LECTURE NOTES ON: GLAESER'S INEQUALITY ON AN INTERVAL

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Abstract. We use elementary methods to find pointwise bounds for the first derivative of a real valued function with a continuous, bounded second derivative on an interval.

1. Introduction

By "Glaeser's inequality," we refer to the elementary calculus theorem that for a function f which is nonnegative on $\mathbb{R}$ and has continuous second derivative bounded above by $M > 0$ on $\mathbb{R}$, the first derivative satisfies the inequalities

$$-\sqrt{2Mf(x)} \leq f'(x) \leq \sqrt{2Mf(x)} \tag{1}$$

at every point $x \in \mathbb{R}$. The surprising part of the statement, and possibly the reason it is not more well-known, is that even though the conclusion is a pointwise inequality, the assumptions are global: f must satisfy the stated conditions on the entire number line $\mathbb{R}$. The inequality $|f'(x)| \leq \sqrt{2Mf(x)}$ is clearly false for nonnegative functions f defined only on an interval (a, b) or ray (a, ∞); the function $f(x) = x$ on the interval $(0, \infty)$ is an easy counterexample.

We refer to the literature ([G], [LN]) for the history and various proofs of Glaeser's inequality, and we note that it and its generalizations to several variables are topics of recent research interest ([CV], [NS]) in the field of partial differential equations. The related problem of estimating the maximum value of $|f'|$ on $\mathbb{R}$ or a subinterval goes back at least to Landau ([CS]).

Our goal here is to find "local" versions of Glaeser's pointwise inequality for f': to replace the domain $\mathbb{R}$ in the theorem with a subinterval of $\mathbb{R}$, possibly at the expense of imposing boundary conditions or getting a weaker pointwise bound.

Some further examples are instructive regarding the need for all the hypotheses in Glaeser's theorem. Given $f(x) \geq 0$ on an interval (a, b), a first attempted approach might be to extend f to a nonnegative function on $\mathbb{R}$, and then apply Glaeser's inequality to the extension. The example $f(x) = x$ on $(0, \infty)$ already shows what goes wrong: f does not extend to a nonnegative differentiable function on $\mathbb{R}$; if it did, then $f'(0)$ would have to be 1 and f would be increasing near 0 and negative for $x \to 0^-$, a contradiction. To avoid this issue, we could try assuming that f' approaches 0 near the left endpoint, but this boundary condition by itself is still not enough for the extension to satisfy either the assumptions or the conclusion of Glaeser's inequality. Considering the example $f(x) = x^{3/2}$ on $(0, \infty)$, f extends to the nonnegative, differentiable function $|x|^{3/2}$ on $\mathbb{R}$, but f'' is unbounded and f' is not bounded pointwise on $(0, \infty)$ by $\mu\sqrt{f}$ for any scalar μ.

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If we consider an extra assumption: that f is increasing on an interval (a,b), in addition to satisfying $f(x) \geq 0$, $\lim_{x\to a^+} f'(x) = 0$, and $f''(x) \leq M$, then we get the inequalities (1) for $x \in (a,b)$ via a one-line proof:

$$\frac{1}{2}(f'(x))^2 = \lim_{w\to a^+} \int_w^x f'(t)f''(t)dt \leq \int_a^x f'(t)M dt \leq Mf(x). \tag{2}$$

However, the "increasing" assumption seems too strong and it would be more satisfactory to find a result that holds even if f' changes sign infinitely many times on an interval.

2. Asymmetric boundary conditions

Our main result, Theorem 2.1, does not assume that f is increasing; the Proof uses a method different from (2), but which is still elementary. One might expect a limit condition on f' at one endpoint, a, to lead to estimates for f' that hold only near a, but it turns out that this condition (and $f''(x) \leq M$ for $a < x < b$) is enough to guarantee the same upper bound as (1) on the whole interval (a,b). Under the further condition that f'' is bounded below by B, we obtain the lower bound $f'(x) \geq -\sqrt{2Mf(x)}$ on a certain subinterval $(a,c]$, where c depends on M and B but not on f.

For simplicity, we consider the interval $\mathbf{I} = (0,C)$; the result clearly can be translated to an interval (a,b).

Theorem 2.1. *Given $f : \mathbf{I} \to \mathbb{R}$, suppose:*

- *$f(x) \geq 0$ for all $x \in \mathbf{I}$;*
- *$f''(x)$ is continuous on $\mathbf{I}$* [we denote this by $f \in \mathcal{C}^2(\mathbf{I})$]*;*
- *there is a constant $M > 0$ so that $f''(x) \leq M$ for all $x \in \mathbf{I}$;*
- *$\lim_{x\to 0^+} f'(x) = 0$.*

Then, for all $x \in \mathbf{I}$,

$$f'(x) \leq \sqrt{2Mf(x)}, \tag{3}$$

$$f'(x) \geq -\sqrt{2Mf(x)} \text{ if } M \geq \frac{2f(x)}{(C-x)^2}, \tag{4}$$

$$f'(x) \geq -\left(\frac{f(x)}{C-x} + \frac{M(C-x)}{2}\right) \text{ if } M < \frac{2f(x)}{(C-x)^2}. \tag{5}$$

If, in addition, there is a constant $B < M$ so that $B \leq f''(x)$ for all $x \in \mathbf{I}$, then f' satisfies:

$$f'(x) \geq \max\left\{Bx, -\sqrt{2Mf(x)}\right\} \text{ if } x \in (0, \tfrac{M}{M-B}C] \cap \mathbf{I} \text{ or } M \geq \frac{2f(x)}{(C-x)^2},$$

$$f'(x) \geq \max\left\{Bx, -\left(\frac{f(x)}{C-x} + \frac{M(C-x)}{2}\right)\right\} \text{ if } x \in (\tfrac{M}{M-B}C, \infty) \cap \mathbf{I} \text{ and } M < \frac{2f(x)}{(C-x)^2}.$$

Proof. Extending f' to a continuous function on $[0,C)$ with $f'(0) = 0$, the Mean Value Theorem applies for $0 < t < C$:

$$\frac{f'(t) - f'(0)}{t - 0} = f''(c_1) \implies f'(t) = f''(c_1)t \implies f'(t) \leq Mt. \tag{6}$$

Similarly, if there is a lower bound $f'' \geq B$, then

$$Bt \leq f'(t). \tag{7}$$

We refer to (6) and (7) as the linear bounds on f'.

Fix $x \in \mathbf{I}$; then, for any $y \in \mathbf{I}$, integration gives:

$$\begin{aligned}
\int_y^x \int_x^t f''(s)dsdt &= \int_y^x (f'(t) - f'(x))dt \\
&= \left[f(t) - tf'(x)\right]_{t=y}^{t=x} \\
&= (f(x) - xf'(x)) - (f(y) - yf'(x)) \\
\implies f(y) &= f(x) - (x-y)f'(x) + \int_y^x \int_t^x f''(s)dsdt \qquad (8) \\
\implies 0 &\leq f(x) - (x-y)f'(x) + \frac{1}{2}(x-y)^2 M. \qquad (9)
\end{aligned}$$

The first line is where the assumption of the continuity of f'' is used, in the Fundamental Theorem of Calculus (so, the $\mathcal{C}^2$ hypothesis may be weakened as in [CS], [NS]). Line (8) is essentially the well-known first-order Taylor polynomial with integral remainder term. The last line uses the upper bound $f'' \leq M$ and the assumption $f(y) \geq 0$. Let $\Delta x = x - y$; then $0 < y < C$ is equivalent to $x - C < \Delta x < x$, and the non-negativity of the quadratic (9) for all $y \in \mathbf{I}$ is equivalent to

$$\frac{1}{2}M(\Delta x)^2 - f'(x)\Delta x + f(x) \geq 0 \tag{10}$$

for all $\Delta x \in (x - C, x)$, which (using $M > 0$) leads to three cases.

Case 1. The quadratic (10) is always positive or has a double root:

$$(-f'(x))^2 - 4 \cdot \frac{1}{2}Mf(x) \leq 0 \implies |f'(x)| \leq \sqrt{2Mf(x)}.$$

Case 2. The quadratic (10) has distinct real zeros, the lesser one is to the right of the interval $(x - C, x)$, so $(f'(x))^2 - 2Mf(x) > 0$ and

$$\begin{aligned}
x &\leq \frac{f'(x) - \sqrt{(f'(x))^2 - 2Mf(x)}}{M} \\
\implies f'(x) &\geq Mx + \sqrt{(f'(x))^2 - 2Mf(x)},
\end{aligned}$$

however, this contradicts the linear bound $f'(x) \leq Mx$.

Case 3. The quadratic (10) has distinct real zeros, the greater one is to the left of the interval $(x - C, x)$, so $(f'(x))^2 - 2Mf(x) > 0$ and

$$\begin{aligned}
x - C &\geq \frac{f'(x) + \sqrt{(f'(x))^2 - 2Mf(x)}}{M} \\
\implies f'(x) &\leq M(x - C) - \sqrt{(f'(x))^2 - 2Mf(x)}, \qquad (11)
\end{aligned}$$

which implies $f'(x) < 0$.

$$\begin{aligned} f'(x) + M(C-x) &\leq -\sqrt{(f'(x))^2 - 2Mf(x)} < 0 \\ \implies (f'(x) + M(C-x))^2 &\geq (f'(x))^2 - 2Mf(x) \\ \implies 2f'(x)M(C-x) + M^2(C-x)^2 &\geq -2Mf(x) \\ \implies f'(x) &\geq -\frac{f(x)}{C-x} - \frac{M(C-x)}{2}. \end{aligned} \tag{12}$$

The inequalities (11) and (12) are consistent only if:

$$\begin{aligned} -\frac{f(x)}{C-x} - \frac{M(C-x)}{2} &< M(x-C) \\ \iff \frac{2f(x)}{(C-x)^2} &> M. \end{aligned}$$

In the case where $f'' \geq B$, from (11) and the linear bound (7) we can conclude

$$Bx \leq f'(x) < M(x-C),$$

and it follows that $x > \frac{M}{M-B}C$. In particular, if $B \geq 0$, Case 3 does not occur.

It follows from the arithmetic-geometric mean inequality that $\sqrt{2Mf(x)} \leq \frac{f(x)}{C-x} + \frac{M(C-x)}{2}$, so the lower bound from (5) is always less than or equal to the lower bound from (4), and they coincide when $M = \frac{2f(x)}{(C-x)^2}$. ∎

The expressions (4) and (5) can be combined into a shorter form (the verification is elementary). Let

$$s(x) = \max\left\{1, \frac{\sqrt{2f(x)}}{\sqrt{M}(C-x)}\right\}.$$

Then (under the conditions of Theorem 2.1), for all $x \in \mathbf{I}$,

$$-\sqrt{2Mf(x)} \cdot \left(\frac{1}{2s(x)} + \frac{s(x)}{2}\right) \leq f'(x) \leq \sqrt{2Mf(x)}. \tag{13}$$

Example 2.2. Considering $f(x) = \frac{1}{2}Mx^2$ on an interval $(0, C)$, the upper bound (3) in the Theorem is sharp, in the sense that for some $f(x)$, equality is achieved at every point x. However, for some other $f(x)$ such as $\frac{1}{2}Mx^2 + 1$, the linear upper bound (6) is better. So (3) could be replaced with the more precise, but equally sharp, expression:

$$f'(x) \leq \min\left\{\sqrt{2Mf(x)}, Mx\right\}.$$

In general, if $f(x) \geq P$ on $(0, C)$ for some positive constant P, substituting $f(x) - P$ for $f(x)$ will still satisfy the hypotheses of the Theorem, and will improve the bounds (3), (4), and (5) without changing f' or M.

Example 2.3. This example shows that the lower bound can also be reached, in this case by a function that vanishes at the endpoint C.

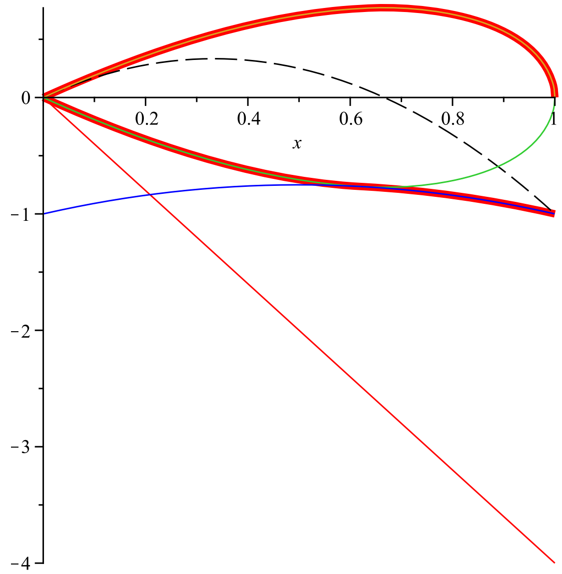

FIGURE 1. The dashed curve is $f'(x)$ from Example 2.3. The thick curves are the Theorem's upper and lower bounds. The linear lower bound is also shown.

Let $f(x) = x^2(1-x) = x^2 - x^3$, on the interval $(0,1)$. Then $f'(0) = 0$ and $f''(x) = 2 - 6x$ is bounded above by $M = 2$ and below by $B = -4$. On the interval $(0, \frac{1}{2}(\sqrt{5}-1)]$, $\frac{2f(x)}{(1-x)^2} \leq M = 2$. The Theorem's bounds (3) and (4) are:

$$-\sqrt{4(x^2 - x^3)} \leq f'(x) = 2x - 3x^2 \leq \sqrt{4(x^2 - x^3)} \tag{14}$$

on this interval. In fact, (14) holds on the larger interval $(0, \frac{8}{9}]$ and the upper bound holds on the whole interval $(0,1)$, but the lower bound from (14) is false for x close to 1. On the interval $(\frac{1}{2}(\sqrt{5}-1), 1)$, $\frac{2f(x)}{(1-x)^2} > M = 2$, and the Theorem's bounds are

$$\max\left\{-4x, -\left(\frac{f(x)}{1-x} + \frac{2(1-x)}{2}\right)\right\} = -x^2 + x - 1 \leq 2x - 3x^2 \leq \sqrt{4(x^2 - x^3)}.$$

As $x \to 1^-$, the inequalities approach $-1 \leq -1 \leq 0$.

Example 2.4. By restricting the domain of the function from the previous Example, we get an example of a function which is positive on $(0, C]$, but the inequality $|f'| \leq \sqrt{2Mf}$ does not hold for x close to C.

Let $f(x) = x^2(1-x) = x^2 - x^3$, on the interval $(0, 0.95)$. Then $f'(0) = 0$ and $f''(x) = 2 - 6x$ is bounded above by $M = 2$ and below by $B = -3.7$. The Theorem's bounds (3) and (4) are:

$$-\sqrt{4(x^2 - x^3)} \leq f'(x) = 2x - 3x^2 \leq \sqrt{4(x^2 - x^3)} \tag{15}$$

on the (approximate) interval $(0, 0.5751]$ where $\frac{2f(x)}{(0.95-x)^2} \leq M = 2$. As in the previous Example, (15) holds on $(0, \frac{8}{9}]$, but the lower bound from (15) is false for x close to 0.95. On the interval

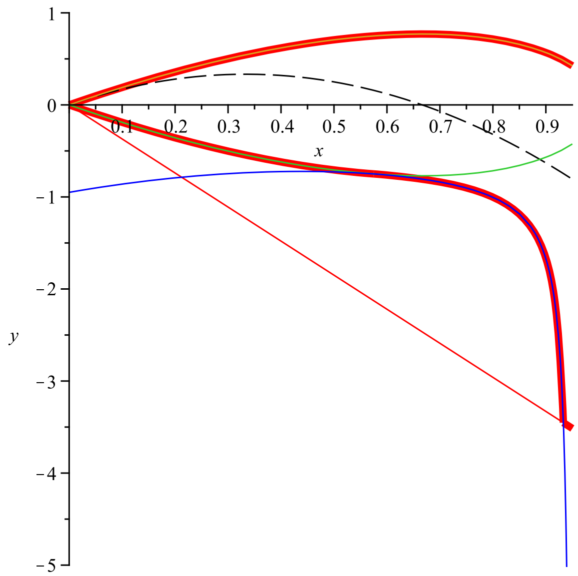

FIGURE 2. The dashed curve is $y = f'(x)$ from Example 2.4.

$(0.5751, 0.95)$, $\frac{2f(x)}{(0.95-x)^2} > M = 2$; as shown in Figure 2, the linear lower bound (7) is better than the rational lower bound (5) for x very close to $C = 0.95$:

$$\begin{aligned} -\left(\frac{f(x)}{0.95-x} + \frac{2(0.95-x)}{2}\right) &\leq 2x - 3x^2 \text{ for } x \in (0.5751, 0.933], \\ -3.7x &\leq 2x - 3x^2 \text{ for } x \in [0.933, 0.95). \end{aligned}$$

Replacing $(0, C)$ by $(0, \infty)$ in Theorem 2.1, an examination of the Proof shows that Case 3 is excluded, leading to the following simple statement.

Proposition 2.5. *Given $f : (0, \infty) \to \mathbb{R}$, suppose:*

- *$f(x) \geq 0$ for all $x \in (0, \infty)$;*
- *$f \in \mathcal{C}^2((0, \infty))$;*
- *there is a constant $M > 0$ so that $f''(x) \leq M$ for all $x \in (0, \infty)$;*
- *$\lim_{x \to 0^+} f'(x) = 0$.*

Then, for all $x \in (0, \infty)$, $-\sqrt{2Mf(x)} \leq f'(x) \leq \sqrt{2Mf(x)}$. ∎

3. Symmetric boundary conditions

If we add another limit condition at the other endpoint of $\mathbf{I} = (0, C)$, $\lim_{x \to C^-} f'(x) = 0$, then we get another linear bound $f'(x) \geq M(x - C)$, which contradicts (11), so Case 3 in the Proof of Theorem 2.1 is excluded.

Proposition 3.1. *Given $f : \mathbf{I} \to \mathbb{R}$, suppose:*

- *$f(x) \geq 0$ for all $x \in \mathbf{I}$;*
- *$f \in \mathcal{C}^2(\mathbf{I})$;*
- *there is a constant $M > 0$ so that $f''(x) \leq M$ for all $x \in \mathbf{I}$;*
- *$\lim_{x \to 0^+} f'(x) = 0$ and $\lim_{x \to C^-} f'(x) = 0$.*

Then, for all $x \in \mathbf{I}$, $-\sqrt{2Mf(x)} \leq f'(x) \leq \sqrt{2Mf(x)}$. ∎

Removing all the boundary conditions on f' leads to this version of the inequality, which we state for any interval (a, b), in the short format as in (13).

Theorem 3.2. *Given $f : (a, b) \to \mathbb{R}$, suppose:*

- *$f(x) \geq 0$ for all $x \in (a, b)$;*
- *$f \in \mathcal{C}^2((a, b))$;*
- *there is a constant $M > 0$ so that $f''(x) \leq M$ for all $x \in (a, b)$.*

Let $s_\ell(x) = \max\left\{1, \frac{\sqrt{2f(x)}}{\sqrt{M}(x-a)}\right\}$ and $s_r(x) = \max\left\{1, \frac{\sqrt{2f(x)}}{\sqrt{M}(b-x)}\right\}$. Then, for all $x \in (a, b)$,

$$-\sqrt{2Mf(x)} \cdot \left(\frac{1}{2s_r(x)} + \frac{s_r(x)}{2}\right) \leq f'(x) \leq \sqrt{2Mf(x)} \cdot \left(\frac{1}{2s_\ell(x)} + \frac{s_\ell(x)}{2}\right).$$

Proof. The inequalities from adapting the Proof of Theorem 2.1 are:

$$f'(x) \leq \sqrt{2Mf(x)} \text{ if } M \geq \frac{2f(x)}{(x-a)^2}, \tag{16}$$

$$f'(x) \leq \frac{f(x)}{x-a} + \frac{M(x-a)}{2} \text{ if } M < \frac{2f(x)}{(x-a)^2}, \tag{17}$$

$$f'(x) \geq -\sqrt{2Mf(x)} \text{ if } M \geq \frac{2f(x)}{(b-x)^2}, \tag{18}$$

$$f'(x) \geq -\left(\frac{f(x)}{b-x} + \frac{M(b-x)}{2}\right) \text{ if } M < \frac{2f(x)}{(b-x)^2}. \tag{19}$$

∎

If there is no right endpoint b, then the lower bound $-\sqrt{2Mf(x)}$ holds on (a, ∞). For $(a, b) = (-R, R)$ with midpoint $x = 0$, this is the same as the result stated in [LN]. Theorem 3.2 has Glaeser's inequality for the domain $\mathbb{R}$ as a limiting case, as $a \to -\infty$, $b \to +\infty$.

Example 3.3. The function $f(x) = \sqrt{1 - x^2}$ on the interval $(-1, 1)$ gives an example where f' is unbounded at both endpoints and f is concave down everywhere. Allowing M to decrease to 0, the estimates (16) and (18) do not apply, only (17) and (19).

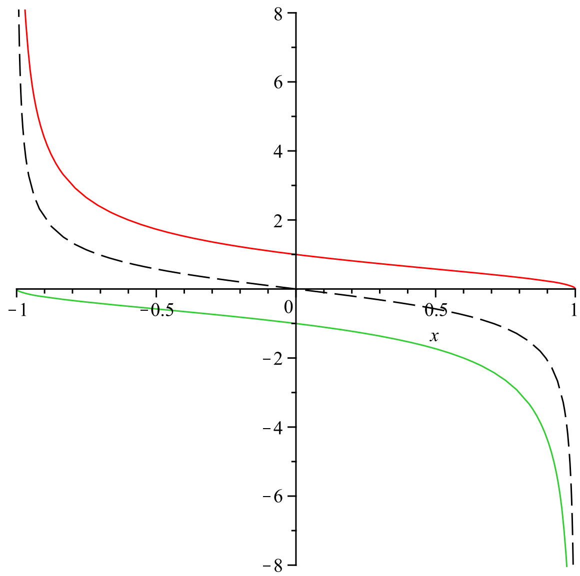

FIGURE 3. $-\frac{f(x)}{b-x} < f'(x) < \frac{f(x)}{x-a}$, from Example 3.3

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