ADDENDUM TO: CR SINGULAR IMMERSIONS OF COMPLEX PROJECTIVE SPACES

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6. Errata

The Zentralblatt Math number for reference [18] should be Zbl 0373.50005.

In reference [15], Mac Lane preferred to spell his name with a space.

7. Updates

Reference [6] will not appear under that title. The relevant material (cited on p. 468, [C_1]) can be found in Reference [4], or in [C_2] instead.

My contact information has changed, and the web address at the end of the article is now obsolete. My current Purdue University Fort Wayne home page is:

`http://users.pfw.edu/CoffmanA/`

8. Citations

The article is cited in these papers: [G], [Slapar], [Starčič].

The following Sections of this addendum are a continuation of the consideration of real manifolds immersed in $\mathbb{C}^n$. They include some of the calculations omitted from the paper [C_1].

9. $\mathbb{C}P^2$ in $\mathbb{C}P^5$, continued

This Section will examine maps from the complex projective plane to $\mathbb{C}^5$, considered as an affine neighborhood in $\mathbb{C}P^5$. The following Example will fill in some of the details from the $t = -1/2$ case of Example 5.2.

Example 9.1. Consider the following coefficient matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1+i & 0 & 0 & 0 & i \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} \end{pmatrix}.$$

The top row is chosen so that $P \circ s \circ \Delta$ will have an image contained in the $Z_0 \neq 0$ neighborhood. Deleting the top row and first, middle, and last columns leaves a

5×6 submatrix, in row-echelon form so that P has rank 6 and $\ker(P)$ is a 3-dimensional subspace of $\mathbb{C}^9$. Its last column (the eighth of nine in P) is chosen so that $k(\ker(P))$, which is the following subspace of $M(3, \mathbb{C})$:

$$\{\begin{pmatrix} -ic_1 - (1+i)c_3 & 0 & \frac{1}{2}c_1 - c_2 \\ \frac{1}{2}c_3 + c_2 & c_3 & c_2 \\ 0 & 0 & c_1 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{C}\},$$

contains no matrices of rank 1, and so $P \circ s$ is defined for all $(z, w) \in \mathbb{C}P^2 \times \mathbb{C}P^2$. Finally, the $-\frac{1}{2}$ entries, which contribute $z_1\bar{z}_1$ and $z_2\bar{z}_2$ terms to the numerators of the parametric functions, are needed so that $P \circ s \circ \Delta$ will not have a triple point.

The composition $P \circ s \circ \Delta : \mathbb{C}P^2 \to \mathbb{C}P^5$ is defined for all of $\mathbb{C}P^2$. By inspection of the parametric map taking $[z_0 : z_1 : z_2]$ to:

$$[z_0\bar{z}_0 + (1+i)z_1\bar{z}_1 + iz_2\bar{z}_2 : (z_0 - \frac{z_1}{2} - z_2)\bar{z}_1 : z_0\bar{z}_2 : z_1\bar{z}_0 : z_1\bar{z}_2 : z_2(\bar{z}_0 + \bar{z}_1 - \frac{\bar{z}_2}{2})],$$

the image of $P \circ s \circ \Delta$ does not meet the $Z_0 = 0$ hyperplane.

The singular locus of $P \circ s$ is a complex algebraic subvariety of the domain $\mathbb{C}P^2 \times \mathbb{C}P^2$. In order to find its intersection with the image of Δ, it will be enough to check the Jacobian matrix of $P \circ s$, considered as a map $\mathbb{C}^4 \to \mathbb{C}^5$ when it is restricted to three of the nine affine charts in the domain, and the $Z_0 \neq 0$ chart in the target. For example, the restriction of $P \circ s$ to the $z_0 \neq 0$, $w_0 \neq 0$ neighborhood defines a map

$$(z_1, z_2, w_1, w_2) \mapsto (\frac{P_1(z, w)}{P_0(z, w)}, \ldots, \frac{P_5(z, w)}{P_0(z, w)}).$$

The locus where the rank drops is the common zero locus of five 4×4 determinants, which will be inhomogeneous rational functions in z_1, z_2, w_1, w_2. Since the image of Δ does not meet the zero locus of the denominators (which are powers of P_0), it is enough to consider the numerators of these rational functions, and re-introduce z_0 and w_0 to get five bihomogeneous polynomials which define a subset of $\{(z, w) \in \mathbb{C}P^2 \times \mathbb{C}P^2 : z_0 \neq 0, w_0 \neq 0, P_0(z, w) \neq 0\}$. Repeating this procedure for the other charts in the domain will give other subsets of $\mathbb{C}P^2 \times \mathbb{C}P^2$, but with significant overlaps, and which satisfy the same bihomogeneous polynomial equations. Using Maple software to assist with the computations, these polynomials are:

$$z_1w_2(2w_2z_2 - 2w_2z_0 + (-2+2i)w_1z_1 + w_2z_1), \tag{9.1}$$

$$z_1w_2((1+i)w_2z_2 - 2w_1z_1 - 2z_1w_0 + w_2z_1), \tag{9.2}$$

$$\begin{aligned} &z_1((-2+2i)z_1w_0^2 + (-4+4i)z_2w_0^2 - 8w_1^2z_1 + (-2+2i)w_2w_0z_0 \\ &+(1-i)w_2z_1w_0 + (4+4i)w_1w_2z_2 + (-10+2i)w_1z_1w_0 \\ &+(4-4i)w_1w_0z_0 + (2-2i)w_2z_2w_0 + 4w_1w_2z_1 + (4-4i)z_0w_0^2), \end{aligned} \tag{9.3}$$

$$\begin{aligned} &w_2(4z_2^2w_2 + 2z_1w_2z_2 + (-4+4i)z_1w_1z_2 - (4+2i)w_2z_2z_0 - iz_0w_2z_1 \\ &+2iw_2z_0^2 + 4iz_2z_0w_0 - 4iw_1z_0^2 + 2iw_1z_1z_0 + 2iz_1z_0w_0 - 4iz_0^2w_0), \end{aligned} \tag{9.4}$$

$$\begin{aligned} &z_0(-2w_0 - 2w_1 + w_2)(2z_0w_0 - z_1w_0 - (2+2i)z_1w_1) \\ &-2z_2((1-2i)z_0w_0w_2 + iz_1w_0w_2 + 2iz_2w_2w_0 - z_1w_1w_2 - 2z_0w_0^2). \end{aligned} \tag{9.5}$$

The real diagonal image of Δ, $[w_0 : w_1 : w_2] = [\bar{z}_0 : \bar{z}_1 : \bar{z}_2]$, meets this locus in a real algebraic variety, which (again, according to Maple) consists of *exactly* three points, $x_1 = \Delta([1 : 0 : 1])$, $x_2 = \Delta([1 : -1 : 0])$, and $x_3 = \Delta([\frac{i}{2} : -i : 1])$.

To verify this, first, it is left to the reader to check that these three points are in the common zero locus of equations (9.1)–(9.5), and are indeed elements of the singular locus of $P \circ s$.

Second, suppose there is some $[z_0 : z_1 : z_2] \in \mathbb{C}P^2$ with $z_0 = 0$ and $z_1 \neq 0$, $z_2 \neq 0$, whose image under Δ satisfies equation (9.5). Then $2z_2 z_1 \bar{z}_1 \bar{z}_2 = 0$, contradicting the non-zero hypothesis, so there are no such points in the singular locus.

The next case is where $z_1 = 0$. Any point $\Delta([z_0 : 0 : z_2])$ satisfies (9.1)–(9.3), and (9.4) then implies

$$\begin{aligned} &\bar{z}_2(4z_2^2\bar{z}_2 - (4+2i)z_0 z_2 \bar{z}_2 + 2iz_0^2\bar{z}_2 + 4iz_0 z_2 \bar{z}_0 - 4iz_0^2\bar{z}_0) \\ &= 2\bar{z}_2(z_0 - z_2)(-2z_2\bar{z}_2 - 2iz_0\bar{z}_0 + iz_0\bar{z}_2) = 0. \end{aligned}$$

One of the solutions is $z_0 = z_2$, which gives the point x_1. Another could be $z_1 = z_2 = 0$, but then (9.5) would imply z_0 is also zero. A third possibility is that the last factor is zero, but in fact that quantity is nonvanishing except at $z_0 = z_2 = 0$. One way to see this is to write it as a sum of squares:

$$\begin{aligned} -2z_2\bar{z}_2 - 2iz_0\bar{z}_0 + iz_0\bar{z}_2 &= \frac{\sqrt{7} - (21 + 8\sqrt{7})i}{21}|z_0 + \frac{-1 + (8 - 3\sqrt{7})i}{4}z_2|^2 \\ &\quad + \frac{-21 - 8\sqrt{7} + \sqrt{7}i}{21}|\frac{-8 + 3\sqrt{7} - i}{4}z_0 + z_2|^2. \end{aligned}$$

The next case is $z_2 = 0$, $z_1 \neq 0$, so that (9.5) becomes

$$2z_0(\bar{z}_0 + \bar{z}_1)(-2z_0\bar{z}_0 + z_1\bar{z}_0 + (2+2i)z_1\bar{z}_1) = 0.$$

One of the solutions is $z_1 = -z_0$, which gives x_2, and another, $z_0 = z_2 = 0$, would imply $z_1 = 0$ when substituted into (9.3). Also, as in the previous case, the last factor is a sum of squares with no nonzero solutions:

$$\begin{aligned} &-2z_0\bar{z}_0 + z_1\bar{z}_0 + (2+2i)z_1\bar{z}_1 \\ = \quad &\frac{-47 + 8\sqrt{47} + \sqrt{47}i}{47}|z_0 + \frac{-1 + (-8 - \sqrt{47})i}{4}z_1|^2 \\ &+ \frac{47 - 9\sqrt{47} + (47 - 7\sqrt{47})i}{47}|\frac{-7 - \sqrt{47} + (9 + \sqrt{47})i}{8}z_0 + z_1|^2. \end{aligned}$$

Finally, the remaining case is that all three projective coordinates are nonzero, so that z_2 can be assumed to be 1, and equating the nonlinear factors of (9.1) and (9.2) to zero gives:

$$\begin{aligned} 0 &= 2 - 2z_0 + (-2+2i)\bar{z}_1 z_1 + z_1, \\ 0 &= (1+i) - 2\bar{z}_1 z_1 - 2z_1\bar{z}_0 + z_1. \end{aligned}$$

This seems to be a difficult system to solve by hand. One approach might be to take the real and imaginary parts, $z_0 = u + iv$, $z_1 = x + iy$, to get four real quadratic equations in u, v, x, y, and then find the real solutions from a (computer assisted) calculation of a standard basis. Another might be to solve the first equation for z_0 and substitute into the second, to get two real cubic equations in x, y. Then inspecting a graph of real cubic curves will show that their only common solution is $(x, y) = (0, -1)$. So, $z_1 = x + iy = -i$ gives the point x_3.

The images of the three points, x_1, x_2, x_3, under $P \circ s$ are

$$X_1 = [1 : 0 : \frac{1-i}{2} : 0 : 0 : \frac{1-i}{4}],$$

$$X_2 = [1 : \frac{-6+3i}{10} : 0 : \frac{i-2}{5} : 0 : 0],$$

$$X_3 = [1 : \frac{-52+12i}{89} : \frac{16+10i}{89} : \frac{-10+16i}{89} : \frac{-32-20i}{89} : \frac{6+26i}{89}],$$

which are the candidates for complex jump points in the image of $\mathbb{C}P^2$. (They are also candidates for differential-topological singularities.)

The real tangent planes at these points are found by considering the restriction of $P \circ s \circ \Delta$ to the $z_0 = 1$ affine neighborhood, so that $P \circ s \circ \Delta : \mathbb{R}^4 \to \mathbb{R}^{10}$ is given by

$$(z_1, z_2) \mapsto (\mathrm{Re}(\frac{P_1}{P_0}), \mathrm{Im}(\frac{P_1}{P_0}), \ldots, \mathrm{Im}(\frac{P_5}{P_0})),$$

and at each point z in the domain, there is a real 10×4 Jacobian matrix D_z of derivatives whose image is a four-dimensional subspace T_z of $\mathbb{R}^{10}$.

It turns out that at each point x_1, x_2, x_3, the real Jacobian matrix has full rank. This is enough to prove that $P \circ s \circ \Delta$ is an immersion.

In $\mathbb{C}^5$, the scalar multiplication map $\vec{v} \mapsto i \cdot \vec{v}$ is real-linear, and induces a complex structure operator J on $\mathbb{R}^{10}$, which is a 10×10 block matrix with five $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ blocks on the diagonal. The concatenation of D_z with $J \cdot \mathrm{D}_z$, a 10×8 matrix, maps $\mathbb{R}^8$ to $\mathbb{R}^{10}$ so that the image subspace is the sum $T_z + JT_z$; it is 8-dimensional at totally real points where T_z and JT_z meet only at the origin, but 6-dimensional at the three complex jump points. So, x_1, x_2, x_3 are not "exceptionally exceptional," that is, none of the tangent spaces is a complex 2-plane, but instead each contains exactly one complex line. In the notation of Section 3, $N_1 = \{x_1, x_2, x_3\}$, and $N_2 = \emptyset$.

To illustrate the idea, the procedure for finding T_z will be recorded here only for $z = x_1$.

$$\mathrm{D}_{x_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & -1/2 \\ 0 & 0 & -1/2 & -1/2 \\ 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 0 & 1/2 \end{pmatrix}$$

has rank 4, but $[\mathrm{D}_{x_1}, J \cdot \mathrm{D}_{x_1}]_{10 \times 8}$ has rank 6. A basis for its kernel is

$$\{(1, -3, 4, -4, 3, 1, 0, 0)^T, (3, 1, 0, 0, -1, 3, -4, 4)^T\},$$

and the following equation shows that the image of D_{x_1} contains a J-invariant subspace:

$$\mathrm{D}_{x_1}\cdot\begin{pmatrix}1\\-3\\4\\-4\end{pmatrix}=J\cdot\mathrm{D}_{x_1}\cdot\begin{pmatrix}-3\\-1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\\-1\\-2\\-1\\-2\\-2\\-1\end{pmatrix}=J\cdot\begin{pmatrix}0\\0\\0\\0\\-2\\1\\-2\\1\\-1\\2\end{pmatrix}.$$

The span of $(0,0,0,0,-1,-2,-1,-2,-2,-1)^T$ and its image under J is the complex line $\{Z_1=Z_2=Z_3-Z_4=(\frac{4}{5}-\frac{3}{5}i)Z_3-Z_5=0\}$. Similar calculations for x_2 and x_3 yield different complex lines tangent to $\mathbb{C}P^2$ in $\mathbb{C}^5$.

It remains to check that $P\circ s\circ\Delta$ is one-to-one except for two double points. First, consider those points in the image such that all three coordinates, z_0, z_1, and z_2, in the domain are nonzero, and $P\circ s\circ\Delta$ restricts to a map:

$$[z_0:z_1:1]\mapsto[\frac{P_0}{z_1}:\frac{P_1}{z_1}:\frac{z_0}{z_1}:\bar{z}_0:1:\frac{P_5}{z_1}].$$

This map is clearly one-to-one from $\{(z_0,z_1)\in\mathbb{C}^2: z_0\neq 0, z_1\neq 0\}$ to $\mathbb{C}^5$. Points on the line $\{[0:z:1]\}$ are mapped to

$$[(1+i)z\bar{z}+i:(-z/2-1)\bar{z}:0:0:z:\bar{z}-\frac{1}{2}],$$

and a calculation will show this restriction is one-to-one, with an image disjoint from the previous image. The next subset of the domain is $\{[z:0:1]\}$, whose points are mapped to

$$[z\bar{z}+i:0:z:0:0:\bar{z}-\frac{1}{2}].$$

This restriction has an image disjoint from the first image, and from the second image, except for their point of intersection $[0:0:1]$ in the domain. However, it is not one-to-one: both values $z=\frac{1}{4}+(-2\pm\frac{3}{4}\sqrt{7})i$ are mapped to the same image point. Another line in the domain is $\{[1:z:0]\}$, whose points are mapped to

$$[1+(1+i)z\bar{z}:(1-z/2)\bar{z}:0:z:0:0];$$

the image is disjoint from the previous three images, but this restriction is also not one-to-one, since $z=\frac{7+\sqrt{47}}{8}+\frac{-9-\sqrt{47}}{8}i$ and $z=\frac{7-\sqrt{47}}{8}+\frac{-9+\sqrt{47}}{8}i$ have the same image. The only remaining point in the domain is $[0:1:0]$, whose image, $[1+i:-\frac{1}{2}:0:0:0:0]$, is not in any of the above images. As in Example 5.1, $P\circ s\circ\Delta$ maps the complex projective lines $\{[z_0:0:z_2]\}$ and $\{[z_0:z_1:0]\}$ in the domain into two-dimensional complex subspaces in the range, falling into the Example 4.9 case of the classification from Theorem 4.3.

Example 9.2. Consider the coefficient matrix from Example 5.3:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 1+i & 0 & 0 & 0 & i \\ -1 & 1 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -1 & \frac{1}{9} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} \end{pmatrix}.$$

The entries differ from the previous P matrix only by contributing some more $z_i\bar{z}_i$ terms to the quadratic polynomials in the parametrization. The composite map $(P \circ s \circ \Delta)([z_0 : z_1 : z_2])$ is:

$$\begin{aligned} [z_0\bar{z}_0 + (1+i)z_1\bar{z}_1 + iz_2\bar{z}_2 \quad &: \quad (z_0 - \frac{1}{2}z_1 - z_2)\bar{z}_1 - z_0\bar{z}_0 + \frac{1}{9}z_2\bar{z}_2 \\ &: \quad z_0\bar{z}_2 \\ &: \quad z_1\bar{z}_0 \\ &: \quad z_1\bar{z}_2 \\ &: \quad z_2(\bar{z}_0 + \bar{z}_1 - \frac{1}{2}\bar{z}_2) - \frac{3}{2}z_0\bar{z}_0]. \end{aligned}$$

It is defined for all of $\mathbb{C}P^2$, and its image does not meet the $Z_0 = 0$ hyperplane. $k(\ker(P))$ is the following subspace of $M(3, \mathbb{C})$:

$$\{\begin{pmatrix} 18c_2 & 0 & -c_3 + (45+7i)c_2 + (2+7i)c_1 \\ c_3 & 18c_1 & c_3 + (-18+2i)c_2 + (-11+2i)c_1 \\ 0 & 0 & 18ic_2 + (-18+18i)c_1 \end{pmatrix} : c_1, c_2, c_3 \in \mathbb{C}\},$$

and it contains no matrices of rank 1, so $P \circ s$ is defined for all $(z, w) \in \mathbb{C}P^2 \times \mathbb{C}P^2$.

The singular locus of $P \circ s$ is defined by bihomogeneous polynomial equations as in the previous examples. Using Maple software to assist with the computations, these polynomials are:

$$z_1w_2((-18+18i)w_1z_1 + (11-2i)w_2z_1 + 18w_2z_2 - 18w_2z_0), \tag{9.6}$$

$$z_1w_2((-1-i)w_2z_2 + 2w_1z_1 - w_2z_1 + 2z_1w_0), \tag{9.7}$$

$$\begin{aligned} &z_1((-36-36i)w_1^2z_1 + (18+18i)w_2w_1z_1 + (36+54i)w_1z_1w_0 \\ &+(-15-51i)w_2z_1w_0 + (18+36i)z_1w_0^2 + (14-90i)w_2z_2w_0 \\ &+36iw_1w_2z_2 - 36z_2w_0^2 + 36w_1w_0z_0 + (54i-18)w_2w_0z_0 + 36w_0^2z_0), \end{aligned} \tag{9.8}$$

$$\begin{aligned} &w_2((-36-36i)z_2w_1z_1 + (4+22i)w_2z_1z_2 + (72+90i)w_1z_1z_0 \\ &+(-15-51i)w_2z_1z_0 + (18+36i)z_1z_0w_0 + (14-126i)w_2z_2z_0 \\ &+36iw_2z_2^2 - 36z_2z_0w_0 + 36w_1z_0^2 + (-18+54i)w_2z_0^2 + 36w_0z_0^2), \end{aligned} \tag{9.9}$$

$$\begin{aligned} &(-14+4i)w_2z_2w_1z_1 + (4+22i)w_2z_1z_2w_0 + (-36-36i)w_1^2z_1z_0 \\ &+(18+18i)w_2w_1z_1z_0 + (36+54i)w_1z_1w_0z_0 + (-15-51i)w_2z_1w_0z_0 \\ &+(18+36i)z_1z_0w_0^2 + 36iw_2z_2^2w_0 + (14-126i)w_2z_2z_0w_0 \\ &-36z_2z_0w_0^2 + 36w_1w_0z_0^2 + (-18+54i)w_2w_0z_0^2 + 36w_0^2z_0^2. \end{aligned} \tag{9.10}$$

The real diagonal image of Δ, $[w_0 : w_1 : w_2] = [\bar{z}_0 : \bar{z}_1 : \bar{z}_2]$, meets this locus in *exactly* three points, $x_1 = \Delta([1:0:3])$, $x_2 = \Delta([1:2:0])$, and $x_3 = \Delta([9+28i : -18-63i : 54-30i])$.

It is easy to check that these three points are in the common zero locus of equations (9.6)–(9.10), and are indeed elements of the singular locus of $P \circ s$.

Suppose there is some $[z_0 : z_1 : z_2] \in \mathbb{C}P^2$ with $z_0 = 0$ and $z_1 \neq 0$, $z_2 \neq 0$, whose image under Δ satisfies equation (9.10). Then $z_1\bar{z}_1z_2\bar{z}_2 = 0$, contradicting the non-zero hypothesis, so there are no such points in the singular locus.

The next case is where $z_1 = 0$. Any point $\Delta([z_0 : 0 : z_2])$ satisfies (9.6)–(9.8), and (9.9) then implies

$$\bar{z}_2(36 + 36i\bar{z}_2z_2^2 + (14 - 126i)\bar{z}_2z_2 - 36z_2 + (-18 + 54i)\bar{z}_2) = 0.$$

z_1 and z_2 cannot both be 0, by (9.10). A solution is $z_2 = 3$, giving the point x_1. It is not obvious on inspection, and the sum of squares trick from the previous Example doesn't immediately apply, but it can be checked that $z_2 = 3$ is the only solution in this case.

The next case is $z_2 = 0$, $z_1 \neq 0$, so that (9.10) becomes

$$18z_0(2\bar{z}_1\bar{z}_0z_0 + (2 + 3i)\bar{z}_0z_1\bar{z}_1 + 2\bar{z}_0^2z_0 + (-2 - 2i)z_1\bar{z}_1^2 + (1 + 2i)\bar{z}_0^2z_1).$$

A solution is $z_0 = 1$, $z_1 = 2$, which gives x_2, and another, $z_0 = z_2 = 0$, would imply $z_1 = 0$ when substituted into (9.8). As in the previous case, the x_2 solution is unique.

Finally, the remaining case is that all three projective coordinates are nonzero, so that z_2 can be assumed to be 1, and equating the nonlinear factors of (9.6) and (9.7) to zero gives:

$$\begin{aligned} 0 &= (-18 + 18i)\bar{z}_1z_1 + (11 - 2i)z_1 + 18 - 18z_0, \\ 0 &= -1 - i + 2\bar{z}_1z_1 - z_1 + 2z_1\bar{z}_0. \end{aligned}$$

This seems to be a difficult system to solve by hand. One approach might be to take the real and imaginary parts, $z_0 = u+iv$, $z_1 = x+iy$, to get four real quadratic equations in u, v, x, y, and then find the real solutions from a (computer assisted) calculation of a standard basis. Another might be to solve the first equation for z_0 and substitute into the second, to get two real cubic equations in x, y. Then inspecting a graph of real cubic curves will show that they meet only once, giving the point x_3.

The images of the three points, x_1, x_2, x_3, under $P \circ s$ are

$$\begin{aligned} X_1 &= [1 + 9i : 0 : 3 : 0 : 0 : -3], \\ X_2 &= [5 + 4i : -1 : 0 : 2 : 0 : -\frac{3}{2}], \\ X_3 &= [10316 + 16218i : -10863 - 7758i : -708 + 3564i \\ &\quad : -3852 - 126i : 1836 - 7884i : -5283 + 4320i], \end{aligned}$$

which are the candidates for complex jump points in the image of $\mathbb{C}P^2$.

The real tangent planes at these points are found by considering the restriction of $P \circ s \circ \Delta$ to the $z_0 = 1$ affine neighborhood, so that $P \circ s \circ \Delta : \mathbb{R}^4 \to \mathbb{R}^{10}$ is given by

$$(z_1, z_2) \mapsto (\mathrm{Re}(\frac{P_1}{P_0}), \mathrm{Im}(\frac{P_1}{P_0}), \ldots, \mathrm{Im}(\frac{P_5}{P_0})),$$

and at each point z in the domain, there is a real 10×4 Jacobian matrix D_z of derivatives whose image is a four-dimensional subspace T_z of $\mathbb{R}^{10}$.

It turns out that at each point x_1, x_2, x_3, the real Jacobian matrix has full rank. This is enough to prove that $P \circ s \circ \Delta$ is an immersion.

The concatenation of D_z with $J \cdot \mathrm{D}_z$ gives a 10×8 matrix, which maps $\mathbb{R}^8$ to $\mathbb{R}^{10}$ so that the image subspace is the sum $T_z + JT_z$; it is 8-dimensional at totally real points where T_z and JT_z meet only at the origin, but 6-dimensional at the three complex jump points. So, x_1, x_2, x_3 are not "exceptionally exceptional," that is, none of the tangent spaces is a complex 2-plane, but instead each contains exactly one complex line. In the notation of Section 3, $N_1 = \{x_1, x_2, x_3\}$, and $N_2 = \emptyset$.

The procedure for finding T_z will be recorded here only for $z = x_1$.

$$\mathrm{D}_{x_1} = \begin{pmatrix} -1/41 & 9/41 & 1/123 & 0 \\ 9/41 & 1/41 & -3/41 & 0 \\ 0 & 0 & -121/3362 & -9/82 \\ 0 & 0 & 351/3362 & -1/82 \\ 1/82 & 9/82 & 0 & 0 \\ -9/82 & 1/82 & 0 & 0 \\ 3/82 & 27/82 & 0 & 0 \\ -27/82 & 3/82 & 0 & 0 \\ 3/82 & -27/82 & 40/1681 & 9/82 \\ -27/82 & -3/82 & 9/1681 & 1/82 \end{pmatrix}$$

has rank 4, but $[\mathrm{D}_{x_1}, J \cdot \mathrm{D}_{x_1}]_{10\times 8}$ has rank 6. A basis for its kernel is

$$\{(1, 0, 6, -54/41, 0, 1, 0, 240/41)^T, (0, -1, 0, -240/41, 1, 0, 6, -54/41)^T\},$$

and the following equation shows that the image of D_{x_1} contains a J-invariant subspace:

$$\mathrm{D}_{x_1} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{240}{41} \end{pmatrix} = J \cdot \mathrm{D}_{x_1} \cdot \begin{pmatrix} 1 \\ 0 \\ 6 \\ -\frac{54}{41} \end{pmatrix} = \begin{pmatrix} \frac{9}{41} \\ \frac{1}{41} \\ -\frac{1080}{1681} \\ -\frac{120}{1681} \\ \frac{9}{82} \\ \frac{1}{82} \\ \frac{27}{82} \\ \frac{3}{82} \\ \frac{1053}{3362} \\ \frac{117}{3362} \end{pmatrix} = J \cdot \begin{pmatrix} \frac{1}{41} \\ -\frac{9}{41} \\ -\frac{120}{1681} \\ \frac{1080}{1681} \\ \frac{1}{82} \\ -\frac{9}{82} \\ \frac{3}{82} \\ -\frac{27}{82} \\ \frac{117}{3362} \\ -\frac{1053}{3362} \end{pmatrix}.$$

At x_2, the complex line in the tangent space is spanned by $\vec{v}_2 = \mathrm{D}_{x_2} \cdot (1, 7, 0, -2)^T$ and $J \cdot \vec{v}_2$, and at x_3, the complex line in the tangent space is spanned by $\vec{v}_3 = \mathrm{D}_{x_2} \cdot (-865/756, 0, 703/504, 1)^T$ and $J \cdot \vec{v}_3$.

It remains to check that $P \circ s \circ \Delta$ is one-to-one. First, consider those points in the image such that all three coordinates, z_0, z_1, and z_2, in the domain are nonzero, and $P \circ s \circ \Delta$ restricts to a map:

$$[z_0 : z_1 : 1] \mapsto [\frac{P_0}{z_1} : \frac{P_1}{z_1} : \frac{z_0}{z_1} : \bar{z}_0 : 1 : \frac{P_5}{z_1}].$$

This map is clearly one-to-one from $\{(z_0, z_1) \in \mathbb{C}^2 : z_0 \neq 0, z_1 \neq 0\}$ to $\mathbb{C}^5$. Points on the line $\{[0 : z : 1]\}$ are mapped to

$$[(1+i)z\bar{z} + i : (-\frac{1}{2}z - 1)\bar{z} + \frac{1}{9} : 0 : 0 : z : \bar{z} - \frac{1}{2}],$$

and points on the line $\{[z : 0 : 1]\}$ are mapped to

$$[z\bar{z} + i : \frac{1}{9} - z\bar{z} : z : 0 : 0 : \bar{z} - \frac{1}{2} - \frac{3}{2}z\bar{z}].$$

The images of these restrictions are disjoint from the previous image, and from each other, excepting their point of intersection $[0:0:1]$ in the domain. Some calculations will show they are one-to-one. Another line in the domain is $\{[1:z:0]\}$, whose points are mapped to

$$[1+(1+i)z\bar{z}:\bar{z}-\frac{1}{2}z\bar{z}-1:0:z:0:-\frac{3}{2}],$$

and this image is also disjoint from the previous images, and obviously one-to-one. The only remaining point in the domain is $[0:1:0]$, whose image, $[1+i:-\frac{1}{2}:0:0:0:0]$, is not in any of the above images.

Example 9.3. This will give a few more details on the matrix from Example 5.4.

$$P=\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The first five rows are the coefficients of Whitney's embedding of $\mathbb{C}P^2$ in $\mathbb{C}^4$, from Example 2.5. Adding the last row makes a rank 6 matrix, and defines a map $P\circ s:\mathbb{C}P^2\times\mathbb{C}P^2\to\mathbb{C}P^5$ taking $([z_0:z_1:z_2],[w_0:w_1:w_2])$ to:

$$[w_0z_0+w_1z_1+w_2z_2:w_2z_1:w_0z_2:w_1z_0:w_0z_0-w_1z_1:w_1z_2].$$

The kernel of P is a three-dimensional subspace of $\mathbb{C}^9$, and the kernel's image under k is the set

$$\{\begin{pmatrix} c_1 & c_3 & 0 \\ 0 & c_1 & 0 \\ c_2 & 0 & -2c_1 \end{pmatrix}:c_1,c_2,c_3\in\mathbb{C}\}.$$

The matrices in this subspace with rank ≤ 1 form exactly two lines, where $c_1=c_2=0$, or $c_1=c_3=0$. The first line is spanned by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\cdot(0,1,0),$$

so $P\circ s$ is not defined at the point $x_0=([0:1:0],[1:0:0])$. The only other point at which $P\circ s$ is not defined is $x_1=([1:0:0],[0:0:1])$.

The composition $P\circ s\circ\Delta:\mathbb{C}P^2\to\mathbb{C}P^5$ is defined for all of $\mathbb{C}P^2$, since x_0 and x_1 are not in the image of Δ. By inspection of the parametric map taking $[z_0:z_1:z_2]$ to:

$$[z_0\bar{z}_0+z_1\bar{z}_1+z_2\bar{z}_2:z_1\bar{z}_2:z_2\bar{z}_0:z_0\bar{z}_1:z_0\bar{z}_0-z_1\bar{z}_1:z_2\bar{z}_1],$$

the image of $P\circ s\circ\Delta$ does not meet the $Z_0=0$ hyperplane, and in the affine neighborhood with the coordinate system $[1:Z_1:Z_2:Z_3:Z_4:Z_5]$, the image is contained in the 7-dimensional real subspace $\{Z_4=\bar{Z}_4,\ Z_5=\bar{Z}_1\}$. $P\circ s\circ\Delta$ is a one-to-one immersion, since it is a smooth graph over Whitney's example.

The singular locus of $P\circ s$ is a complex analytic subvariety of the domain $(\mathbb{C}P^2\times\mathbb{C}P^2)\setminus\{x_0,x_1\}$. In order to find its intersection with the image of Δ, it will be enough to check the Jacobian matrix of $P\circ s$, restricted to three of the nine affine charts in the domain, and the $Z_0 \neq 0$ chart in the target. The bihomogeneous polynomial defining equations are

$$-z_0w_0w_2z_2 - w_1z_1w_2z_2 + 2z_0w_0w_1z_1,$$
$$w_1z_2(w_2z_2 - w_1z_1),$$
$$w_0z_2(w_2z_2 - 2w_1z_1),$$
$$-z_0w_1(w_2z_2 - 2w_1z_1),$$
$$-w_1^2z_2^2.$$

The real diagonal image of Δ meets this locus at exactly three points, $x_2 = \Delta([1 : 0 : 0])$, $x_3 = \Delta([0 : 1 : 0])$, and $x_4 = \Delta([0 : 0 : 1])$. The images of these three points under $P \circ s$ are

$$X_2 = [1 : 0 : 0 : 0 : 1 : 0],$$
$$X_3 = [1 : 0 : 0 : 0 : -1 : 0],$$
$$X_4 = [1 : 0 : 0 : 0 : 0 : 0],$$

which are the candidates for complex jump points in the embedded $\mathbb{C}P^2$.

The real tangent planes at these points are found by considering the image of the real jacobian map of restrictions to affine neighborhoods, $P \circ s \circ \Delta : \mathbb{R}^4 \to \mathbb{R}^{10}$. At X_3, the tangent 4-plane is $\{Z_2 = Z_4 = 0, Z_5 = \bar{Z}_1\}$, which contains the Z_3-axis. At X_4, the tangent 4-plane is $\{Z_3 = Z_4 = 0, Z_5 = \bar{Z}_1\}$, which contains the Z_2-axis. The unusual point is X_2, where the tangent space is the complex 2-plane $\{Z_1 = Z_4 = Z_5 = 0\}$, which, by the codimension formula for complex tangents, is a topologically unstable phenomenon; this submanifold is not in general position.

10. Embedding S^4 in $\mathbb{C}^n$

There is a more direct, algebraic approach to finding complex tangents of real algebraic varieties, without looking at the complexification. The idea is to work with the implicit equations (which could be derived from a parametric map), and compute the equations for the tangent bundle and its image under J. The intersection will be the union of the variety and the complex tangent lines. This method will be demonstrated for some standard embeddings of spheres in affine space. Let $\mathbb{R}^n$ have coordinates $\{e_1, \dots, e_n\}$, and consider it as the $e_0 = 1$ neighborhood of $\mathbb{R}P^n$.

Example 10.1. S^4, as a smooth hypersurface in $\mathbb{R}^5$, is given by the equation $\sum_{i=1}^{5} e_i^2 = 1$. As a real projective algebraic variety, it is the zero set of the quadric

$$-e_0^2 + e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_5^2.$$

$\mathbb{R}^5$, considered as the subspace $e_6 = 0$ of the space

$$F^3 = (\mathbb{R}^6, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}),$$

contains, like any other hyperplane, a J-invariant subspace of real dimension 4, in this case spanned by $\{e_1, e_2, e_3, e_4\}$.

Let $\{r_1, \ldots, r_6\}$ denote the fiber coordinates in the real tangent bundle TF of F; TF has the obvious almost complex structure, also denoted J. S^4 is the affine variety defined by the ideal $(\sum_{i=1}^{5} e_i^2 - 1, e_6)$.

The tangent bundle TS^4 is a subbundle of the restriction of TF to S^4. Generic fibers of TS^4 will have 1 complex direction. The expected codimension of the set of CR singular points is 4; at such singular points p, T_pS^4 is a complex 2-subspace of (T_pF, J_p).

TS^4 is also an affine variety in TF, given by the ideal (f_1, f_2, g_1, g_2):

$$\begin{aligned} f_1 &= \sum_{i=1}^{5} e_i^2 - 1 \\ f_2 &= e_6 \\ g_1 &= 2e_1r_1 + 2e_2r_2 + 2e_3r_3 + 2e_4r_4 + 2e_5r_5 \\ g_2 &= r_6. \end{aligned}$$

In general, the $\{f_i\}$ are the equations (in $e_1, \ldots, e_{2n}$) defining the variety, and g_i is the polynomial $\left(\dfrac{\partial f_i}{\partial e_b}\right) \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}$.

The operator J acts on each fiber of TS^4 and the image JTS^4 is another affine variety in TF, given by the ideal (f_1, f_2, g_1', g_2'):

$$\begin{aligned} g_1' &= -2e_2r_1 + 2e_1r_2 - 2e_4r_3 + 2e_3r_4 + 2e_5r_6 \\ g_2' &= -r_5. \end{aligned}$$

In general, g_i' is the polynomial $\left(\dfrac{\partial f_i}{\partial e_b}\right) J \begin{pmatrix} r_1 \\ \vdots \\ r_{2n} \end{pmatrix}$.

The intersection $TS^4 \cap JTS^4$ is the variety given by the ideal generated by the concatenation of the generating sets. Some simplification occurs when Macaulay computes a standard (Gröbner) basis of the ideal. In particular, redundant polynomials are removed, and terms involving r_5 and r_6 drop out since these variables are already in the ideal. The ideal defining $TS^4 \cap JTS^4$ is given by the basis

$$\begin{aligned} f_1 &= \sum_{i=1}^{5} e_i^2 - 1 \\ f_2 &= e_6 \\ g_1'' &= e_1r_1 + e_2r_2 + e_3r_3 + e_4r_4 \\ g_2'' &= e_2r_1 - e_1r_2 + e_4r_3 - e_3r_4 \\ g_3'' &= r_5 \\ g_4'' &= r_6. \end{aligned}$$

Obviously, the intersection contains $S^4 \times \{\vec{0}\}$. In a given fiber T_pF, the g'' equations define four planes meeting transversely, and so in a 2-dimensional intersection, except when $e_1 = e_2 = e_3 = e_4 = 0$, in which case the intersection is 4-dimensional. These exceptional planes are tangent to S^4 at $(0,0,0,0,\pm 1,0)$, and are the J-invariant planes parallel to the complex subspace $e_5 = e_6 = 0$ in F. The orientations of these two planes, inherited from any given orientation on S^4, are opposite. One is complex, the other "anticomplex."

Example 10.2. For a generic embedding of S^4 in $\mathbb{C}^4$, the previous embedding in $\mathbb{C}^3$ could be composed with a real-linear map into $\mathbb{C}^4$ so that the image of the original ambient $\mathbb{R}^5$ has one complex direction in $\mathbb{C}^4$. One could also consider $\mathbb{R}^5$ with some operator giving it the structure $\mathbb{C} \oplus \mathbb{R}^3$. The following approach will instead include $\mathbb{R}^5$ in $\mathbb{R}^5 \oplus \mathbb{R}^3 \cong \mathbb{C}^4$, and then use a complex structure operator so that the space spanned by $\{e_1, \ldots, e_5\}$ is generic with respect to the new operator. For example, let

$$F^4 = (\mathbb{R}^8, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}).$$

The defining ideal for TS^4 is essentially the same as before:

$$\begin{aligned} f_1 &= \sum_{i=1}^{5} e_i^2 - 1 \\ f_2 &= e_6 \\ f_3 &= e_7 \\ f_4 &= e_8 \\ g_1 &= 2e_1r_1 + 2e_2r_2 + 2e_3r_3 + 2e_4r_4 + 2e_5r_5 \\ g_2 &= r_6 \\ g_3 &= r_7 \\ g_4 &= r_8. \end{aligned}$$

The ideal for JTS^4 is generated by (f_i, g_i'), with:

$$\begin{aligned} g_1' &= -2e_2r_1 + 2e_1r_2 + 2e_5r_6 + 2e_3r_7 + 2e_4r_8 \\ g_2' &= -r_5 \\ g_3' &= -r_3 \\ g_4' &= -r_4. \end{aligned}$$

The standard basis of the ideal defining their intersection is

$$(f_1, e_6, e_7, e_8, r_3, r_4, r_5, r_6, r_7, r_8, e_1r_1 + e_2r_2, e_2r_1 - e_1r_2).$$

Most tangent planes are totally real; at each $p \in S^4$, the equations in the fiber coordinates are eight intersecting planes. At CR singular points, the intersection is not transverse; the planes intersect in a J-invariant 2-plane exactly when $e_1 = e_2 =$

0. Along this set N_1, the complex tangents are the planes spanned by $\{r_1, r_2\}$, parallel to the complex subspace spanned by $\{(1,0,\ldots,0),(0,1,0,\ldots,0)\}$ in F^4. This is a trivial bundle, consistent with the formula of [Webster] for the chern class: $\int_{N_1} c_1 H^1 = p_1 S^4 = 0$.

Using the analogous embedding in higher dimensions and a similar complex structure operator, a sphere S^n in $\mathbb{C}^n$ has complex tangent locus N_1 equal to the $(n-2)$-sphere, $e_1 = e_2 = 0$, and $N_2 = \emptyset$. H^1 is still trivial, and normal to the tangent bundle TN_1, again consistent with characteristic class formulas, as is the nonexistence of higher-order complex tangents.

Example 10.3. Any submanifold immersed in a totally real $\mathbb{R}^5$ is totally real. The computations in the previous examples could be carried out again for S^4 and a suitable complex structure on $\mathbb{C}^5$, and then the varieties TS^4 and JTS^4 would intersect exactly in $S^4 \times \{\vec{0}\}$. Similarly, S^n embeds as a totally real, real algebraic submanifold of $\mathbb{C}^{n+1}$ for $n \geq 0$.

11. Immersed Spheres

For purposes of comparison to Example 4.9, these examples will review some totally real immersions of spheres. The alternative method for detecting complex tangents will work on these examples also.

Given the n-sphere $x_1^2 + \ldots + x_n^2 + a^2 = 1$ in $\mathbb{R}^{n+1}$, [Weinstein] gave the following totally real immersion of this sphere into $\mathbb{C}^n$:

$$j(x_1, \ldots, x_n, a) = (x_1(1+2ia), \ldots, x_n(1+2ia)).$$

The image has a single self-intersection at the origin. By eliminating the a variable from the immersion map, the image can be described as a real algebraic variety. The cases $n = 1$ and $n = 2$ are reviewed here, the remaining being similar. $\mathbb{C}^n$ will have coordinates $z_r = x_r + iy_r$.

Example 11.1. Eliminating a from the equations $x_1^2 + a^2 = 1$ and $y_1 = 2ax_1$ gives the single equation $y_1^2 = 4x_1^2(1 - x_1^2)$, an "eight curve" in $\mathbb{C}^1$ ([L]). This is the prototypical transverse self-intersection for the higher dimensional spheres.

Example 11.2. The equations are homogenized by introducing x_0. Eliminating a from the equations $x_1^2 + x_2^2 + a^2 = x_0^2$, $x_0 y_1 = 2ax_1$, and $x_0 y_2 = 2ax_2$ gives the following ideal:

$$\begin{gathered}
x_0(x_2 y_1 - x_1 y_2) \\
x_0^2 y_1^2 - 4x_1^2(x_0^2 - x_1^2 - x_2^2) \\
x_0^2 y_2^2 - 4x_2^2(x_0^2 - x_1^2 - x_2^2) \\
x_0^2 y_1 y_2 - 4x_1 x_2(x_0^2 - x_1^2 - x_2^2) \\
(x_2 y_1 - x_1 y_2)(x_1^2 + x_2^2).
\end{gathered}$$

Considering only the affine subset $x_0 = 1$, the following three polynomials define the same zero set:

$$\begin{aligned}
f_1 &= x_2 y_1 - x_1 y_2 \\
f_2 &= y_1^2 - 4x_1^2(1 - x_1^2 - x_2^2) \\
f_3 &= y_2^2 - 4x_2^2(1 - x_1^2 - x_2^2).
\end{aligned}$$

Banchoff and Farris (reference [2] of [C_1]) use a similar immersion in the $n = 2$ case, but give it in terms of an analytic parametrization with domain $0 \leq u \leq 2\pi$, $-\frac{\pi}{2} \leq v \leq \frac{\pi}{2}$:

$$\begin{aligned} x_1 &= \cos(u)\cos(v) \\ y_1 &= -\cos(u)\sin(v)\cos(v) \\ x_2 &= \sin(u)\cos(v) \\ y_2 &= -\sin(u)\cos(v)\sin(v). \end{aligned}$$

Let $\{r_1, \ldots, r_4\}$ denote the fiber coordinates in the real tangent bundle TF of $F = \mathbb{C}^2$ The tangent planes and their images under J are given by the equations

$$\begin{aligned} g_1 &= -y_2r_1 + y_1r_2 + x_2r_3 - x_1r_4 \\ g_2 &= 16x_1^3r_1 + 8x_1x_2^2r_1 + 8x_1^2x_2r_2 - 8x_1r_1 + 2y_1r_3 \\ g_3 &= 8x_1x_2^2r_1 + 8x_1^2x_2r_2 + 16x_2^3r_2 - 8x_2r_2 + 2y_2r_4 \\ g_1' &= -x_2r_1 + x_1r_2 - y_2r_3 + y_1r_4 \\ g_2' &= 16x_1^3r_3 + 8x_1x_2^2r_3 + 8x_1^2x_2r_4 - 2y_1r_1 - 8x_1r_3 \\ g_3' &= 8x_1x_2^2r_3 + 8x_1^2x_2r_4 + 16x_2^3r_4 - 2y_2r_2 - 8x_2r_4. \end{aligned}$$

At the self-intersection at the origin, these six equations are all zero. However, by homogenizing and computing the quotient ideal $(f_1, \ldots, g_3') : (x_1, y_1, x_2, y_2)$, it can be seen that the tangent bundle is totally real outside this point. In particular, the polynomials $x_0^4(r_1^2 + r_3^2)^2$ and $x_0^4(r_2^2 + r_4^2)^2$ are in the quotient ideal, so in the $x_0 = 1$ neighborhood, all the r coordinates must be zero. At the origin, the tangent cone is the union of the two totally real planes $(y_1 - 2x_1, y_2 - 2x_2)$, $(y_1 + 2x_1, y_2 + 2x_2)$.

This method of finding complex tangents, or demonstrating the "totally real" property, by computing the implicit equations for the image of a parametric map, and then the equations for the tangent bundle and its image under J, seems computationally intractable for more complicated varieties. Finding singularities in the complexification is an easier computation.

12. Some implicit equations

The method of "eliminating parameters" takes a parametric map and finds implicit equations for the smallest algebraic variety containing the image of the parametrization.

Example 12.1. Recall Whitney's embedding of $\mathbb{C}P^2$ into $\mathbb{C}^3 \oplus \mathbb{R}$:

$$[z_1 : z_2 : z_3] \mapsto \frac{1}{|z_1|^2 + |z_2|^2 + |z_3|^2}(z_2\overline{z}_3, z_3\overline{z}_1, z_1\overline{z}_2, |z_1|^2 - |z_2|^2).$$

Expanding the real and imaginary parts of these rational functions in terms of $z_k = x_k + iy_k$ defines a quadratically parameterized map $\mathbb{R}P^5 \to \mathbb{R}P^7$ with image diffeomorphic to $\mathbb{C}P^2$, contained in the affine neighborhood $e_0 = 1$:

$$
\begin{aligned}
e_0 &= x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 \\
e_1 &= x_2x_3 + y_2y_3 \\
e_2 &= y_2x_3 - x_2y_3 \\
e_3 &= x_1x_3 + y_1y_3 \\
e_4 &= x_1y_3 - y_1x_3 \\
e_5 &= x_1x_2 + y_1y_2 \\
e_6 &= y_1x_2 - x_1y_2 \\
e_7 &= x_1^2 + y_1^2 - x_2^2 - y_2^2.
\end{aligned}
$$

Let I be the ideal in $\mathbb{R}[e_0, \ldots, e_7, x_1, \ldots, y_3]$ generated by the polynomials

$$
\begin{array}{lcl}
e_0 & - & (x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2) \\
e_1 & - & (x_2x_3 + y_2y_3) \\
 & \vdots & \\
e_7 & - & (x_1^2 + y_1^2 - x_2^2 - y_2^2).
\end{array}
$$

A standard basis of this ideal has 97 elements. Twenty of them (listed below) are polynomials in only the e_k's.

The ideal I_{20} generated by these twenty elements, considered as elements of $\mathbb{R}[e_0, \ldots, e_7]$, defines a projective variety V_{20} of real codimension three. The affine part $V_{20} \cap \{e_0 = 1\}$ is the smallest affine algebraic variety containing the image of the parameterization. It is not clear whether this is exactly the image of Whitney's embedding; there may be points in $\mathbb{R}P^7$ that satisfy all twenty equations, but that are not of the form $\frac{1}{|z_1|^2+|z_2|^2+|z_3|^2}(z_2\overline{z}_3, z_3\overline{z}_1, z_1\overline{z}_2, |z_1|^2 - |z_2|^2)$. Such points all lie in a subvariety of strictly larger codimension.

$$
\begin{aligned}
f_1 &= e_3^2e_4 + e_4^3 + e_0e_2e_5 + 2e_4e_5^2 + e_0e_1e_6 + 2e_4e_6^2 - e_0e_4e_7 - e_2e_5e_7 - e_1e_6e_7 + e_4e_7^2\\
f_2 &= e_3^3 + e_3e_4^2 - e_0e_1e_5 + 2e_3e_5^2 + e_0e_2e_6 + 2e_3e_6^2 - e_0e_3e_7 + e_1e_5e_7 - e_2e_6e_7 + e_3e_7^2\\
f_3 &= e_2e_3e_5 + e_1e_4e_5 + e_1e_3e_6 - e_2e_4e_6\\
f_4 &= e_2e_3e_4 + e_1e_4^2 + e_0e_4e_6 + 2e_2e_5e_6 + 2e_1e_6^2 - e_4e_6e_7\\
f_5 &= e_2e_3^2 + e_2e_4^2 + e_0e_4e_5 + 2e_2e_5^2 + e_0e_3e_6 + 2e_2e_6^2 - e_4e_5e_7 - e_3e_6e_7\\
f_6 &= e_2^2e_5 - e_4^2e_5 + e_1e_2e_6 - e_3e_4e_6 - e_2e_4e_7\\
f_7 &= e_2^2e_3 + e_1e_2e_4 + e_0e_2e_6 + 2e_4e_5e_6 + 2e_3e_6^2 + e_2e_6e_7\\
f_8 &= e_1e_3e_4 - e_2e_4^2 - e_0e_4e_5 - 2e_2e_5^2 - 2e_1e_5e_6 + e_4e_5e_7\\
f_9 &= e_1e_3^2 + e_1e_4^2 - e_0e_3e_5 + 2e_1e_5^2 + e_0e_4e_6 + 2e_1e_6^2 + e_3e_5e_7 - e_4e_6e_7\\
f_{10} &= e_1e_2e_5 + e_3e_4e_5 - e_2^2e_6 + e_3^2e_6 + e_2e_3e_7\\
f_{11} &= e_1e_2e_3 - e_2^2e_4 - e_0e_2e_5 - 2e_4e_5^2 - 2e_3e_5e_6 - e_2e_5e_7\\
f_{12} &= e_1^2e_6 + e_2^2e_6 - e_3^2e_6 - e_4^2e_6 - e_2e_3e_7 - e_1e_4e_7\\
f_{13} &= e_1^2e_5 - e_3^2e_5 - e_1e_2e_6 + e_3e_4e_6 + e_1e_3e_7\\
f_{14} &= e_1^2e_4 + e_2^2e_4 + e_0e_2e_5 + 2e_4e_5^2 + e_0e_1e_6 + 2e_4e_6^2 + e_2e_5e_7 + e_1e_6e_7\\
f_{15} &= e_1^2e_3 - e_1e_2e_4 - e_0e_1e_5 + 2e_3e_5^2 - 2e_4e_5e_6 - e_1e_5e_7\\
f_{16} &= e_1^2e_2 + e_2^3 + e_0e_4e_5 + 2e_2e_5^2 + e_0e_3e_6 + 2e_2e_6^2 + e_0e_2e_7 + e_4e_5e_7 + e_3e_6e_7 + e_2e_7^2\\
f_{17} &= e_1^3 + e_1e_2^2 - e_0e_3e_5 + 2e_1e_5^2 + e_0e_4e_6 + 2e_1e_6^2 + e_0e_1e_7 - e_3e_5e_7 + e_4e_6e_7 + e_1e_7^2\\
f_{18} &= e_0e_2e_3 + e_0e_1e_4 + e_0^2e_6 - 2e_3^2e_6 - 2e_4^2e_6 - 4e_5^2e_6 - 4e_6^3 - e_2e_3e_7 - e_1e_4e_7 - e_6e_7^2\\
f_{19} &= e_0e_1e_3 - e_0e_2e_4 - e_0^2e_5 + 2e_3^2e_5 + 2e_4^2e_5 + 4e_5^3 + 4e_5e_6^2 - e_1e_3e_7 + e_2e_4e_7 + e_5e_7^2\\
f_{20} &= e_0e_1^2 + e_0e_2^2 - e_0e_3^2 - e_0e_4^2 + (e_0^2 - e_1^2 - e_2^2 - e_3^2 - e_4^2 - 4e_5^2 - 4e_6^2 - e_7^2)e_7.
\end{aligned}
$$

It is also not clear whether the ideal can be generated by fewer polynomials. Define a smaller ideal I_4 generated by the second, fourth, fifth, and sixth polynomials in the list of twenty; it also forms a codimension three variety, V_4. In fact, V_{20} is a dense subvariety of V_4; a computation shows that the Zariski closure of the difference $V_4 \setminus V_{20}$ is a codimension four variety. For example, the 3-planes $\{e_3 = e_4 = e_5 = e_6 = 0\}$, $\{e_1 = e_2 = e_3 = e_4 = 0\}$ are contained in V_4 but not in V_{20}.

Attempting this procedure on a parametrization where the denominator is not real-valued (say, Example 5.3) will be more complicated, since the real and imaginary parts will (in general) define a quartic parametrization $\mathbb{R}P^5 \to \mathbb{R}P^{10}$.

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