 = 1$, and $(f^{-1} \circ f)(z) = z$ for all $z \in \mathbb{C}$.)
- Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection in the xy-plane, $R(X,Y,Z) = (X,Y,-Z)$. Show that $f^{-1} \circ R \circ f$ is inversion in the unit circle.

Exercise 8.5. (This is a modification of Exercise 9.36 of [G3] and [G4]; see Section 9 below.) Given complex constants $\{a, b, c, d\} \subseteq \mathbb{C}$, assume $\delta = ad - bc \neq 0$ and $c \neq 0$. Define linear fractional transformations $T_1(z) = z + a/c$, $T_2(z) = (-\delta/c^2)z$, $T_3(z) = z^{-1}$, and $T_4(z) = z + d/c$. Show that

$$(T_1 \circ T_2 \circ T_3 \circ T_4)(z) = \frac{az + b}{cz + d}.$$

Exercise 8.6. (This is a modification of Exercise 9.38 of [G3] and [G4].) Show that if $|z| < 1$, then $\text{Re}(\frac{i+z}{i-z}) > 0$.

Hint. First, Simplify $\frac{i+z}{i-z}$, by multiplying numerator and denominator by the conjugate of the denominator:

$$\frac{i + z}{i - z} = \frac{i + z}{i - z} \cdot \frac{-i - \bar{z}}{-i - \bar{z}} = \frac{(i + z) \cdot (-i - \bar{z})}{|i - z|^2}.$$

The new denominator is real and positive, so expand the numerator, and separate the result into the real and imaginary parts. Finally, apply the hypothesis $|z| < 1$ to get the inequality. ∎

Exercise 8.7. Let γ_r be the circle with radius r, and center $c = i \cdot r$ on the imaginary y-axis. Draw a picture of γ_r in the xy-plane, to show it's tangent to the real x-axis. The formula for the inversion of z in γ_r is

$$F(z) = \frac{r^2}{\bar{z} + ir} + ir.$$

Assuming z is constant, show that

$$\bar{z} = \lim_{r \to +\infty} \left(\frac{r^2}{\bar{z} + ir} + ir \right).$$

Hint. First, simplify the quantity by adding fractions. Then compute the limit. Geometrically, this Exercise shows that the conjugation function $f(z) = \bar{z}$, which is the reflection in the x-axis, is a limit of inversion of circles γ_r, where the radius approaches $+\infty$. ∎

9 Errata for the textbook

On p. 91 of [G3], Proposition 3.21(c) should start "If $\angle P > \angle Q \ldots$".

On p. 354 of [G3], the numerator of the linear fractional transformation should be a function of z, not x.

On p. 377 of [G3] and p. 463 of [G4], the composite in Exercise 9.36 seems to be in the wrong order. See Exercise 8.5, above.

In [G4], the First Printing of the Fourth Edition has Exercise 12. on page 93; there is an erroneous claim made by part 12.c., about affine planes. This error is fixed in the Fourth Printing, by deleting the old Exercise 12.c. and replacing it with the old 12.d., so that the new Exercise 12.c. is correctly given as "Exhibit a model of incidence geometry in which parallel lines exist but parallelism is not transitive."

References

[C] A. Coffman, *Real congruence of complex matrix pencils and complex projections of real Veronese varieties*, Linear Algebra and its Applications **370** (2003), 41–83.

[E] Euclid, *Elements*, transl. T. Heath, Second edition, Dover, 1956.

[Golos] E. Golos, *Foundations of Euclidean and Non-Euclidean Geometry*, Holt, Reinhart, and Winston, 1968.

[G3] M. Greenberg, *Euclidean and Non-Euclidean Geometries, Development and History*, Third edition, Third printing (1996), Freeman, 1993.

[G4] M. Greenberg, *Euclidean and Non-Euclidean Geometries, Development and History*, Fourth edition, Fourth printing, Freeman, 2008.