COMMENTS ON: THE ALGEBRA AND GEOMETRY OF STEINER AND OTHER QUADRATICALLY PARAMETRIZABLE SURFACES

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1. Errata

The following typos appear in the published paper, [CSS].

- (p. 270) The derivative, equation (34), should read
$$\frac{\partial f_3}{\partial x} = [x_2(x_3^2+x_4^2) - 2x_1x_4^2, x_1(x_3^2+x_4^2),$$
$$2x_1x_2x_3 - 4x_3(x_3^2+x_4^2), 2x_1x_2x_4 - 2x_1^2x_4 - 4(x_3^2+x_4^2)x_4].$$
This corrects the lower right 2×2 block of the Hessian, equation (35), to
$$\begin{matrix} 2x_1x_2 - 12x_3^2 - 4x_4^2 & -8x_3x_4 \\ -8x_3x_4 & 2x_1x_2 - 2x_1^2 - 12x_4^2 - 4x_3^2 \end{matrix}\ \Bigg].$$
- (p. 273) Below equation (58) should read "As in the case of Σ_7 and Σ_8"
- (p. 279) **Theorem 7.** should read "*The order of* Σ *is* $4-\nu_\Sigma$."
- (p. 281) Above equation (105) should read "Since $\mathcal{P}(\Sigma) \not\subseteq \mathbb{M}_2$, $e \neq 0$."
- (p. 284) Step (2) in Section **6.** should read "$\det\{\lambda\mathbf{M} + \mu\mathbf{N}\}$."

2. Updates

The references mention unpublished notes of A. Schwartz (1932–2024 [CN]) and C. Stanton — a version from 1988 is available from me, on request. Another pre-1996 source on this topic, with some details on the matrix calculations leading to the classification theorem, is [C_1], which I made available online in 2018.

It should have been mentioned that Coffman's research was supported in part by a National Science Foundation Research Experience for Undergraduates program in the summer of 1990.

Since publication, the following related article has come to our attention: [D]. The topic of projections of the real Veronese variety has more recently been considered in [C_3]. Also see my web page on Steiner surfaces, currently at this address: [C_2].

3. Citations

Our article is cited in these academic papers: [A_1], [A_2], [A_3], [ABB], [AMT], [AS_1], [AS_2], [BJKL], [BOR], [BCF], [BEG], [CFRV], [EGL_1], [EGL_2], [GS], [G], [HJS], [H], [HW_1], [HK], [HW_2], [HW_3], [KO], [LG], [L], [M], [PA], [PL], [PO], [POS], [PR], [PT], [P], [RJ], [S], [SPS], [VMD], [WG], [WC], [WCD], [Y], [Zanella], [Z_1], [Z_2], [Z_3], as well as these books: [F], [KI], [OSG], and this computer technical manual: [T].

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