Notes on Differential Topology and Almost Complex Structures

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1 Linear Algebra: Complex Structure Operators

Definition 1.1. Given a real vector space V, a real linear map $J : V \to V$ such that $J \circ J = -Id_V$ is called a "complex structure operator," or more briefly a CSO.

Example 1.2. The "standard" CSO on the space $\mathbb{R}^{2n}$ is the $2n \times 2n$ block matrix

$$J_{std} = \begin{pmatrix} 0 & -1 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & -1 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & -1 \\ & & & & & 1 & 0 \end{pmatrix}.$$

Lemma 1.3. *Given V, if J is a CSO, then:*

- *$-J$ is also a CSO on V;*
- *For any involution $N : V \to V$ commuting with J (i.e., $N \circ N = Id_V$, $N \circ J = J \circ N$), the composite $N \circ J$ is also a CSO on V.*

- *For any invertible real linear map $A : U \to V$, the composite $A^{-1} \circ J \circ A$ is a CSO on U.*

$\blacksquare$

Lemma 1.4. *Given a vector space V with a CSO J_V, another vector space U with a CSO J_U, and a real linear map $A : U \to V$, the following are equivalent:*

- $J_V \circ A = A \circ J_U$;
- $A + J_V \circ A \circ J_U = 0$;
- *For any real scalars a, b, $(a \cdot Id_V + b \cdot J_V) \circ A = A \circ (a \cdot Id_U + b \cdot J_U)$.*

$\blacksquare$

A map A satisfying any of the above equivalent properties is called c-linear with respect to J_U and J_V (or more briefly when clear, just c-linear). A map is a-linear with respect to J_U and J_V if it is c-linear with respect to the CSOs $-J_U$ and J_V.

Lemma 1.5. *Given vector spaces U, V, with CSOs J_U, J_V as in the previous Lemma, the space of linear maps $\mathrm{Hom}(U, V)$ admits a direct sum decomposition*

$$\mathrm{Hom}(U, V) = \mathrm{Hom}_c(U, V) \oplus \mathrm{Hom}_a(U, V).$$

Any $A \in \mathrm{Hom}(U, V)$ can be written uniquely as a sum of a c-linear map and an a-linear map. The projection operators are

$$P_c(A) = \frac{1}{2}(A - J_V \circ A \circ J_U), \qquad P_a(A) = \frac{1}{2}(A + J_V \circ A \circ J_U),$$

so that $A = P_c(A) + P_a(A)$, and $P_c(A)$ is c-linear. $\blacksquare$

Consider $V = \mathbb{R}^{2n}$ with the usual basis, and let $\mathcal{J}_n$ be the subset of $GL(2n, \mathbb{R})$ consisting of all CSOs on V. By the Theorem on Jordan Canonical Form over $\mathbb{R}$, the smooth map $S : GL(2n, \mathbb{R}) \to \mathcal{J} : G \mapsto G^{-1} \circ J_{std} \circ G$ is onto. (Lemma 1.11 gives a proof of this special case of JCF.) Let $GL(n, \mathbb{C})$ denote the subgroup of elements $A \in GL(2n, \mathbb{R})$ such that A is c-linear with respect to J_{std}: $A \circ J_{std} = J_{std} \circ A$. Then $S(G) = S(A \circ G)$ for any $A \in GL(n, \mathbb{C})$, so S induces a well-defined map from the coset space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$

onto $\mathcal{J}_n$. Since $G^{-1} \circ J_{std} \circ G = H^{-1} \circ J_{std} \circ H \implies H \circ G^{-1} \in GL(n, \mathbb{C})$, the induced map is also one-to-one. The conclusion is that $\mathcal{J}_n$ is diffeomorphic to the homogeneous space $GL(2n, \mathbb{R})/GL(n, \mathbb{C})$, which has real dimension $(2n)^2 - 2n^2 = 2n^2$.

Lemma 1.6. *Given V and any two CSOs J_1, J_2, if $J_1 + J_2$ is invertible, then $(J_1 + J_2)^{-1} \circ (J_1 - J_2)$ is a-linear with respect to J_1, J_1 and also with respect to J_2, J_2.*

Hint. First, for any two CSOs J_1, J_2, the map $J_1 + J_2$ is c-linear with respect to J_1 and J_2: $(J_1 + J_2) \circ J_1 = J_2 \circ (J_1 + J_2)$.

Consider

$$(J_1 + J_2)^{-1} \circ (J_1 - J_2) \circ J_1 + J_1 \circ (J_1 + J_2)^{-1} \circ (J_1 - J_2)$$

and multiply by $J_1 + J_2$ to get

$$\begin{aligned}
& (J_1 - J_2) \circ J_1 + (J_1 + J_2) \circ J_1 \circ (J_1 + J_2)^{-1} \circ (J_1 - J_2) \\
= \; & (J_1 - J_2) \circ J_1 + J_2 \circ (J_1 + J_2) \circ (J_1 + J_2)^{-1} \circ (J_1 - J_2) \\
= \; & -Id_V - J_2 \circ J_1 + J_2 \circ J_1 + Id_V = 0_{\mathrm{End}(V)}.
\end{aligned}$$

The calculation showing that $(J_1 + J_2)^{-1} \circ (J_1 - J_2)$ anticommutes with J_2 is similar. ∎

For CSOs J, J_0 on $V = \mathbb{R}^{2n}$ such that $J + J_0$ is invertible as in Lemma 1.6, the following identity is easily checked:

$$(J + J_0)^{-1} \circ (J - J_0) = -\frac{1}{2}(Id - \frac{1}{2}J_0 \circ (J - J_0))^{-1} \circ J_0 \circ (J - J_0). \qquad (1)$$

In view of this identity (which shows the first-order approximation in $J - J_0$), and also Lemma 1.6, if J_0 is fixed, then the mapping

$$J \mapsto (J + J_0)^{-1} \circ (J - J_0)$$

is a local diffeomorphism from a neighborhood of J_0 in $\mathcal{J}_n$ (so that $J - J_0$ is small in some matrix norm) to a neighborhood of the origin of $\mathrm{Hom}_a(V, V)$ (the real vector space of endomorphisms of V which are a-linear with respect to J_0). This is consistent with the earlier calculation that the real dimension is $\frac{1}{2}(2n)^2 = 2n^2$, and the mapping gives an explicit local coordinate chart around J_0 in $\mathcal{J}_n$.

It will be more convenient later to switch the sign and consider the transformation

$$J \mapsto A = (J + J_0)^{-1} \circ (J_0 - J). \tag{2}$$

Then it is elementary ([C$_2$] §5.1, [R]) to check that this transformation has inverse (for J near J_0 and $A \in \mathrm{Hom}_a(V, V)$ with $Id + A$ invertible):

$$A \mapsto J = (Id + A) \circ J_0 \circ (Id + A)^{-1}. \tag{3}$$

Lemma 1.7. *Given a $2n \times 2$ real matrix A, if A is c-linear with respect to the standard 2×2 and $2n \times 2n$ J_{std} CSO matrices, and rank$(A) < 2$, then A is the zero matrix.*

Proof. First consider the 2×2 case; a quick calculation shows

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

implies $b = -c$ and $a = d$, so $\det(A) = a^2 + b^2$. If A is singular, then A is the 2×2 zero matrix. In the $2n \times 2$ case, a similar calculation with J_{std} shows A is a column of n 2×2 blocks of that form, so if the rank is less than 2, then all the blocks must be zero. ∎

Example 1.8. Given endomorphisms J_1, J_2 on $\mathbb{R}^{2m}$, $\mathbb{R}^{2n}$, respectively, the block matrix

$$J = \begin{pmatrix} J_1 & B \\ 0 & J_2 \end{pmatrix} \tag{4}$$

is a CSO on $\mathbb{R}^{2m+2n}$ if and only if J_1 and J_2 are both CSOs and $B_{2m\times 2n}$ is a-linear, i.e., $J_1 \cdot B = -B \cdot J_2$. The matrix J is similar to the block matrix

$$J_0 = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}, \tag{5}$$

via the relation $J = G^{-1} \cdot J_0 \cdot G$, where

$$\begin{aligned} G &= \begin{pmatrix} Id_{2m\times 2m} & \frac{1}{2} B \cdot J_2 \\ 0 & Id_{2n\times 2n} \end{pmatrix}, \\ G^{-1} &= \begin{pmatrix} Id_{2m\times 2m} & -\frac{1}{2} B \cdot J_2 \\ 0 & Id_{2n\times 2n} \end{pmatrix}. \end{aligned}$$

Example 1.9. Suppose J is a CSO of the form (4), and that J_3 is a CSO such that J_2+J_3 is invertible. There exists G_1, from Example 1.8, such that $G_1^{-1}\cdot J_0\cdot G_1=J$, where J_0 is in the block form (5). From (3) with $J_0=J_3$, there exists

$$G_2=\left(\begin{array}{cc} Id & 0 \\ 0 & (Id+A)^{-1}\end{array}\right)$$

so that $G_2^{-1}\cdot\left(\begin{array}{cc} J_1 & 0 \\ 0 & J_3\end{array}\right)\cdot G_2=J_0$. The composite transformation and its inverse are:

$$\begin{aligned}
G_3 &= G_2G_1=\left(\begin{array}{cc} Id & \frac{1}{2}B\cdot J_2 \\ 0 & (Id+A)^{-1}\end{array}\right), \qquad (6)\\
G_3^{-1} &= \left(\begin{array}{cc} Id & -\frac{1}{2}B\cdot J_2\cdot(Id+A) \\ 0 & Id+A\end{array}\right),\\
A &= (J_2+J_3)^{-1}\cdot(J_3-J_2),\\
\left(\begin{array}{cc} J_1 & 0 \\ 0 & J_3\end{array}\right) &\mapsto G_3^{-1}\cdot\left(\begin{array}{cc} J_1 & 0 \\ 0 & J_3\end{array}\right)\cdot G_3=\left(\begin{array}{cc} J_1 & B \\ 0 & J_2\end{array}\right)=J,\\
J=\left(\begin{array}{cc} J_1 & B \\ 0 & J_2\end{array}\right) &\mapsto G_3\cdot J\cdot G_3^{-1}=\left(\begin{array}{cc} J_1 & 0 \\ 0 & J_3\end{array}\right).
\end{aligned}$$

It can be checked that doing the steps in the other order — (3) then (5) — gives the same matrix G_3.

Lemma 1.10. *Given V with CSO J and $\vec{v}_1,\ldots,\vec{v}_\ell\in V$, if*

$$(\vec{v}_1,J(\vec{v}_1),\vec{v}_2,J(\vec{v}_2),\ldots,\vec{v}_{\ell-1},J(\vec{v}_{\ell-1}),\vec{v}_\ell)$$

is a linearly independent list, then so is

$$(\vec{v}_1,J(\vec{v}_1),\vec{v}_2,J(\vec{v}_2),\ldots,\vec{v}_{\ell-1},J(\vec{v}_{\ell-1}),\vec{v}_\ell,J(\vec{v}_\ell)).$$

Proof. Except for a re-ordering of the lists, this Lemma is recalled from ([C$_2$] §5.1). ∎

Lemma 1.11. *Given $n\geq 1$, $\mathbb{R}^{2n}$ with CSO $J_{2n\times 2n}$, there exists $G\in GL(n,\mathbb{R})$ such that $G\cdot J\cdot G^{-1}=J_{std}$.*

Proof. If $J+J_{std}$ is invertible, then (3) can be used. The following method is less canonical but works for any J, not requiring that $J+J_{std}$ is invertible.

Pick any non-zero $\vec{v}_1 \in \mathbb{R}^{2n}$; then the pair $(\vec{v}_1, J(\vec{v}_1))$ is an independent list by Lemma 1.10. If this is a basis, stop; otherwise, it does not span $\mathbb{R}^{2n}$, so there is some $\vec{v}_2$ not in $\operatorname{span}\{\vec{v}_1, J(\vec{v}_1)\}$, so $(\vec{v}_1, J(\vec{v}_1), \vec{v}_2)$ is an independent list, and by Lemma 1.10, $(\vec{v}_1, J(\vec{v}_1), \vec{v}_2, J(\vec{v}_2))$ is an independent list. This can be continued, repeating the arbitrary choice of $\vec{v}_k$ and adding $J(\vec{v}_k)$, until the list spans $\mathbb{R}^{2n}$ and is a basis (this gives a proof that the dimension must be even). Let H be the $2n \times 2n$ matrix formed by stacking the basis vectors as columns:

$$H = [\vec{v}_1, J(\vec{v}_1), \vec{v}_2, J(\vec{v}_2), \ldots, \vec{v}_{n-1}, J(\vec{v}_{n-1}), \vec{v}_n, J(\vec{v}_n)],$$

so by construction, H has linearly independent columns and is invertible. Let $(\vec{e}_1, \ldots, \vec{e}_{2n})$ be the standard basis of $\mathbb{R}^{2n}$, so that $H \cdot \vec{e}_{2k-1} = \vec{v}_k$ and $H \cdot \vec{e}_{2k} = J(\vec{v}_k)$. Let $G = H^{-1}$; then the matrix $G \cdot J \cdot G^{-1} = H^{-1} \cdot J \cdot H$ satisfies:

$$\begin{aligned} H^{-1} \cdot J \cdot H \cdot \vec{e}_{2k-1} &= H^{-1} \cdot J \cdot \vec{v}_k = \vec{e}_{2k}, \\ H^{-1} \cdot J \cdot H \cdot \vec{e}_{2k} &= H^{-1} \cdot J \cdot J \cdot \vec{v}_k = -H^{-1} \cdot \vec{v}_k = -\vec{e}_{2k-1}. \end{aligned}$$

The conclusion is that $G \cdot J \cdot G^{-1} = J_{std}$. ∎

Example 1.12. Let C denote the usual complex conjugation on $\mathbb{C}^2$, so $C(z_1, z_2) = (\bar{z}_1, \bar{z}_2)$. C is a-linear with respect to the standard CSO J_{std}. With respect to real coordinates $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, C has 4×4 matrix representation

$$C = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}.$$

Any a-linear function $A : \mathbb{R}^4 \to \mathbb{R}^4$ is of the form $A = B_1 \circ C = C \circ B_2$, for some c-linear functions B_1 and B_2; specifically, one can choose $B_1 = A \circ C$ and $B_2 = C \circ A$. If the c-linear function B_1 has matrix representation $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with respect to complex coordinates z_1, z_2 on $\mathbb{C}^2$, and where $\alpha = a_1 + ia_2$, etc., then the following real linear transformations have matrix representations:

$$B_1 = \begin{bmatrix} a_1 & -a_2 & b_1 & -b_2 \\ a_2 & a_1 & b_2 & b_1 \\ c_1 & -c_2 & d_1 & -d_2 \\ c_2 & c_1 & d_2 & d_1 \end{bmatrix}, \quad A = B_1 \circ C = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_2 & -a_1 & b_2 & -b_1 \\ c_1 & c_2 & d_1 & d_2 \\ c_2 & -c_1 & d_2 & -d_1 \end{bmatrix}.$$

2 Differential Topology: Coordinate charts and the tangent bundle

2.1 Manifolds

We begin by following some notation of [H]. Let M be a $\mathcal{C}^r$ ($r \geq 0$, or $r = \infty$, or $r = \omega$) manifold of dimension n, so that M is covered by open sets with coordinate charts $\phi_j : U_j \to \mathbb{R}^n$, and for two charts, $\phi_j \circ \phi_k^{-1} : \phi_k(U_j \cap U_k) \to \mathbb{R}^n$ is $\mathcal{C}^r$.

If $0 \leq r < s$ and M is a $\mathcal{C}^s$ manifold, then M is also a $\mathcal{C}^r$ manifold, trivially. There is a non-trivial converse (see Proposition 2.2 below). Given a $\mathcal{C}^s$ manifold M, some more coordinate charts can be added so that $\phi_j \circ \phi_k^{-1} : \phi_k(U_j \cap U_k) \to \mathbb{R}^n$ is $\mathcal{C}^r$ but not $\mathcal{C}^s$; this changes M to a $\mathcal{C}^r$ manifold which is not a $\mathcal{C}^s$ manifold. The topological ($\mathcal{C}^0$) structure is the same but the differential structure has changed.

For a $\mathcal{C}^r$ manifold M and another $\mathcal{C}^{r'}$ manifold M' with charts $\psi_{k'} : V_{k'} \to \mathbb{R}^{n'}$, consider a map $u : M' \to M$. Suppose that for every point x of M', there are some neighborhoods $x \in V$, $u(V) \subseteq U$, so that $\phi \circ u \circ \psi^{-1} : \psi(V) \to \mathbb{R}^{n'}$ is a $\mathcal{C}^{r''}$ map. Then, for any coordinate charts, $\phi_k \circ u \circ \psi_{k'}^{-1}$ is $\mathcal{C}^{r'''}$ where it is defined, where $r''' = \min\{r, r', r''\}$. This follows from the equality of composites:

$$\phi_k \circ u \circ \psi_{k'}^{-1} = \phi_k \circ (\phi^{-1} \circ \phi) \circ u \circ (\psi^{-1} \circ \psi) \circ \psi_{k'}^{-1} = (\phi_k \circ \phi^{-1}) \circ (\phi \circ u \circ \psi^{-1}) \circ (\psi \circ \psi_{k'}^{-1}).$$

So, the only coordinate-independent notion of a $\mathcal{C}^{r''}$ map $u : M' \to M$ is where $r'' \leq \min\{r, r'\}$.

Definition 2.1. Given a $\mathcal{C}^r$ manifold M, a subset $A \subseteq M$ is a $\underline{\mathcal{C}^r\ k\text{-submanifold}}$ means: at each point x of A there exists a neighborhood U of x in M and a coordinate chart $\phi : U \to \mathbb{R}^n$ such that $U \cap A = \phi^{-1}(\phi(U) \cap \mathbb{R}^k)$, where $\mathbb{R}^k = \{(x_1, \ldots, x_k, 0, \ldots, 0)\} \subseteq \mathbb{R}^n$.

A $\mathcal{C}^r$ k-submanifold is a k-dimensional $\mathcal{C}^r$ manifold with charts $\phi|_{U \cap A}$. For $0 \leq r < s$, it is possible that M is a $\mathcal{C}^s$ manifold, and A is not a $\mathcal{C}^s$ k-submanifold, but by adding more coordinate charts to make M a $\mathcal{C}^r$ manifold, A is a $\mathcal{C}^r$ k-submanifold of the $\mathcal{C}^r$ manifold M.

Proposition 2.2. *Given $1 \leq r < s \leq \infty$, let M be a $\mathcal{C}^r$ manifold with open covering U_α and coordinate charts ϕ_α. Let $\mathbf{\Psi} = \{(U_\beta, \phi_\beta)\}$ be a <u>maximal atlas</u>, adding all possible open subsets of M and all maps ϕ_β which have $\mathcal{C}^r$ overlaps with the given charts on M. Then, there exists a subset of $\mathbf{\Psi}$ which is a $\mathcal{C}^s$ differential structure for M.*

Sketch of Proof. The construction is non-trivial ([H] §2.2) and uses the property that the coordinate charts in the $\mathcal{C}^r$ structure can be approximated by $\mathcal{C}^s$ charts in a compatible way. ∎

2.2 Bundles

Let B, E, and F be topological spaces. A function $p : E \to B$ is a <u>fiber bundle</u> means that for neighborhoods U_k in B, the map p looks like a projection $U_k \times F \to U_k$. More precisely, there exists a covering of B by open subsets U_k, so that for each k, there is a homeomorphism Φ_k from the open set $p^{-1}(U_k) \subseteq E$ to $U_k \times F$, satisfying $\pi_k \circ \Phi_k = p|_{p^{-1}(U_k)}$, where $\pi_k : U_k \times F \to U_k$ is the usual projection onto the first factor.

It follows from the above definition that p is onto and continuous. The inverse image of a point, $p^{-1}(\{x\})$, is a "fiber," and as a subspace of E, it is homeomorphic to F, as follows. The restriction of Φ_k to $p^{-1}(\{x\})$ is a continuous, one-to-one function, with image contained in $U_k \times F$. Given $y \in p^{-1}(\{x\})$, $\Phi_k(y)$ satisfies $\pi_k(\Phi_k(y)) = p(y) = x$, so $\Phi_k(y) \in \{x\} \times F$. If $w \in \{x\} \times F$, then $w = \Phi_k(y)$ for some $y \in p^{-1}(U_k)$, and $x = \pi_k(w) = \pi_k(\Phi_k(y)) = p(y)$, so $y \in p^{-1}(\{x\})$. So, the image of $\Phi_k|_{p^{-1}(\{x\})}$ is exactly $\{x\} \times F$. The inverse of $\Phi_k|_{p^{-1}(\{x\})}$ is equal to the restriction of Φ_k^{-1} to the subspace $\{x\} \times F$, so $\Phi_k|_{p^{-1}(\{x\})}$ has a continuous inverse. The composite of $\Phi_k|_{p^{-1}(\{x\})}$ with the projection $\pi_F : \{x\} \times F \to F$ is a homeomorphism, which can be denoted:

$$\pi_F \circ \left(\Phi_k|_{p^{-1}(\{x\})} \right) = \Phi_{k,x} : p^{-1}(\{x\}) \to F.$$

A <u>section</u> of $p : E \to B$ is a continuous function $s : B \to E$ such that $p \circ s$ is the identity map on B. Such a map could be called a "global" section to distinguish from a "local" section $s : V \to E$ with $(p \circ s)(x) = x$ for $x \in V \subseteq B$.

2.3 Vector Bundles

Consider the special case of a fiber bundle where $F = \mathbb{R}^m$, B is a $\mathcal{C}^0$ manifold, and B is covered by coordinate charts $\phi_k : U_k \to \mathbb{R}^n$ as in Subsection 2.1. Then the topological space E is a $\mathcal{C}^0$ manifold of dimension $n+m$. The open sets $p^{-1}(U_k)$ are a covering of E by coordinate neighborhoods, with charts $(\phi_k \times Id_{\mathbb{R}^m}) \circ \Phi_k : p^{-1}(U_k) \to \mathbb{R}^{n+m}$. The composites

$$\begin{aligned} (\phi_j \times Id_{\mathbb{R}^m}) \circ \Phi_j \circ ((\phi_k \times Id_{\mathbb{R}^m}) \circ \Phi_k)^{-1} \quad &: \\ ((\phi_k \times Id_{\mathbb{R}^m}) \circ \Phi_k)(p^{-1}(U_k) \cap p^{-1}(U_j)) \quad &\to \quad \mathbb{R}^{n+m} \end{aligned} \tag{7}$$

are continuous.

A fiber bundle $p : E \to B$ as above is a vector bundle means that on each intersection of charts in B, $x \in U_j \cap U_k$,

$$\Phi_{j,x} \circ \Phi_{k,x}^{-1} : \mathbb{R}^m \to \mathbb{R}^m$$

is linear (and invertible), and as a function of x,

$$g_{jk} : U_j \cap U_k \to GL(m, \mathbb{R}) : g_{jk}(x) = \Phi_{j,x} \circ \Phi_{k,x}^{-1}$$

is a continuous function. By construction, these transition functions satisfy the cocycle identities: $g_{kk}(x) = Id_{\mathbb{R}^m}$ and $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$. Here, $F = \mathbb{R}^m$ is not just an abstract vector space, it is the actual Cartesian m-space of column m-vectors, with the standard basis $\vec{e}_1, \ldots, \vec{e}_m$. $g_{jk}(x)$ is not just an abstract linear map, but a size $m \times m$ matrix where the entries are real valued functions depending on x.

Conversely, given B, a coordinate chart covering U_k, and transition functions g_{jk} on $U_j \cap U_k$ satisfying the cocycle identities, a vector bundle $p : E \to B$ can be constructed, using a quotient space. Before describing the construction, we will need two point-set topological Lemmas.

Lemma 2.3. *Given topological spaces* Π *and* Z*, an onto function* $Q : \Pi \to E$*, and the quotient topology induced by* Q *on* E*, suppose* V *is an open subset of* E *and* $f : V \to Z$ *is any function. The following are equivalent:*

(i) f *is continuous;*

(ii) $f \circ (Q|_{Q^{-1}(V)}) : Q^{-1}(V) \to Z$ *is continuous.*

Further, if $\tilde{f} : Q^{-1}(V) \to Z$ *is any continuous function which is constant on subsets of the form* $Q^{-1}(\{v\})$ *for* $v \in V$*, then there exists a unique continuous function* $f : V \to Z$ *such that* $f \circ (Q|_{Q^{-1}(V)}) = \tilde{f}$*.*

Proof. By definition of quotient topology, a set U is open in E iff $Q^{-1}(U)$ is open in Π. It follows that Q is continuous and $Q^{-1}(V)$ is open in Π. For $(i) \implies (ii)$, the map in (ii) is the composite of the continuous function f with the restriction of the continuous function Q to the subspace $Q^{-1}(V)$.

For $(ii) \implies (i)$, let U be any open set in Z; we want to show $f^{-1}(U)$ is open in V. By hypothesis, $(f \circ (Q|_{Q^{-1}(V)}))^{-1}(U)$ is open in $Q^{-1}(V)$. It follows that $(Q|_{Q^{-1}(V)})^{-1}(f^{-1}(U)) = W \cap Q^{-1}(V)$ for some W open in Π, and this set equals $\{x \in Q^{-1}(V) : (Q|_{Q^{-1}(V)})(x) \in f^{-1}(U)\}$. If y is any element of Π with $Q(y) \in f^{-1}(U) \subseteq V$, then $y \in Q^{-1}(V)$ and $Q(y) = (Q|_{Q^{-1}(V)})(y)$, so the above expression simplifies to $\{x \in \Pi : Q(x) \in f^{-1}(U)\} = Q^{-1}(f^{-1}(U))$. This is an open set in Π, so by definition of quotient topology, $f^{-1}(U)$ is open in E, and contained in V, so it is open in V.

Now suppose $\tilde{f}$ is given, and for $v \in V$, let $f(v) = \tilde{f}(x)$ for any $x \in Q^{-1}(\{v\})$; f is well-defined by hypothesis. For $x \in Q^{-1}(V)$, with $Q(x) = v$, $(f \circ (Q|_{Q^{-1}(V)}))(x) = f(Q(x)) = f(v) = \tilde{f}(x)$, so if $\tilde{f}$ is continuous, then f is continuous by the previous paragraph. For uniqueness, if $h \circ (Q|_{Q^{-1}(V)}) = f \circ (Q|_{Q^{-1}(V)}) = \tilde{f}$, let $v \in V$ with $Q(x) = v$, so $h(v) = (h \circ (Q|_{Q^{-1}(V)}))(x) = \tilde{f}(x) = (f \circ (Q|_{Q^{-1}(V)}))(x) = f(v)$. ∎

Lemma 2.4. *Given topological spaces* Z*,* Π*,* E*, and a continuous map* $Q : \Pi \to E$*, suppose there is an open covering* U_α *of* Z*, and a collection of functions* $f_\alpha : U_\alpha \to \Pi$ *such that each* $Q \circ f_\alpha : U_\alpha \to E$ *is continuous. If* $Q(f_\alpha(z)) = Q(f_\beta(z))$ *for all* $z \in U_\alpha \cap U_\beta$*, then there is a continuous map* $f : Z \to E$ *with* $Q(f_\alpha(z)) = f(z)$ *for all* α *and* $z \in U_\alpha$*.*

Proof. The functions f_α need not be continuous. Define $f(z) = Q(f_\alpha(z))$ for any α with $z \in U_\alpha$; f is well-defined by hypothesis. Let V be any open set in E; then $f^{-1}(V) = \cup f^{-1}(V) \cap U_\alpha$. Each set $f^{-1}(V) \cap U_\alpha$ is equal to $\{z \in Z : f(z) \in V \text{ and } z \in U_\alpha\} = \{z \in U_\alpha : f(z) \in V\} = \{z \in U_\alpha : Q(f_\alpha(z)) \in V\} = (Q \circ f_\alpha)^{-1}(V)$, which is open in Z by hypothesis, so $f^{-1}(V)$ is a union of open sets. ∎

Let Λ be the index set $\{k\}$ for the given covering of B by coordinate charts U_k, with the discrete topology, and consider this disjoint union as a topological space:

$$\Pi = \bigcup_{k \in \Lambda} U_k \times \{k\} \times \mathbb{R}^m.$$

Define a relation on the set of triples: $(x, k, \vec{a}) \sim (y, j, \vec{b})$ means: $x = y \in U_j \cap U_k$ and $g_{jk}(x) : \vec{a} \mapsto \vec{b}$. This is an equivalence relation by the cocycle identities. The equivalence class of any point $(x, k, \vec{a}) \in U_k \times \{k\} \times \mathbb{R}^m$ is denoted $[x, k, \vec{a}]$, and satisfies:

$$[x, k, \vec{a}] = \bigcup_{j \in \Lambda} \begin{cases} \{(x, j, g_{jk}(x)(\vec{a}))\} & \text{if } x \in U_j \cap U_k \\ \emptyset & \text{if } x \notin U_j \cap U_k \end{cases}$$

Let E be the set of all equivalence classes. Define the onto function $Q : \Pi \to E : (x, k, \vec{a}) \mapsto [x, k, \vec{a}]$, and let E have the quotient topology as in Lemma 2.3. By definition, a set V is open in E if and only if $Q^{-1}(V)$ is open in Π.

Fix k and a coordinate chart $\phi_k : U_k \to \mathbb{R}^n$ for B. Then the set $U_k \times \{k\} \times \mathbb{R}^m$ is open in Π. Q is one-to-one on this open set: if $(y, k, \vec{b}) \sim (x, k, \vec{a})$ then $y = x$ and $g_{kk}(x) : \vec{a} \mapsto \vec{a}$. $Q^{-1}(Q(U_k \times \{k\} \times \mathbb{R}^m))$ is the set of points in Π that are equivalent to points in $U_k \times \{k\} \times \mathbb{R}^m$:

$$Q^{-1}(Q(U_k \times \{k\} \times \mathbb{R}^m)) = \bigcup_{j \in \Lambda} (U_j \cap U_k) \times \{j\} \times \mathbb{R}^m,$$

so it is a union of open sets and is open in Π. By definition of quotient topology, $Q(U_k \times \{k\} \times \mathbb{R}^m)$ is open in E; E is covered by open sets of this form.

The main consequence of the quotient topology is that Lemma 2.3 gives the following criterion for a function to be continuous on E. Let Z be any topological space; a function $f : E \to Z$ is continuous if and only if there is a continuous function $\tilde{f} : \Pi \to Z$ such that $\tilde{f} = f \circ Q$. This $\tilde{f}$ must be constant on equivalence classes: if $(x, k, \vec{a}) \sim (y, j, \vec{b})$ then $\tilde{f}((x, k, \vec{a})) = f([x, k, \vec{a}]) = f([y, j, \vec{b}]) = \tilde{f}((y, j, \vec{b}))$, and for any such $\tilde{f}$, there is a unique induced map f. So, to define a continuous function $f : E \to Z$, it is enough to define a continuous $\tilde{f}$ on all the open sets $U_k \times \{k\} \times \mathbb{R}^m$ and then check that for all j, if $x \in U_j \cap U_k$, then $\tilde{f}((x,k,\vec{a})) = \tilde{f}((x,j,g_{jk}(x)(\vec{a})))$. Then, for any equivalence class $[x,k,\vec{a}] = Q((x,k,\vec{a}))$, $f([x,k,\vec{a}]) = f(Q((x,k,\vec{a}))) = \tilde{f}((x,k,\vec{a}))$, independent of the choice of representative $(x,k,\vec{a})$.

In the other direction, to define a continuous function $f : Z \to E$, it is enough to cover Z by open sets V_α and apply Lemma 2.4 to $Q : \Pi \to E$. If there is a collection of continuous functions $f_\alpha : V_\alpha \to \Pi$, then each $Q \circ f_\alpha$ is continuous, and $f(z) = Q(f_\alpha(z))$ is a well-defined continuous function $Z \to E$ (not depending on α) if $f_\alpha(z) \sim f_\beta(z)$ for all $z \in V_\alpha \cap V_\beta$. Equivalently, if $f_\alpha(z) = (f_\alpha^1(z), j, \vec{f}_\alpha^2(z))$ and $f_\beta(z) = (f_\beta^1(z), k, \vec{f}_\beta^2(z))$, then $f_\alpha^1(z) = f_\beta^1(z) \in U_j \cap U_k$ and $g_{jk}(f_\alpha^1(z)) : \vec{f}_\beta^2(z) \mapsto \vec{f}_\alpha^2(z)$.

Define $\tilde{p} : \Pi \to B : (x,k,\vec{a}) \mapsto x$. Then $\tilde{p}$ is constant on equivalence classes: if $(x,k,\vec{a}) \sim (y,j,\vec{b})$, then $x = y$ so $\tilde{p}((x,k,\vec{a})) = \tilde{p}((y,j,\vec{b})) = x = y$. Also, $\tilde{p}$ is continuous on each subset $U_k \times \{k\} \times \mathbb{R}^m$, so it induces the continuous function $p : E \to B : p([x,k,\vec{a}]) = x$ by Lemma 2.3.

To show that $p : E \to B$ defines a vector bundle, we need to define the functions $\Phi_k : p^{-1}(U_k) \to U_k \times \mathbb{R}^m$. Lemma 2.3 applies to the open set $p^{-1}(U_k)$ in E. The set $Q^{-1}(p^{-1}(U_k))$ is equal to

$$(p \circ Q)^{-1}(U_k) = \tilde{p}^{-1}(U_k) = \bigcup_{j \in \Lambda} (U_k \cap U_j) \times \{j\} \times \mathbb{R}^m. \tag{8}$$

For $(y,j,\vec{b})$ in this set, define

$$\tilde{\Phi}_k : (y,j,\vec{b}) \mapsto (y, (g_{jk}(y))^{-1}(\vec{b})) \in U_k \times \mathbb{R}^m,$$

then $\tilde{\Phi}_k$ is continuous and if $(y',j',\vec{b}') \sim (y,j,\vec{b})$ for $y \in U_k \cap U_j \cap U_{j'}$, then $y' = y$, $g_{j'j}(y) : \vec{b} \to \vec{b}'$, and

$$\begin{aligned} \tilde{\Phi}_k((y',j',\vec{b}')) &= (y', (g_{j'k}(y'))^{-1}(\vec{b}')) = (y, (g_{j'k}(y'))^{-1}((g_{j'j}(y))(\vec{b}))) \\ &= (y, (g_{jk}(y))^{-1}(\vec{b})) = \tilde{\Phi}_k((y,j,\vec{b})). \end{aligned}$$

So, there is an induced continuous map $\Phi_k : p^{-1}(U) \to U_k \times \mathbb{R}^m$, with $\Phi_k \circ Q|_{Q^{-1}(p^{-1}(U_k))} = \tilde{\Phi}_k$, and for any $x \in U_k \cap U_j$, and $\vec{b} \in \mathbb{R}^m$,

$$\Phi_k([x,j,\vec{b}]) = (x, (g_{jk}(x))^{-1}(\vec{b})). \tag{9}$$

In particular, for any $x \in U_k$ and $\vec{a} \in \mathbb{R}^m$, $[x,k,\vec{a}] \in p^{-1}(U_k)$, and $\Phi_k([x,k,\vec{a}]) = (x,\vec{a})$. By construction, $(\pi_k \circ \Phi_k)([x,k,\vec{a}]) = x = (p|_{p^{-1}(U_k)})([x,k,\vec{a}])$.

To show that Φ_k is a homeomorphism, we need a continuous inverse. Define a collection of continuous functions indexed by $j \in \Lambda$,

$$\begin{array}{rcl} \Psi_{jk} : (U_k \cap U_j) \times \mathbb{R}^m & \to & (U_k \cap U_j) \times \{j\} \times \mathbb{R}^m \subseteq \Pi \\ (y, \vec{b}) & \mapsto & (y, j, (g_{jk}(y))(\vec{b})). \end{array}$$

For $y \in U_k \cap U_j \cap U_{j'}$,

$$\begin{array}{rcl} \Psi_{j'k}((y, \vec{b})) & = & (y, j', (g_{j'k}(y))(\vec{b})) \\ & \sim & (y, j, g_{jj'}(y)((g_{j'k}(y))(\vec{b}))) \\ & = & (y, j, (g_{jk}(y))(\vec{b})) = \Psi_{jk}((y, \vec{b})), \end{array}$$

so these functions satisfy $Q \circ \Psi_{j'k} = Q \circ \Psi_{jk}$ on $(U_k \cap U_j \cap U_{j'}) \times \mathbb{R}^m$, and the sets $(U_k \cap U_j) \times \mathbb{R}^m$ are an open cover of $U_k \times \mathbb{R}^m$, so the function $\Psi_k : U_k \times \mathbb{R}^m \to E$ defined by $\Psi_k(x, \vec{a}) = Q(\Psi_{kk}((x, \vec{a}))) = [x, k, \vec{a}]$ is continuous by Lemma 2.4, and a two-sided inverse of Φ_k. (We could have defined Ψ_{kk} only and then $\Psi_k = Q \circ \Psi_{kk}$, but the above application of Lemma 2.4 shows that Ψ_k can be defined in a coordinate-independent way.) This is enough to show that $p : E \to B$ is a fiber bundle with fiber $\mathbb{R}^m$.

To show that this construction (being given g_{jk} and constructing p and Φ_k) gives a vector bundle with transition functions agreeing with the given data, consider $x \in U_j \cap U_k$. Then $p^{-1}(\{x\}) = \{[x, j, \vec{b}] : \vec{b} \in \mathbb{R}^m\}$, and $\Phi_k|_{p^{-1}(\{x\})} : p^{-1}(\{x\}) \to \{x\} \times \mathbb{R}^m$, with

$$(\Phi_k|_{p^{-1}(\{x\})})([x, j, \vec{b}]) = (x, (g_{jk}(x))^{-1}(\vec{b}))$$

as in (9). So $(\pi_{\mathbb{R}^m} \circ (\Phi_k|_{p^{-1}(\{x\})}))^{-1} : \mathbb{R}^m \to p^{-1}(\{x\})$ is defined by

$$\vec{a} \mapsto [x, j, (g_{jk}(x))(\vec{a})]. \tag{10}$$

Again using (9), $\Phi_j|_{p^{-1}(\{x\})}([x, j, \vec{b}]) = (x, \vec{b})$, so

$$(\pi_{\mathbb{R}^m} \circ (\Phi_j|_{p^{-1}(\{x\})})) \circ (\pi_{\mathbb{R}^m} \circ (\Phi_k|_{p^{-1}(\{x\})}))^{-1} = g_{jk}(x) : \mathbb{R}^m \to \mathbb{R}^m,$$

which shows $p : E \to B$ is a vector bundle with transition functions g_{jk}.

Returning to (7), the coordinate change functions on the manifold E can be expressed in terms of g_{jk}. Consider an element $x \in U_i \cap U_j \cap U_k$, and an element $[x, i, \vec{b}] \in p^{-1}(U_j) \cap p^{-1}(U_k)$. Then, from (9),

$$(\phi_k \times Id_{\mathbb{R}^m}) \circ \Phi_k : [x, i, \vec{b}] \mapsto (\phi_k(x), (g_{ik}(x))^{-1}(\vec{b})) \in \mathbb{R}^{n+m} \tag{11}$$

has inverse

$$
\begin{aligned}
((\phi_k \times Id_{\mathbb{R}^m}) \circ \Phi_k)^{-1} : \mathbb{R}^{n+m} &\to p^{-1}(U_j) \cap p^{-1}(U_k) \\
(\vec{v}, \vec{a}) &\mapsto [\phi_k^{-1}(\vec{v}), i, (g_{ik}(\phi_k^{-1}(\vec{v})))(\vec{a})]. \qquad (12)
\end{aligned}
$$

So, the composite in (7) maps $(\vec{v}, \vec{a})$ to:

$$
\begin{aligned}
(\phi_j \times Id_{\mathbb{R}^m}) \circ \Phi_j \quad &: \quad [\phi_k^{-1}(\vec{v}), i, (g_{ik}(\phi_k^{-1}(\vec{v})))(\vec{a})] \\
&\mapsto \big(\phi_j(\phi_k^{-1}(\vec{v})), (g_{ij}(\phi_k^{-1}(\vec{v})))^{-1}(g_{ik}(\phi_k^{-1}(\vec{v}))(\vec{a}))\big) \\
&= \big((\phi_j \circ \phi_k^{-1})(\vec{v}), (g_{jk}(\phi_k^{-1}(\vec{v})))(\vec{a})\big)\,. \qquad (13)
\end{aligned}
$$

For an open set $V \subseteq B$, a local section $s : V \to E$ can be defined using Lemma 2.4. Using the coordinate charts $U_k \subseteq B$, V has an open cover $V \cap U_k$. On $V \cap U_k$, denote $s_k : V \cap U_k \to \Pi$ by $s_k(x) = (s_k^1(x), s_k^2(x), \vec{s}_k^{\,3}(x))$, which can be any function where $Q \circ s_k : V \cap U_k \to E$ is continuous. Suppose further that for $x \in V \cap U_k \cap U_{k'}$, $Q(s_k(x)) = Q(s_{k'}(x))$, which by construction means $s_k^1(x) = s_{k'}^1(x)$ and $(g_{s_{k'}^2(x) s_k^2(x)}(s_k^1(x)))(\vec{s}_k^{\,3}(x)) = \vec{s}_{k'}^{\,3}(x)$. Then Lemma 2.4 defines a continuous function $s : V \to E$ at any point $x \in V \cap U_k$ by $s(x) = Q(s_k(x))$. The definition of section requires $p(s(x)) = x$ for $x \in V$, and by construction of p, $p(s(x)) = p([s_k^1(x), s_k^2(x), \vec{s}_k^{\,3}(x)]) = s_k^1(x) = x$. So, on $V \cap U_k$, $s_k(x) = (x, s_k^2(x), \vec{s}_k^{\,3}(x))$, which can be replaced by $s_k'(x) = (x, k, (g_{k s_k^2(x)}(x))(\vec{s}_k^{\,3}(x))) \sim s_k(x)$ without changing s. It follows that if $\vec{s}_k : V \cap U_k \to \mathbb{R}^m$ is any collection of continuous functions such that for $x \in V \cap U_k \cap U_j$, $(g_{jk}(x))(\vec{s}_k(x)) = \vec{s}_j(x)$, then the formula $s(x) = [x, k, \vec{s}_k(x)]$ defines a continuous local section $s : V \to E$.

2.4 Hom bundles

Let $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^q)$ denote the real vector space of $q \times m$ real matrices. For invertible matrices $A_{m\times m}$ and $B_{q\times q}$, the matrix product function $C_{q\times m} \mapsto B \cdot C \cdot A^{-1}$ is an invertible linear transformation of $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^q)$.

Now let B be a $\mathcal{C}^0$ manifold, and suppose there are two vector bundles on B using the same coordinate charts U_k (this can always be achieved by a "refinement" of two open covers). First, $p_1 : E_1 \to B$ has transition functions $g_{jk}^1(x) : \mathbb{R}^m \to \mathbb{R}^m$, and second, $p_2 : E_2 \to B$ has transition functions $g_{jk}^2(x) : \mathbb{R}^q \to \mathbb{R}^q$. Let $GL(q \times m, \mathbb{R})$ denote the set of invertible linear transformations of $\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^q)$, embedded as an open subset of $\mathbb{R}^{(qm)^2}$ via matrix representation. Define a new function $g^3_{jk}: U_j \cap U_k \to GL(q\times m, \mathbb{R})$, by the formula

$$g^3_{jk}(x) : C_{q\times m} \mapsto g^2_{jk}(x)\cdot C \cdot (g^1_{jk}(x))^{-1}. \tag{14}$$

The collection of $g^3_{jk}(x)$ functions satisfies the cocycle identities (this is easily checked) so they are transition functions for a new bundle with base B. Let $\mathrm{Hom}(E_1, E_2)$ be the bundle with base B, fiber $F = \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^q)$, and transition functions g^3_{jk} on the charts U_k, so $\mathrm{Hom}(E_1, E_2)$ can be constructed as in Section 2.3, as a quotient of

$$\Pi = \bigcup_{k\in\Lambda} U_k \times \{k\} \times \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^q).$$

For $V \subseteq B$, a local section $S : V \to \mathrm{Hom}(E_1, E_2)$ is defined as a collection of matrix valued functions on coordinate charts. If $S_k : V \cap U_k \to \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^q)$ is any collection of continuous functions such that for $x \in V \cap U_k \cap U_j$,

$$g^2_{jk}(x)\cdot S_k(x)\cdot (g^1_{jk}(x))^{-1} = S_j(x), \tag{15}$$

then a continuous local section $S : V \to \mathrm{Hom}(E_1, E_2)$ is defined on $V \cap U_k$ by $S(x) = [x, k, S_k(x)]$.

Suppose s^1 is a local section $V \to E_1$, defined by $s^1(x) = [x, k, \vec{s}^1_k(x)]$ as in Section 2.3. Then S acts on s^1 as follows: define $\vec{s}_k^{\,2} : V \cap U_k \to \mathbb{R}^q$ by multiplying matrix times column vector: $\vec{s}_k^{\,2}(x) = S_k(x)\cdot \vec{s}^1_k(x)$. If $x \in V \cap U_k \cap U_j$, then, using (14):

$$\begin{aligned}
(g^2_{jk}(x))(\vec{s}_k^{\,2}(x)) &= (g^2_{jk}(x))(S_k(x)\cdot \vec{s}^1_k(x)) \\
&= g^2_{jk}(x)\cdot S_k(x)\cdot (g^1_{jk}(x))^{-1}\cdot g^1_{jk}(x)\cdot \vec{s}^1_k(x) \\
&= S_j(x)\cdot \vec{s}^1_j(x) = \vec{s}_j^{\,2}(x).
\end{aligned}$$

This shows $\vec{s}_k^{\,2}(x)$ defines a local section $V \to E_2$, which can be denoted

$$s^2(x) = S(x)\cdot s^1(x) = [x, k, \vec{s}_k^{\,2}(x)] = [x, k, S_k(x)\cdot \vec{s}^1_k(x)]. \tag{16}$$

2.5 Maps of bundles

Given two fiber bundles $p_1 : E_1 \to B_1$, $p_2 : E_2 \to B_2$ as in Section 2.2, a continuous map $\Gamma : E_1 \to E_2$ is a fiber map means: there exists a continuous $f : B_1 \to B_2$ such that $p_2 \circ \Gamma = f \circ p_1$. A fiber map satisfies $p_1(y) = x \implies p_2(\Gamma(y)) = f(p_1(y)) = f(x)$, so Γ maps the fiber $p_1^{-1}(\{x\}) \subseteq E_1$ to the fiber $p_2^{-1}(\{f(x)\}) \subseteq E_2$.

In the special case where E_1 is a vector bundle with fiber $\mathbb{R}^{m_1}$ and local trivializations Φ_k^1, and E_2 is a vector bundle with fiber $\mathbb{R}^{m_2}$ and local trivializations Φ_α^2, $\Gamma : E_1 \to E_2$ is a morphism of vector bundles means: Γ is a fiber map, and for each $x \in U_k \subseteq B_1$, $\Phi^2_{\alpha,f(x)} \circ \Gamma|_{p_1^{-1}(\{x\})} \circ (\Phi^1_{k,x})^{-1} : \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ is linear. Γ is a bimorphism means: the above linear map is invertible for every x, and Γ is an isomorphism means: Γ is a bimorphism, $B_1 = B_2$, and $f : B_1 \to B_2$ is the identity map.

To see how this is related to the construction with transition functions, we need another point-set topology lemma.

Lemma 2.5. *Let Π_1, Π_2, E_2 be topological spaces, let $Q_1 : \Pi_1 \to E_1$ be an onto function so that E_1 has the quotient topology, and let $Q_2 : \Pi_2 \to E_2$ be continuous. Suppose there is a covering of E_1 by open sets V_α, and there is a collection of functions $f_\alpha : Q_1^{-1}(V_\alpha) \to \Pi_2$ such that:*

- *$Q_2 \circ f_\alpha : Q_1^{-1}(V_\alpha) \to E_2$ is continuous;*
- *for $x_1 \in Q_1^{-1}(V_\alpha)$, $x_2 \in Q_1^{-1}(V_\beta)$, if $Q_1(x_1) = Q_1(x_2)$ then $Q_2(f_\alpha(x_1)) = Q_2(f_\beta(x_2))$.*

Then, there exists a continuous function $f : E_1 \to E_2$ such that for any $y \in V_\alpha \cap V_\beta$, if $x_1 \in Q_1^{-1}(V_\alpha)$, $x_2 \in Q_1^{-1}(V_\beta)$ satisfy $Q_1(x_1) = Q_1(x_2) = y$, then $f(y) = Q_2(f_\alpha(x_1)) = Q_2(f_\beta(x_2))$.

Proof. By definition of quotient topology, $Q_1^{-1}(V_\alpha)$ is open, so the collection $Q_1^{-1}(V_\alpha)$ is an open covering of Π_1. For $z \in (Q_1^{-1}(V_\alpha)) \cap (Q_1^{-1}(V_\beta))$, $Q_2(f_\alpha(z)) = Q_2(f_\beta(z))$ by hypothesis, so Lemma 2.4 applies with $Z = \Pi_1$. There is a continuous map $\tilde{f} : \Pi_1 \to E_2$ with $Q_2(f_\alpha(z)) = \tilde{f}(z)$.

Now, for any $v \in V_\alpha \cap V_\beta$, if $x_1 \in Q_1^{-1}(\{v\}) \subseteq Q_1^{-1}(V_\alpha)$ and $x_2 \in Q_1^{-1}(\{v\}) \subseteq Q_1^{-1}(V_\beta)$, $Q_1(x_1) = Q_1(x_2) = v$, so by hypothesis,

$$\tilde{f}(x_1) = Q_2(f_\alpha(x_1)) = Q_2(f_\beta(x_2)) = \tilde{f}(x_2),$$

showing $\tilde{f}$ is constant on each set $Q_1^{-1}(\{v\})$. By Lemma 2.3 applied to $Z = E_2$, there is a continuous function $f : E_1 \to E_2$ such that $f \circ Q_1 = \tilde{f}$. The conclusion is that for any $y \in V_\alpha \cap V_\beta$, if $x_1 \in Q_1^{-1}(V_\alpha)$, $x_2 \in Q_1^{-1}(V_\beta)$ satisfy $Q_1(x_1) = Q_1(x_2) = y$, then $f(y) = f(Q_1(x_1)) = \tilde{f}(x_1) = Q_2(f_\alpha(x_1)) = Q_2(f_\beta(x_2))$. ∎

Suppose B_1 and B_2 are manifolds, with transition functions g^1_{jk} on an open cover U_k for B_1, and transition functions $g^2_{\beta\alpha}$ on an open cover V_α for B_2, defining vector bundles $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$ as in Section 2.3. We want to apply Lemma 2.5 to see what sort of local expressions define a vector bundle morphism $E_1 \to E_2$.

E_1 is covered by open sets $p_1^{-1}(U_k)$, and as in (8),

$$Q_1^{-1}(p_1^{-1}(U_k)) = (p_1 \circ Q_1)^{-1}(U_k) = \bigcup_{j \in \Lambda_1} (U_k \cap U_j) \times \{j\} \times \mathbb{R}^m.$$

A function $f_k : Q_1^{-1}(p_1^{-1}(U_k)) \to \Pi_2$ can be defined piecewise, $f_k((x, j, \vec{b})) = f_{kj}((x, j, \vec{b}))$, on the pieces of the domain:

$$\begin{aligned} f_{kj} : (U_k \cap U_j) \times \{j\} \times \mathbb{R}^{m_1} &\to \Pi_2 = \bigcup_{\alpha \in \Lambda_2} V_\alpha \times \{\alpha\} \times \mathbb{R}^{m_2} \\ (x, j, \vec{b}) &\mapsto (f^1_{kj}((x, j, \vec{b})), f^2_{kj}((x, j, \vec{b})), \vec{f}^{\,3}_{kj}((x, j, \vec{b}))). \end{aligned} \tag{17}$$

$Q_2 \circ f_k$ is continuous if and only if every $Q_2 \circ f_{kj}$ is continuous. To satisfy the other hypothesis of Lemma 2.5, consider $(x_1, j_1, \vec{b}_1) \in Q_1^{-1}(p_1^{-1}(U_k))$ and $(x_2, j_2, \vec{b}_2) \in Q_1^{-1}(p_1^{-1}(U_i))$. $Q_1((x_1, j_1, \vec{b}_1)) = Q_1((x_2, j_2, \vec{b}_2))$ means $(x_1, j_1, \vec{b}_1) \sim_1 (x_2, j_2, \vec{b}_2)$, so $x_1 = x_2 \in U_k \cap U_i \cap U_{j_1} \cap U_{j_2}$ and $g^1_{j_2 j_1}(x_1) : \vec{b}_1 \to \vec{b}_2$. Functions f_k satisfying $f_k((x_1, j_1, \vec{b}_1)) \sim_2 f_i((x_2, j_2, \vec{b}_2))$ when $(x_1, j_1, \vec{b}_1) \sim_1 (x_2, j_2, \vec{b}_2)$ will satisfy:

$$\begin{aligned} f^1_{kj_1}((x_1, j_1, \vec{b}_1)) &= f^1_{ij_2}((x_1, j_2, g^1_{j_2 j_1}(x_1)(\vec{b}_1))) \\ \vec{f}^{\,3}_{kj_1}((x_1, j_1, \vec{b}_1)) &= G \cdot \vec{f}^{\,3}_{ij_2}((x_1, j_2, g^1_{j_2 j_1}(x_1)(\vec{b}_1))) \\ G &= g^2_{f^2_{kj_1}((x_1, j_1, \vec{b}_1)) f^2_{ij_2}((x_1, j_2, g^1_{j_2 j_1}(x_1)(\vec{b}_1)))}(f^1_{kj_1}((x_1, j_1, \vec{b}_1))). \end{aligned}$$

By Lemma 2.5, a collection f_k satisfying the above identities defines a continuous map $\Gamma : E_1 \to E_2 : [x, j, \vec{b}] \mapsto [f^1_{kj}((x, j, \vec{b})), f^2_{kj}((x, j, \vec{b})), \vec{f}^{\,3}_{kj}((x, j, \vec{b}))]$. For this to be a fiber map, there must be some continuous function f :

$B_1 \to B_2$ so that for $x \in U_k \cap U_j$, $p_2(\Gamma([x, j, \vec{b}])) = f^1_{kj}((x, j, \vec{b}))$ matches $f(p_1([x, j, \vec{b}])) = f(x)$, so f^1_{kj} depends only on x. By refining, if necessary, the covering of B_1 (using more and smaller open coordinate neighborhoods), we can assume that for each k there is some $\alpha = f^2(k)$ so that $f(U_k) \subseteq V_\alpha$. Then $f^2_{kj}((x, j, \vec{b}))$ can be replaced by $f^2(k)$, and $\vec{f}^{\,3}_{kj}((x, j, \vec{b}))$ can be replaced by $g^2_{f^2(k) f^2_{kj}((x,j,\vec{b}))}(f(x))\vec{f}^{\,3}_{kj}((x, j, \vec{b}))$ without changing Γ. It follows that if $f : B_1 \to B_2$ is a continuous map with $f(U_k) \subseteq V_{f^2(k)}$, and there are continuous functions $\vec{f}^{\,3}_{kj} : (U_k \cap U_j) \times \{j\} \times \mathbb{R}^{m_1} \to \mathbb{R}^{m_2}$ such that for $x \in U_k \cap U_i \cap U_{j_1} \cap U_{j_2}$,

$$\vec{f}^{\,3}_{kj_1}((x, j_1, \vec{b}_1)) = g^2_{f^2(k)f^2(i)}(f(x))\vec{f}^{\,3}_{ij_2}((x, j_2, g^1_{j_2 j_1}(x)(\vec{b}_1))), \tag{18}$$

then the collection

$$\begin{aligned} f_{kj}((x, j, \vec{b})) &= \left(f(x), f^2(k), \vec{f}^{\,3}_{kj}((x, j, \vec{b}))\right) &(19)\\ &= \left(f(x), f^2(k), \vec{f}^{\,3}_{kk}((x, k, g^1_{kj}(x)(\vec{b})))\right) &(20)\end{aligned}$$

defines a fiber map $E_1 \to E_2$, for $x \in U_k \cap U_j$, so by (18), all these expressions are equal:

$$\begin{aligned} \Gamma([x, k, \vec{a}]) &= \Gamma([x, j, g^1_{jk}(x)(\vec{a})]) \\ &= \left[f(x), f^2(k), \vec{f}^{\,3}_{kk}((x, k, \vec{a}))\right] \\ &= \left[f(x), f^2(k), \vec{f}^{\,3}_{kj}((x, j, g^1_{jk}(x)(\vec{a})))\right] \\ &= \left[f(x), f^2(j), g^2_{f^2(j)f^2(k)}(f(x))\vec{f}^{\,3}_{kj}((x, j, g^1_{jk}(x)(\vec{a})))\right] \\ &= \left[f(x), f^2(j), \vec{f}^{\,3}_{jk}((x, k, \vec{a}))\right] \\ &= \left[f(x), f^2(j), \vec{f}^{\,3}_{jj}((x, j, g^1_{jk}(x)(\vec{a})))\right]. \end{aligned}$$

To check whether Γ is a morphism of vector bundles, using Equations (9) and (10), for $x \in U_k \cap U_j$,

$$(\Phi^1_{k,x})^{-1} : \mathbb{R}^{m_1} \to p_1^{-1}(\{x\}) : \vec{a} \mapsto [x, j, (g^1_{jk}(x))(\vec{a})] = [x, k, \vec{a}].$$

This is mapped by Γ to

$$\left[f(x), f^2(k), \vec{f}^{\,3}_{kk}((x, k, \vec{a}))\right]$$

and then by $\Phi^2_{\alpha,f(x)} : p_2^{-1}(\{f(x)\}) \to \mathbb{R}^{m_2}$ to

$$g^2_{\alpha,f^2(k)}(f(x))(\vec{f}^{\,3}_{kk}((x,k,\vec{a}))) = g^2_{\alpha,f^2(k)}(f(x))(\vec{f}^{\,3}_{kj}((x,j,g^1_{jk}(x)(\vec{a})))).$$

Since $g^1_{jk}(x)$ and $g^2_{\alpha,f^2(k)}(f(x))$ are linear, Γ will be a morphism of vector bundles if for each fixed x, j, the transformation $\mathbb{R}^{m_1} \to \mathbb{R}^{m_2} : \vec{b} \mapsto \vec{f}^{\,3}_{kj}((x,j,\vec{b}))$ is linear. So, $\vec{f}^{\,3}_{kj}$ can be represented as a $m_2 \times m_1$ matrix with entries depending on x, subject to the transformation rule (18). Considering (20), the j index only appears at one point in the RHS, so the following notation can be introduced: $\vec{f}^{\,3}_{kj}((x,j,\vec{b})) = F^3_k(x) \cdot g^1_{kj}(x) \cdot \vec{b}$, for a $m_2 \times m_1$ matrix $F^3_k(x)$ with entries depending continuously on x, indexed by k only. It follows that $\vec{f}^{\,3}_{kk}((x,k,\vec{a})) = F^3_k(x) \cdot \vec{a}$. The transformation rule (18) applied to F^3_k, after a brief computation, becomes:

$$F^3_k(x) = g^2_{f^2(k)f^2(i)}(f(x)) \cdot F^3_i(x) \cdot g^1_{ik}(x). \tag{21}$$

The conclusion here is that a vector bundle morphism can be expressed in the following simple form — a matrix representation. Given $f : B_1 \to B_2$, and any collection of functions $F^3_k(x) : U_k \to \mathrm{Hom}(\mathbb{R}^{m_1}, \mathbb{R}^{m_2})$ satisfying (21) for $x \in U_k \cap U_i$, the following formula defines a vector bundle morphism:

$$\Gamma([x,k,\vec{a}]) = [f(x), f^2(k), F^3_k(x) \cdot \vec{a}]. \tag{22}$$

To see the local coordinate expression for a morphism of vector bundles, use coordinate charts $\phi_k : U_k \to \mathbb{R}^{m_1}$ for B_1 and $\psi_\alpha : V_\alpha \to \mathbb{R}^{m_2}$ for B_2, with $f(U_k) \subseteq V_{f^2(k)}$ as above. Let $p_1^{-1}(U_k) \cap p_1^{-1}(U_j)$ be an open set in E_1, so that as in (7),

$$((\phi_k \times Id_{\mathbb{R}^{m_1}}) \circ \Phi^1_k)(p_1^{-1}(U_k) \cap p_1^{-1}(U_j)) \subseteq \mathbb{R}^{n_1+m_1}$$

is a coordinate neighborhood, where as in (12),

$$\begin{aligned} ((\phi_k \times Id_{\mathbb{R}^{m_1}}) \circ \Phi^1_k)^{-1} : \mathbb{R}^{n_1+m_1} &\to p_1^{-1}(U_j) \cap p_1^{-1}(U_k) \\ (\vec{v},\vec{a}) &\mapsto [\phi_k^{-1}(\vec{v}), j, (g^1_{jk}(\phi_k^{-1}(\vec{v})))(\vec{a})] \\ &= [\phi_k^{-1}(\vec{v}), k, \vec{a}]. \end{aligned}$$

This is mapped by Γ to:

$$\begin{aligned} &\left[f(\phi_k^{-1}(\vec{v})), f^2(k), \vec{f}^{\,3}_{kk}((\phi_k^{-1}(\vec{v}), k, \vec{a}))\right] \\ = &\left[f(\phi_k^{-1}(\vec{v})), f^2(k), \vec{f}^{\,3}_{kj}((\phi_k^{-1}(\vec{v}), j, (g^1_{jk}(\phi_k^{-1}(\vec{v})))(\vec{a})))\right] \\ = &\left[f(\phi_k^{-1}(\vec{v})), f^2(k), F^3_k(\phi_k^{-1}(\vec{v})) \cdot \vec{a}\right], \end{aligned}$$

and then, as in (11), by $(\psi_\alpha \times Id_{\mathbb{R}^{m_2}}) \circ \Phi^2_\alpha$ to:

$$\begin{aligned} & \Big(\psi_\alpha(f(\phi_k^{-1}(\vec{v}))), (g^2_{f^2(k)\alpha}(f(\phi_k^{-1}(\vec{v}))))^{-1}(\vec{f}^{\,3}_{kk}((\phi_k^{-1}(\vec{v}), k, \vec{a})))\Big) \qquad (23) \\ = \; & \Big(\psi_\alpha(f(\phi_k^{-1}(\vec{v}))), (g^2_{\alpha f^2(k)}(f(\phi_k^{-1}(\vec{v}))))(\vec{f}^{\,3}_{kj}((\phi_k^{-1}(\vec{v}), j, (g^1_{jk}(\phi_k^{-1}(\vec{v})))(\vec{a}))))\Big) \\ = \; & \Big(\psi_\alpha(f(\phi_k^{-1}(\vec{v}))), g^2_{\alpha f^2(k)}(f(\phi_k^{-1}(\vec{v}))) \cdot F^3_k(\phi_k^{-1}(\vec{v})) \cdot \vec{a}\Big). \end{aligned}$$

A special case of a vector bundle morphism is that a section $S : B \to \mathrm{Hom}(E_1, E_2)$ can define a morphism $\Gamma_S : E_1 \to E_2$ by the formula, for $x \in U_k$,

$$[x, k, \vec{a}] \mapsto [x, k, S_k(x) \cdot \vec{a}]. \qquad (24)$$

More precisely, recall S is defined on U_k by matrix valued functions $S_k(x)$, so for $(x, j, \vec{a}) \in Q_1^{-1}(p_1^{-1}(U_k)) \subseteq \Pi_1$, let $f_k((x, j, \vec{a})) = f_{kj}((x, j, \vec{a})) = (x, j, S_j(x) \cdot \vec{a}) \in \Pi_2$ as in Lemma 2.5 and (17). Converting from j to k coordinates using (15),

$$(x, j, S_j(x) \cdot \vec{a}) \sim (x, k, g^2_{kj}(x) \cdot S_j(x) \cdot \vec{a}) = (x, k, S_k(x) \cdot g^1_{kj}(x) \cdot \vec{a}),$$

so $f(x) = x$, $f^2(k) = k$, and the expression

$$\vec{f}^{\,3}_{kj}((x, j, \vec{a})) = S_k(x) \cdot g^1_{kj}(x) \cdot \vec{a}$$

satisfy the transformation rule (18). This is a special case of the previous F^3_k construction, with $F^3_k(x) = S_k(x)$, and where the transformation rules (15) and (21) are equivalent. In the $(\vec{v}, \vec{a})$ local coordinates as in (23), the formula for Γ_S is

$$\begin{aligned} (\vec{v}, \vec{a}) \; & \mapsto \; \big((\phi_\alpha \circ \phi_k^{-1})(\vec{v}), g^2_{\alpha k}(\phi_k^{-1}(\vec{v})) \cdot S_k(\phi_k^{-1}(\vec{v})) \cdot \vec{a}\big) \qquad (25) \\ & = \; \big((\phi_\alpha \circ \phi_k^{-1})(\vec{v}), S_\alpha(\phi_k^{-1}(\vec{v})) \cdot g^1_{\alpha k}(\phi_k^{-1}(\vec{v})) \cdot \vec{a}\big). \end{aligned}$$

The action of S on a section $s^1 : V \to E_1$ as in (16) from Section 2.4 is the same as composing $\Gamma_S : E_1 \to E_2$ with s^1:

$$S(x) \cdot s^1(x) = [x, k, S_k(x) \cdot \vec{s}^{\,1}_k(x)] = (\Gamma_S \circ s^1)(x).$$

Definition 2.6. Given a continuous function $f : B_1 \to B_2$ and a vector bundle $E \to B_2$ with fiber $\mathbb{R}^n$, open cover $V_k \subseteq B_2$, and transition functions g_{jk}, the open sets $f^{-1}(B_2)$ are an open cover of B_1, and the functions $g_{jk} \circ f$ satisfy the cocycle identities on $f^{-1}(V_j) \cap f^{-1}(V_k)$, so they define a bundle with base B_1 and fiber $\mathbb{R}^n$: the <u>pullback bundle</u> $f^*E \to B_1$.

There is a canonical bimorphism $\varepsilon : f^*E \to E$. Since f maps $f^{-1}(V_k)$ to V_k, define f^2 from (19) by $f^2(k) = k$. Let $F_k^3(x)$, as in (22), be the constant matrix $Id_{\mathbb{R}^n}$. Then, the transformation rule (21) is satisfied, so

$$[x, k, \vec{a}] \mapsto [f(x), k, \vec{a}]$$

is a well-defined morphism of vector bundles $f^*E \to E$.

Conversely, if $\Gamma : E_1 \to E_2$ is a morphism of the form $[f(x), f^2(k), F_k^3(x) \cdot \vec{a}]$, then there is a morphism $\gamma : E_1 \to f^*E_2$ such that $\varepsilon \circ \gamma = \Gamma$. As previously assumed, $f(U_k) \subseteq V_{f^2(k)}$. For $x \in U_k \subseteq f^{-1}(V_{f^2(k)})$, define $\gamma : [x, k, \vec{a}] \mapsto [x, f^2(k), F_k^3(x) \cdot \vec{a}]$; for $x \in f^{-1}(V_{f^2(j)})$, define $\varepsilon : [x, f^2(j), \vec{b}] \mapsto [f(x), f^2(j), \vec{b}]$. γ is a well-defined morphism, satsifying the transformation rule (21), using the transition functions $g_{jk}^2 \circ f$ from Definition 2.6. If Γ is a bimorphism, then γ is an isomorphism.

2.6 Regularity for bundles

Let B be a $\mathcal{C}^r$ manifold as in Section 2.1, and let E be a vector bundle with open cover $U_k \subseteq B$ as in Section 2.3. Expression (7) shows E is a manifold with at least $\mathcal{C}^0$ regularity. If E is a $\mathcal{C}^s$ manifold, then by (13), $\phi_j \circ \phi_k^{-1}$ is $\mathcal{C}^{s'}$ with $s' \geq s$, so B is a $\mathcal{C}^{s'}$ manifold; if the given regularity of B is $r < s'$, then r can be replaced by s' and then $r \geq s$. The remaining case is $r \geq s' \geq s$, so in either case, given E and B, we can assume $0 \leq s \leq r$.

If E is a $\mathcal{C}^s$ manifold, then by (13), every function $g_{jk} \circ \phi_k^{-1} : \phi_k(U_j \cap U_k) \to GL(m, \mathbb{R})$ is $\mathcal{C}^{s'}$ with $s' \geq s$. Conversely, if every $g_{jk} \circ \phi_k^{-1}$ is $\mathcal{C}^{s'}$ then E is a $\mathcal{C}^s$ manifold with $s = \min\{r, s'\}$.

If E_1 and E_2 are two bundles with base B as in Section 2.4, and E_1 is a $\mathcal{C}^{s_1}$ manifold and E_2 is a $\mathcal{C}^{s_2}$ manifold, then $s_1 \leq r$, $s_2 \leq r$, every $g_{jk}^1 \circ \phi_k^{-1}$ is $\mathcal{C}^{s_1'}$ with $s_1' \geq s_1$, and every $g_{jk}^2 \circ \phi_k^{-1}$ is $\mathcal{C}^{s_2'}$ with $s_2' \geq s_2$. By (14), every $g_{jk}^3 \circ \phi_k^{-1}$ is $\mathcal{C}^{s_3'}$ with $s_3' \geq \min\{s_1', s_2'\} \geq \min\{s_1, s_2\}$, so $\mathrm{Hom}(E_1, E_2)$ is a $\mathcal{C}^{\min\{s_1, s_2\}}$ manifold.

Because a section is a continuous map $s : B \to E$, the *a priori* regularity is at most $\mathcal{C}^s$, as in Section 2.1. In general, a section $s(x) = [x, k, \vec{s}_k(x)]$ is $\mathcal{C}^t$, $0 \leq t \leq s \leq r$, if on open sets U_k,

$$\begin{array}{rcl} (\phi_k \times Id_{\mathbb{R}^m}) \circ \Phi_k \circ s \circ \phi_k^{-1} : \phi_k(U_k) & \to & \mathbb{R}^{n+m} \\ \vec{v} & \mapsto & (\vec{v}, \vec{s}_k(\phi_k^{-1}(\vec{v}))), \end{array}$$

or equivalently $\vec{s}_k \circ \phi_k^{-1} : \mathbb{R}^n \to \mathbb{R}^m$, is $\mathcal{C}^{t'}$, $t' \geq t$.

If S is a $\mathcal{C}^{t_0}$ section of $\mathrm{Hom}(E_1, E_2)$ and s^1 is a $\mathcal{C}^{t_1}$ section of E_1, then by (16), $s^2(x) = S(x) \cdot s^1(x)$ is a $\mathcal{C}^{\min\{t_0,t_1\}}$ section of E_2.

For two vector bundles E_1, E_2, so that E_1 is a $\mathcal{C}^{s_1}$ manifold, B_1 is a $\mathcal{C}^{r_1}$ manifold, E_2 is a $\mathcal{C}^{s_2}$ manifold, and B_2 is a $\mathcal{C}^{r_2}$ manifold, consider a morphism $\Gamma : E_1 \to E_2$ with $p_2 \circ \Gamma = f \circ p_1$. Γ can be a $\mathcal{C}^{s_3}$ map with $s_3 \leq \min\{s_1, s_2\}$, and f can be a $\mathcal{C}^{r_3}$ map with $r_3 \leq \min\{r_1, r_2\}$; there is nothing in the first component of the local coordinate formula (23) that raises or lowers the regularity of f, since $\psi_\alpha \circ f \circ \phi_k^{-1}$ is exactly the local coordinate formula for f as a map $B_1 \to B_2$. The expressions in the second component of (23) are $g^2_{\alpha f^2(k)} \circ f \circ \phi_k^{-1} = (g^2_{\alpha f^2(k)} \circ \psi_\alpha^{-1}) \circ (\psi_\alpha \circ f \circ \phi_k^{-1})$ and $g^1_{jk} \circ \phi_k^{-1}$, which already appear in the local coordinate expressions for E_2, f, and E_1, and $(\vec{v}, \vec{a}) \mapsto \vec{f}^{\,3}_{kk}((\phi_k^{-1}(\vec{v}), k, \vec{a})) = (F^3_k \circ \phi_k^{-1})(\vec{v}) \cdot \vec{a}$, which is linear in $\vec{a}$, but $\mathcal{C}^{s_3}$ in the $\vec{v}$ coordinates.

If S is a $\mathcal{C}^{t_0}$ section of $\mathrm{Hom}(E_1, E_2)$, so that $S_k \circ \phi_k^{-1}$ is $\mathcal{C}^{t_0}$ on $\phi_k(U_k)$, and S defines a morphism $\Gamma_S : E_1 \to E_2$ as in (24), then by (25), Γ_S is a $\mathcal{C}^{t_0}$ map from the $\mathcal{C}^{s_1}$ manifold E_1 to the $\mathcal{C}^{s_2}$ manifold E_2.

For a $\mathcal{C}^{r_1}$ manifold B_1, vector bundle $E \to B_2$, so that E is a $\mathcal{C}^s$ manifold and B_2 is a $\mathcal{C}^{r_2}$ manifold, with $0 \leq s \leq r_2$, consider a $\mathcal{C}^{r_3}$ map $f : B_1 \to B_2$, with $r_3 \leq \min\{r_1, r_2\}$. Then the pullback bundle f^*E, as in Definition 2.6, with transition functions $g_{jk} \circ f$, is a $\mathcal{C}^{\min\{r_3,s\}}$ manifold, and the canonical bimorphism $\varepsilon : f^*E \to E$ is a $\mathcal{C}^{\min\{r_3,s\}}$ map.

2.7 The tangent bundle

Let M be a $\mathcal{C}^r$ manifold with $r \geq 1$ and coordinate charts $\phi_k : U_k \to \mathbb{R}^n$. For $x \in U_k \cap U_j$, denote by $\mathrm{D}_{\phi_k(x)}(\phi_j \circ \phi_k^{-1})$ the $n \times n$ Jacobian matrix of first derivatives of $\phi_j \circ \phi_k^{-1}$, evaluated at $\phi_k(x) \in \phi_k(U_k)$. The functions $g_{jk}(x) = \mathrm{D}_{\phi_k(x)}(\phi_j \circ \phi_k^{-1})$ satisfy the cocycle identities: $g_{kk}(x) = Id_{\mathbb{R}^n}$ and $g_{ij}(x)g_{jk}(x) = g_{ik}(x)$, by the Chain Rule, so they define a vector bundle $TM \to M$ with fiber $\mathbb{R}^n$. The composites $g_{jk} \circ \phi_k^{-1}$ are $\mathcal{C}^{r-1}$ functions, so TM is a $\mathcal{C}^{r-1}$ manifold.

Elements of TM are, as in Section 2.3, equivalence classes of ordered triples, where $x \in U_k \subseteq M$, $\vec{a} \in \mathbb{R}^n$, and $[x, k, \vec{a}]$ is the equivalence class of $(x, k, \vec{a})$ under the relation

$$(x, k, \vec{a}) \sim (y, j, \vec{b}) \iff x = y \text{ and } \mathrm{D}_{\phi_k(x)}(\phi_j \circ \phi_k^{-1}) \cdot \vec{a} = \vec{b}. \qquad (26)$$

We could call x a "base point" and $\vec{a}$ a "tangent vector." A vector field on M (or an open subset) is a section of TM, so it can be defined as in Section 2.3 by the formula $v(x) = [x, k, \vec{v}_k(x)]$, where $\vec{v}_k : U_k \to \mathbb{R}^n$ is any collection of functions subject to the coordinate change rule $\vec{v}_j(x) = \mathrm{D}_{\phi_k(x)}(\phi_j \circ \phi_k^{-1}) \cdot \vec{v}_k(x)$ on $U_k \cap U_j$. A vector field is $\mathcal{C}^t$, $0 \le t \le r-1$, if on open sets U_k, $\vec{v}_k \circ \phi_k^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, is $\mathcal{C}^{t'}$, $t' \ge t$.

Let M' be another $\mathcal{C}^{r'}$ manifold with $r' \ge 1$ and coordinate charts $\psi_{k'} : V_{k'} \to \mathbb{R}^{n'}$, as in Section 2.1. Suppose $u : M' \to M$ is a $\mathcal{C}^{r''}$ map, and there is an expression $f^2(k')$ so that $u(V_{k'}) \subseteq U_{f^2(k')}$, as in (19).

A map from one tangent bundle to another, of the form $\Gamma : TM' \to TM$, defined as in Section 2.5 by a formula of the form

$$\Gamma([x', k', \vec{a}]) = [u(x'), f^2(k'), F^3_{k'}(x') \cdot \vec{a}],$$

is well-defined on the whole space if and only if it respects the equivalence relation (26); if $(x', k', \vec{a}) \sim (x', j', \vec{b})$, then

$$(u(x'), f^2(k'), F^3_{k'}(x') \cdot \vec{a}) \sim (u(x'), f^2(j'), F^3_{j'}(x') \cdot \vec{b}),$$

that is:

$$\begin{aligned}
\vec{b} &= \mathrm{D}_{\psi_{k'}(x')}(\psi_{j'} \circ \psi_{k'}^{-1}) \cdot \vec{a}, \\
F^3_{j'}(x') \cdot \vec{b} &= F^3_{j'}(x') \cdot \mathrm{D}_{\psi_{k'}(x')}(\psi_{j'} \circ \psi_{k'}^{-1}) \cdot \vec{a} \\
&= \mathrm{D}_{\phi_{f^2(k')}(u(x'))}(\phi_{f^2(j')} \circ \phi^{-1}_{f^2(k')}) \cdot F^3_{k'}(x') \cdot \vec{a}, \\
F^3_{k'}(x') &= (\mathrm{D}_{\phi_{f^2(k')}(u(x'))}(\phi_{f^2(j')} \circ \phi^{-1}_{f^2(k')}))^{-1} \cdot F^3_{j'}(x') \cdot \mathrm{D}_{\psi_{k'}(x')}(\psi_{j'} \circ \psi_{k'}^{-1}) \\
&= \mathrm{D}_{\phi_{f^2(j')}(u(x'))}(\phi_{f^2(k')} \circ \phi^{-1}_{f^2(j')}) \cdot F^3_{j'}(x') \cdot \mathrm{D}_{\psi_{k'}(x')}(\psi_{j'} \circ \psi_{k'}^{-1}). \quad (27)
\end{aligned}$$

The transformation rule (27) exactly matches rule (21).

Example 2.7. Given an open set $U \subseteq \mathbb{R}^n$, the product $U \times \mathbb{R}^m$ can be considered a trivial vector bundle as follows. U admits an open covering by one open set, itself, so $\Lambda = \{1\}$, with one coordinate chart $Id : U \to \mathbb{R}^n$, giving U a $\mathcal{C}^\omega$ differential structure. Let there be one transition function $g_{11}(x) = Id \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m)$. The equivalence classes in $\Pi = U \times \{1\} \times \mathbb{R}^m$ are singletons, $\{(x, 1, \vec{a})\} = [x, 1, \vec{a}]$, so $Q : \Pi \to E$ is a homeomorphism, E is a vector bundle with projection $p : E \to U$ and a homeomorphism $\Phi_1 : E \to U \times \mathbb{R}^m : [x, 1, \vec{a}] \mapsto (x, \vec{a})$. When $m = n$, this construction matches the definition of tangent bundle, and there is no information lost by identifying $[x, 1, \vec{a}] \in TU$ with $(x, \vec{a}) \in U \times \mathbb{R}^n$. This TU is a $\mathcal{C}^\omega$ manifold.

For $u : M' \to M$ as in Section 2.1, denote by $\mathrm{D}_{\vec{x}}(\phi \circ u \circ \psi^{-1})$ the $n \times n'$ Jacobian matrix of first derivatives, evaluated at $\vec{x} \in \mathbb{R}^{n'}$. For a fixed map u, and fixed point $x' \in M'$, but different charts $\psi_{j'}$, $\psi_{k'}$, ϕ_j, ϕ_k, the Jacobians $\mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1})$ and $\mathrm{D}_{\psi_{k'}(x')}(\phi_k \circ u \circ \psi_{k'}^{-1})$ are related by the chain rule:

$$\begin{aligned}
& \mathrm{D}_{\psi_{k'}(x')}(\phi_k \circ u \circ \psi_{k'}^{-1}) \\
= \ & \mathrm{D}_{\psi_{k'}(x')}(\phi_k \circ \phi_j^{-1} \circ \phi_j \circ u \circ \psi_{j'}^{-1} \circ \psi_{j'} \circ \psi_{k'}^{-1}) \\
= \ & \mathrm{D}_{\phi_j(u(x'))}(\phi_k \circ \phi_j^{-1}) \cdot \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot \mathrm{D}_{\psi_{k'}(x')}(\psi_{j'} \circ \psi_{k'}^{-1}) \quad (28)
\end{aligned}$$

Notation 2.8. Corresponding to the previously considered map $u : M' \to M$, with charts $u(V_{j'}) \subseteq U_{f^2(j')}$, abbreviate $j = f^2(j')$ and $k = f^2(k')$. Then, in view of the above transformation rule (28) for Jacobians, the matrix expression

$$F^3_{j'}(x') = \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) = \mathrm{D}_{\psi_{j'}(x')}(\phi_{f^2(j')} \circ u \circ \psi_{j'}^{-1})$$

satisfies (27), so the map on trivializations defined by the formula:

$$(x', j', \vec{b}) \mapsto (u(x'), j, \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot \vec{b}) \quad (29)$$

respects the equivalence relation (26), and the following differential map $du : TM' \to TM$ is a well-defined morphism of vector bundles:

$$du : [x', j', \vec{b}] \mapsto [u(x'), j, \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot \vec{b}].$$

For a $\mathcal{C}^{r''}$ map $u : M' \to M$, by (29), the morphism du is a $\mathcal{C}^{r''-1}$ map from the $\mathcal{C}^{r'-1}$ manifold TM' to the $\mathcal{C}^{r-1}$ manifold TM. The composite of vector bundle morphisms is another morphism, and the differential map of a composite satisfies $d(u \circ v) = (du) \circ (dv)$, by the chain rule.

Remark 2.9. The linear map $\mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1})$, and therefore the differential du, can be defined even if u is merely differentiable, not necessarily $\mathcal{C}^1$.

Example 2.10. As a special case, consider the $\mathcal{C}^r$ manifold M, and choose just one of its coordinate charts, $\phi_k : U_k \to \mathbb{R}^n$. Also, consider the open set $\phi_k(U_k) \subseteq \mathbb{R}^n$ as a $\mathcal{C}^\omega$ manifold with one coordinate chart $Id : \phi_k(U_k) \to \mathbb{R}^n$, as in Example 2.7, so that the tangent bundle of $\phi_k(U_k)$ is trivial, with a homeomorphism $\Phi_1 : T(\phi_k(U_k)) \to \phi_k(U_k) \times \mathbb{R}^n : [\vec{v}, 1, \vec{a}] \mapsto (\vec{v}, \vec{a})$. The differential of the map $\phi_k : U_k \to \phi_k(U_k)$ is:

$$d\phi_k : [p, k, \vec{b}] \mapsto [\phi_k(p), 1, \mathrm{D}_{\phi_k(p)}(Id \circ \phi_k \circ \phi_k^{-1}) \cdot \vec{b}] = [\phi_k(p), 1, \vec{b}].$$

So, the differential map of the coordinate chart, in the k coordinates on the open set U_k, is represented by the identity matrix on tangent vectors.

Example 2.11. The composite $\phi_j \circ u \circ \psi_{j'}^{-1}$ from Notation 2.8, but now considered as a map from the manifold $\psi_{j'}(V_j) \subseteq \mathbb{R}^{n'}$ (with trivial tangent bundle as in Example 2.7) to the manifold $\phi_j(U_j)$ (as in Example 2.10), has differential map $d(\phi_j \circ u \circ \psi_{j'}^{-1}) : T(\psi_{j'}(V_j)) \to T(\phi_j(U_j))$:

$$\begin{aligned}
[\psi_j(x'), 1, \vec{b}] &\mapsto [\phi_j(u(x')), 1, \mathrm{D}_{Id'(\psi_{j'}(x'))}(Id \circ (\phi_j \circ u \circ \psi_{j'}^{-1}) \circ (Id')^{-1}) \cdot \vec{b}] \\
&= [\phi_j(u(x')), 1, \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot \vec{b}],
\end{aligned}$$

which is the same matrix and vector expression as (29), but with base points $\psi_j(x')$, $\phi_j(u(x'))$. For $\vec{v} \in \psi_{j'}(V_{j'})$ and $x' = \psi_{j'}^{-1}(\vec{v}) \in V_{j'}$, the above expression becomes:

$$[\vec{v}, 1, \vec{b}] \mapsto [(\phi_j \circ u \circ \psi_{j'}^{-1})(\vec{v}), 1, \mathrm{D}_{\vec{v}}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot \vec{b}],$$

Definition 2.12. A $\mathcal{C}^1$ map $u : M' \to M$ is an <u>immersion</u> means: du is one-to-one on fibers; that is, $\mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1})$ has rank $n' \leq n$ at every point x' (the rank does not depend on coordinate charts).

Definition 2.13. A map $u : M' \to M$ is an <u>embedding</u> means: u is an immersion and u is a homeomorphism onto its image $u(M')$.

Proposition 2.14. *Given $r'' \geq 1$ and a $\mathcal{C}^{r''}$ embedding $u : M' \to M$, the image $u(M)$ is a $\mathcal{C}^{r''}$ submanifold of M. Conversely, a $\mathcal{C}^{r''}$ submanifold of M is the image of a $\mathcal{C}^{r''}$ embedding.*

Sketch of Proof. Assuming there is an embedding $u : M' \to M$, the existence of submanifold charts in a $\mathcal{C}^{r''}$ structure on M as in Definition 2.1 uses the Implicit Function Theorem. The converse is that the inclusion map of a submanifold is a $\mathcal{C}^{r''}$ embedding. See ([H] Theorem 1.3.1.). ∎

Proposition 2.15. *If $u : M' \to M$ is a $\mathcal{C}^r$ embedding and a homeomorphism, then the inverse u^{-1} is also a $\mathcal{C}^r$ embedding and a homeomorphism, by the Inverse Function Theorem* ([H]). ∎

Definition 2.16. Let A be a $\mathcal{C}^r$ submanifold of M. A <u>$\mathcal{C}^{s'}$ tubular neighborhood</u> of A is an open set $f(E)$ with $M \subseteq f(E) \subseteq V$, where E is a $\mathcal{C}^s$ vector bundle with base A and $f : E \to M$ is a $\mathcal{C}^{s'}$ embedding (so $0 \leq s' \leq s \leq r$) such that for $x \in A$, $f([x, k, \vec{0}]) = x$.

Proposition 2.17. *If $1 \leq r \leq \infty$ and A is a $\mathcal{C}^r$ submanifold of M, then there exists a $\mathcal{C}^r$ tubular neighborhood of A.*

Sketch of Proof. The definition of k-submanifold (Definition 2.1) is that there is a local version of a tubular neighborhood: at a point in A, there is a $\mathcal{C}^r$ map ϕ^{-1} from a neighborhood of the trivial bundle $\mathbb{R}^k \times \mathbb{R}^{n-k}$ to a neighborhood of the point in M.

The claimed global existence, with $s' = s = r$, is stated but not proved as Exercise 4.6.1. of [H]. One construction sets the bundle E from Definition 2.16 equal to the normal bundle of A ([H] §§4.2, 4.5.), a subbundle of TA — however, TA is a $\mathcal{C}^{r-1}$ bundle, so $s \leq r-1$ in this case. Some $\mathcal{C}^r$ approximation to the embedding of the normal bundle must be used instead, as in [P]. ∎

3 Almost complex structures

3.1 Representation in local coordinates

Continuing with a $\mathcal{C}^r$ manifold M, let $\dim M = 2n$ and $r \in [1, \infty]$, so TM is a $\mathcal{C}^{r-1}$ manifold; denote the tangent space at the point x by $p^{-1}(x) = T_xM$. The bundle $\mathrm{Hom}(TM, TM)$ (from Section 2.4) is also a $\mathcal{C}^{r-1}$ manifold, and a section $J : M \to \mathrm{Hom}(TM, TM)$ is defined by matrix valued functions on open sets in M, $J_k : U_k \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$, satisfying (15):

$$\begin{aligned} J_k(x) &= \mathrm{D}_{\phi_j(x)}(\phi_k \circ \phi_j^{-1}) \cdot J_j(x) \cdot \mathrm{D}_{\phi_k(x)}(\phi_j \circ \phi_k^{-1}) \\ &= (\mathrm{D}_{\phi_k(x)}(\phi_j \circ \phi_k^{-1}))^{-1} \cdot J_j(x) \cdot \mathrm{D}_{\phi_k(x)}(\phi_j \circ \phi_k^{-1}). \end{aligned} \tag{30}$$

so $J(x) = [x, k, J_k(x)]$ is well-defined. If $J_k(x)$ is a CSO on $\mathbb{R}^{2n}$ ($J_k(x) \cdot J_k(x) = -Id_{\mathbb{R}^{2n}}$), then, because (30) is a similarity transformation, so is $J_j(x)$ for any j (Lemma 1.3). For $0 \leq s \leq r-1$, a $\mathcal{C}^s$ section $J : M \to \mathrm{Hom}(TM, TM)$ such that each matrix $J_k(x)$ is a CSO is an "almost complex structure" of regularity $\mathcal{C}^s$ on M. As in (24), J also defines a $\mathcal{C}^s$ homeomorphism $TM \to TM$:

$$[x, k, \vec{a}] \mapsto [x, k, J_k(x) \cdot \vec{a}],$$

which satisfies the transformation rule (27), as shown by (30). This vector bundle morphism can also be denoted J; the regularity condition is that each $J_k \circ \phi_k^{-1} : \mathbb{R}^{2n} \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is $\mathcal{C}^{s'}$ with $s' \geq s$.

Example 3.1. For $u : M' \to M$ as in Notation 2.8, suppose u is an invertible $\mathcal{C}^{r''}$ embedding with $0 \leq r'' \leq \min\{r, r'\}$, and there are open covers so that $u(V_{j'}) = U_j$. Let J be a $\mathcal{C}^s$ almost complex structure on M, $0 \leq s \leq r - 1$. Using (29), the vector bundle morphism

$$(du)^{-1} \circ J \circ du = d(u^{-1}) \circ J \circ du : TM' \to TM'$$

is defined in local coordinates by

$$\begin{aligned} & (x', j', \vec{b}) \\ \mapsto\ & (x', j', D_{\phi_j(u(x'))}(\psi_{j'} \circ u^{-1} \circ \phi_j^{-1}) \cdot J_j(u(x')) \cdot D_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_j^{-1}) \cdot \vec{b}), \end{aligned}$$

so the matrix expression

$$J'_{j'}(x') = (D_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_j^{-1}))^{-1} \cdot J_j(u(x')) \cdot D_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_j^{-1})$$

is a CSO similar to $J_j(u(x'))$, and defines a $\mathcal{C}^{\min\{s, r''-1\}}$ almost complex structure $J'(x')$ on M'.

Example 3.2. As a special case of Example 3.1, let $u = \phi_k^{-1} : \phi_k(U_k) \to U_k$, as in Example 2.10. An almost complex structure J on M restricts to an almost complex structure on the open set U_k. Then

$$(d(\phi_k^{-1}))^{-1} \circ J \circ d(\phi_k^{-1}) : [\vec{v}, 1, \vec{b}] \mapsto [\vec{v}, 1, J_k(\phi_k^{-1}(\vec{v})) \cdot \vec{b}] \tag{31}$$

is an almost complex structure on $\phi_k(U_k) \subseteq \mathbb{R}^{2n}$, where the matrix-valued function $\vec{v} \mapsto J_k(\phi_k^{-1}(\vec{v}))$ is the same as the local formula for J in the k coordinate chart U_k and has the same $\mathcal{C}^s$ regularity.

Example 3.3. Let M be a $\mathcal{C}^r$ manifold with $\mathcal{C}^s$ almost complex structure J, $0 \leq s \leq r - 1$, defined on charts $\phi_k : V_k \to \mathbb{R}^{2n}$ by $J_k(x)$. Consider an open subset V of M, and a map $\phi : V \to \mathbb{R}^{2n}$ so that ϕ is a homeomorphism onto its image $U = \phi(V)$, and $\phi \circ \phi_k^{-1}$ is $\mathcal{C}^\rho$ for all k (with $1 \leq \rho$, so, ϕ could be a chart, added to the $\mathcal{C}^r$ atlas of M, but we are not assuming any local formula for J on this chart). U has an open cover $U_0 = U$, and $U_k = \phi(V \cap V_k)$. The coordinate chart on U_0 is the inclusion $\phi_0 : U_0 \to \mathbb{R}^{2n}$. The tangent bundle of U has a (global) trivialization $[x, 0, \vec{a}] = (x, \vec{a}) \in U \times \mathbb{R}^{2n}$, and local trivializations with transition functions $g_{jk}(x)$ depending on the coordinate charts for U_k. U has an almost complex structure $J' = d\phi \circ J \circ d(\phi^{-1})$ as in Example 3.1 with $u = \phi^{-1}$. J' has some matrix representation in the neighborhood U_0 with the globally trivial tangent bundle: to calculate it, we will first find the matrix representation in the U_k neighborhoods, and then use formula (30) to convert from U_k coordinates to U_0 coordinates.

Because U is an open subset of $\mathbb{R}^{2n}$, there are two different ways to assign coordinate charts to the open sets U_k. It will turn out that the matrix representation of J' in U_0 does not depend on the method.

Method 1. Assign to U_k the chart equal to the composite $\phi_k \circ \phi^{-1}|_{U_k} : U_k \to \mathbb{R}^{2n}$. The coordinate change functions on U from U_j to U_k are $(\phi_k \circ \phi^{-1}) \circ (\phi_j \circ \phi^{-1})^{-1} = \phi_k \circ \phi_j^{-1}$, which are $\mathcal{C}^r$ functions, and from U_0 to U_k are $\phi_k \circ \phi^{-1} \circ (\phi_0|_{U_k})^{-1}$, which are $\mathcal{C}^\rho$ functions (by hypothesis and Proposition 2.15), so with these charts, U is a $\mathcal{C}^{\min\{r,\rho\}}$ manifold. The coordinate representation for ϕ in the V_k, U_k neighborhoods is $(\phi_k \circ \phi^{-1}) \circ \phi \circ \phi_k^{-1}$, which is the identity on $\phi_k(V \cap V_k)$. By construction (similar to Example 2.10), in the V_k and U_k neighborhoods,

$$d\phi : [x, k, \vec{a}] \mapsto [\phi(x), k, \vec{a}],$$

and similarly for $d(\phi^{-1})$, so the matrix representation of $d\phi \circ J \circ d(\phi^{-1})$ in the U_k neighborhood is $[x, k, \vec{a}] \mapsto [x, k, J_k(\phi^{-1}(x)) \cdot \vec{a}]$. Using (30) to convert from U_k coordinates to U_0 coordinates, $J_k(\phi^{-1}(x))$ transforms to

$$J_0'(x) = (\mathrm{D}_x(\phi_k \circ \phi^{-1}))^{-1} \cdot J_k(\phi^{-1}(x)) \cdot \mathrm{D}_x(\phi_k \circ \phi^{-1}). \tag{32}$$

This matrix expression is a $\mathcal{C}^{\min\{s,\rho-1\}}$ function of x, and by (30), does not depend on k (replacing k with j gives the same matrix).

Method 2. Assign to U_k the chart equal to the inclusion $\phi_0|_{U_k} : U_k \to \mathbb{R}^{2n}$. The coordinate change functions on U from U_j to U_k are identity maps on $U_k \cap U_j$, so with these charts, U is a $\mathcal{C}^\omega$ manifold. The coordinate representation for ϕ in the V_k, U_k neighborhoods is $\phi_0|_{U_k} \circ \phi \circ \phi_k^{-1} = \phi \circ \phi_k^{-1}$. By construction, in the V_k and U_k neighborhoods,

$$\begin{aligned} d\phi : [x, k, \vec{a}] &\mapsto [\phi(x), k, \mathrm{D}_{\phi(x)}(\phi \circ \phi_k^{-1}) \cdot \vec{a}], \\ d(\phi^{-1}) : [\phi(x), k, \vec{b}] &\mapsto [x, k, \mathrm{D}_x(\phi_k \circ \phi^{-1}) \cdot \vec{b}]. \end{aligned}$$

So, the matrix representation of $d\phi \circ J \circ d(\phi^{-1})$ in the U_k neighborhood is

$$(\mathrm{D}_x(\phi_k \circ \phi^{-1}))^{-1} \cdot J_k(\phi^{-1}(x)) \cdot \mathrm{D}_x(\phi_k \circ \phi^{-1}).$$

Using (30) to convert from U_k coordinates to U_0 coordinates, the matrix representation does not change, so $J_0'(x)$ is exactly the same as (32).

3.2 Pointwise normalization

Given any chart on M, $\phi_j : U_j \to \mathbb{R}^{2n}$, the matrix $J_j(\phi_j^{-1}(\vec{0}))$ is a CSO on $T_{\phi_j^{-1}(\vec{0})}M$, which can be temporarily denoted J^0. There exists some $G \in GL(2n, \mathbb{R})$ such that $J^0 = G^{-1} \cdot J_{std} \cdot G$. We may consider a new chart on M, $\phi_k : U_k \to \mathbb{R}^{2n}$, where $U_k = U_j$ and $\phi_k = G \circ \phi_j$. Then, $\phi_k^{-1}(\vec{0}) = \phi_j^{-1}(\vec{0})$, and by the transformation rule (30),

$$\begin{aligned} J_k(\phi_k^{-1}(\vec{0})) &= (\mathrm{D}_{\phi_k(\phi_k^{-1}(\vec{0}))}(\phi_j \circ \phi_k^{-1}))^{-1} \cdot J_j(\phi_j^{-1}(\vec{0})) \cdot \mathrm{D}_{\phi_k(\phi_k^{-1}(\vec{0}))}(\phi_j \circ \phi_k^{-1}) \\ &= (\mathrm{D}_{\vec{0}}(G^{-1}))^{-1} \cdot J^0 \cdot \mathrm{D}_{\vec{0}}(G^{-1}) \\ &= G \cdot J^0 \cdot G^{-1} = J_{std}. \end{aligned} \tag{33}$$

The conclusion is that at any point $x \in M$, there is some chart ϕ_k on M so that $J_k(x)$, the matrix representation of J at the one point x in the k coordinate system, is equal to J_{std}.

By the continuity of J, and considering the inverse formula appearing in Equation (1), there is some possibly smaller neighborhood $U_\ell \subseteq U_k$ of x on which $J_k + J_{std}$ is invertible at every point of U_ℓ. The neighborhood U_ℓ has coordinate chart ϕ_ℓ equal to just the restriction of ϕ_k to U_ℓ, so $J_\ell(x) = J_{std}$ still works. Later, it will be convenient to assume that coordinate charts in M are always chosen with these two properties (the normalization at the point x, and the invertibility of the sum on the neighborhood).

There is an even stronger normalization possible in the $n = 1$ case.

Proposition 3.4 (Korn, Lichtenstein)**.** *If M is a $\mathcal{C}^{1+\alpha}$ real surface with $\mathcal{C}^\alpha$ almost complex structure J, $0 < \alpha$, then around each point x_0 there is some chart $\phi_k : U_k \to \mathbb{R}^2$ so that the matrix representation is constant: $J_k(x) = J_{std}$ for all $x \in U_k$.*

Sketch of Proof. See [Chern], [NN], Theorems III.3.16–III.3.20 of [MP], pp. 77, 78. The proof in [MS_2] assumes J is $\mathcal{C}^{1+\alpha}$. ∎

4 Pseudoholomorphic maps

Given a $\mathcal{C}^r$ manifold M with a $\mathcal{C}^s$ $(0 \le s \le r-1)$ almost complex structure J, and similarly (M', J'), a $\mathcal{C}^{r''}$ map $u : M' \to M$ is pseudoholomorphic with respect to J', J, if the map $du : TM' \to TM$ satisfies:

$$J \circ du = du \circ J'.$$

Example 4.1. As a trivial example, if u is a diffeomorphism and M' has the induced almost complex structure $(du)^{-1} \circ J \circ du$ from Example 3.1, then u is pseudoholomorphic.

Example 4.2. As even more trivial examples, the identity map u on (M, J) is pseudoholomorphic, and any constant map $M' \to M$, so that du has matrix representation $\equiv 0$, is also pseudoholomorphic.

In terms of local charts as in Notation 2.8, the morphism $du \circ J'$ is defined by:

$$(x', j', \vec{b}) \quad \mapsto \quad (u(x'), j, \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot J'_{j'}(x') \cdot \vec{b})$$

and the map $J \circ du$ by:

$$(x', j', \vec{b}) \quad \mapsto \quad (u(x'), j, J_j(u(x')) \cdot \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot \vec{b}), \tag{34}$$

so u is pseudoholomorphic if and only if there are pairs of charts covering M' and M such that $u(V_{j'}) \subseteq U_j$, on which the following matrix-valued functions of x' are equal:

$$\mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot J'_{j'}(x') = J_j(u(x')) \cdot \mathrm{D}_{\psi_{j'}(x')}(\phi_j \circ u \circ \psi_{j'}^{-1}). \tag{35}$$

For $\vec{v} \in \psi_{j'}(V_{j'})$ and $x' = \psi_{j'}^{-1}(\vec{v}) \in V_{j'}$, LHS of (35) is:

$$\mathrm{D}_{\vec{v}}(\phi_j \circ u \circ \psi_{j'}^{-1}) \cdot (J'_{j'} \circ \psi_{j'}^{-1})(\vec{v}) \tag{36}$$

and RHS is:

$$(J_j \circ \phi_j^{-1})((\phi_j \circ u \circ \psi_{j'}^{-1})(\vec{v})) \cdot \mathrm{D}_{\vec{v}}(\phi_j \circ u \circ \psi_{j'}^{-1}). \tag{37}$$

The regularity of the LHS expression (36) is $\mathcal{C}^\lambda$, $\lambda \ge \min\{r'' - 1, s'\}$, and of the RHS (37) is $\mathcal{C}^\rho$, $\rho \ge \min\{r'' - 1, s\}$; the equality LHS=RHS does not immediately give any information about or restrictions on s, s' or r''. The composite $J_j \circ u \circ \phi_{j'}^{-1}$ from (37) is a local coordinate representation of the composite $J \circ u$, which will appear in Section 6.2.

Considering Examples 2.11 and 3.2, the equality (35) is equivalent to each composite map $\phi_j \circ u \circ \psi_{j'}^{-1} : \psi_{j'}(V_{j'}) \to \phi_j(U_j)$ being pseudoholomorphic with respect to the induced almost complex structures $J'_{j'} \circ \psi_{j'}^{-1}$ on $\psi_{j'}(V_{j'}) \subseteq \mathbb{R}^{2n'}$ and $J_j \circ \phi_j^{-1}$ on $\phi_j(U_j) \subseteq \mathbb{R}^{2n}$. So, for maps u, the pseudoholomorphic property can be checked locally by comparing the above matrix functions depending on $\vec{v}$. The local analysis or geometry of pseudoholomorphic maps can be considered, without loss of generality, by only looking at a $\mathcal{C}^{r''}$ function from an open set in $\mathbb{R}^{2n'}$ to $\mathbb{R}^{2n}$, its Jacobian matrix of first derivatives, and $\mathcal{C}^s$ (respectively $\mathcal{C}^{s'}$) matrices $J(\vec{w})$ and $J'(\vec{v})$.

Of course, Equation (35) is exactly the statement that the differential is c-linear at each point with respect to the CSOs at that point. The equality of matrices can be called the generalized Cauchy-Riemann equations. In analogy with Lemmas 1.4, 1.5, we could define an operator

$$\overline{\partial}_J(u) = \frac{1}{2}(du + J \circ du \circ J'), \tag{38}$$

(so, it projects du to its a-linear part) and then u is pseudoholomorphic if and only if $\overline{\partial}_J(u) = 0$.

5 J-holomorphic curves

Notation 5.1. For $r > 0$ and $z_0 \in \mathbb{C}$ (or $\mathbb{R}^2$), let $D(z_0, r)$ denote the Euclidean open disk in the plane with center z_0 and radius r, and as the special case with $z_0 = 0$, abbreviate $D(0, r) = D_r$.

The D_r notation need not be confused with the already used Jacobian determinant notation D.

The notation for a ball in higher dimensions is similar.

Notation 5.2. For $r > 0$ and z_0 in some normed vector space, let $B(z_0, r)$ denote the open ball with center z_0 and radius r. As special cases with $z_0 = \vec{0}$, abbreviate $B(\vec{0}, r) = B_r$ and $B(\vec{0}, 1) = B$.

5.1 Local formulation

For the local analysis of pseudoholomorphic maps $u : M' \to M$ near the points $x' \mapsto p = u(x')$, in the case where M' is a real surface, the following set up is convenient.

M is a $\mathcal{C}^r$ $2n$-manifold, $n \geq 1$, $r \geq 1$, with a $\mathcal{C}^s$ almost complex structure, J, $0 \leq s \leq r-1$. There is a coordinate chart U_j (not depending on the map u) so that $\phi_j(p) = \vec{0}$, the matrix representation of J in the local trivialization satisfies $J_j(p) = J_{std}$ (by Equation (33)), and $J_j + J_{std}$ is invertible at every point of U_j. The coordinate chart image in $\mathbb{R}^{2n}$ can be chosen to be the unit ball, $B = \phi_j(U_j)$, centered at $\vec{0}$ with radius 1.

M' is a $\mathcal{C}^{r'}$ real surface, $r' > 1$, with $\mathcal{C}^{s'}$ almost complex structure J', $0 < s' \leq r'-1$, and $u : M' \to M$ is $\mathcal{C}^\rho$, $1 \leq \rho \leq \min\{r', r\}$. By the continuity of u, there is some neighborhood V of x' so that

$$u(V) \subseteq U_j, \tag{39}$$

and by Proposition 3.4, there is a $\mathcal{C}^{s'+1}$ differential structure on M' ($s'+1 \leq r'$), and some chart $V_{j'} \subseteq V$ so that $\psi_{j'} : V_{j'} \to D_1 \subseteq \mathbb{C} = \mathbb{R}^2$, $\psi_{j'}(x') = 0$, and the induced almost complex structure on the unit disk D_1 is the constant matrix J_{std}.

In these neighborhoods, the local geometry of a map u can be reduced to the equivalent analysis of the $\mathcal{C}^{\min\{\rho, s'+1\}}$ map

$$f = \phi_j \circ u \circ \psi_{j'}^{-1} : D_1 \to B, \tag{40}$$

where $f(0) = \vec{0}$ and $B = \phi_j(U_j)$ is a neighborhood of $\vec{0}$ in $\mathbb{R}^{2n}$ with a $\mathcal{C}^s$ almost complex structure (on the trivialized tangent bundle $B \times \mathbb{R}^{2n}$):

$$J_B : B \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n}), \qquad J_B(\vec{x}) = J_j(\phi_j^{-1}(\vec{x}))$$

(as in (31)) satisfying $J_B(\vec{0}) = J_{std}$. Since the almost complex structure on the domain is always the standard complex structure, we can refer to $f : D_1 \to B$ as a J-holomorphic map (or J-holomorphic curve) if it is pseudoholomorphic with respect to J_{std} and J_B.

Let $z = (x, y)$ be the coordinate on D_1, and $(z, \vec{b})$ the coordinates on the (trivial) tangent bundle TD_1, so the differential maps (34) have the following form:

$$\begin{aligned} df \circ J' : (z, \vec{b}) &\mapsto (f(z), \mathrm{D}_z(f) \cdot J_{std} \cdot \vec{b}) \\ J_B \circ df : (z, \vec{b}) &\mapsto (f(z), J_B(f(z)) \cdot \mathrm{D}_z(f) \cdot \vec{b}), \end{aligned}$$

and the generalized Cauchy-Riemann equations are, for $f(z) = f(x,y) = (f^1, \ldots, f^{2n})^T$ (real column $2n$-vector):

$$J_B(f(z)) \cdot \begin{pmatrix} \frac{df^1}{dx} & \frac{df^1}{dy} \\ \vdots & \vdots \\ \frac{df^{2n}}{dx} & \frac{df^{2n}}{dy} \end{pmatrix} = \begin{pmatrix} \frac{df^1}{dx} & \frac{df^1}{dy} \\ \vdots & \vdots \\ \frac{df^{2n}}{dx} & \frac{df^{2n}}{dy} \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{df^1}{dy} & -\frac{df^1}{dx} \\ \vdots & \vdots \\ \frac{df^{2n}}{dy} & -\frac{df^{2n}}{dx} \end{pmatrix}. \tag{41}$$

This is equivalent to looking at just one column:

$$J_B(f(z)) \cdot \frac{df}{dx} = \frac{df}{dy}, \tag{42}$$

since multiplying both sides by $J_B(f(z))$ gives the other column in the matrix equation.

Notation 5.3. For $f : \mathbb{R}^2 \to \mathbb{R}^{2n}$, $f(z) = f(x,y) = (f^1, \ldots, f^{2n})$, the following two derivative expressions are each a real column $2n$-vector of functions of x, y:

$$\partial f = \frac{df}{dz} = \frac{1}{2} \cdot (\frac{d}{dx} - J_{std} \cdot \frac{d}{dy}) f = \frac{1}{2} \cdot \frac{df}{dx} - \frac{1}{2} \cdot J_{std} \cdot \frac{df}{dy},$$

$$\overline{\partial} f = \frac{df}{d\bar{z}} = \frac{1}{2} \cdot (\frac{d}{dx} + J_{std} \cdot \frac{d}{dy}) f = \frac{1}{2} \cdot \frac{df}{dx} + \frac{1}{2} \cdot J_{std} \cdot \frac{df}{dy}.$$

These identities follow as a consequence: $\frac{df}{dx} = (\partial + \overline{\partial}) f$, $\frac{df}{dy} = J_{std} \cdot (\partial - \overline{\partial}) f$. The generalized Cauchy-Riemann equations can then be re-expressed:

$$\begin{aligned} \frac{df}{dy} &= J_B(f(z)) \cdot \frac{df}{dx} \\ J_{std} \cdot (\partial - \overline{\partial}) f &= J_B(f(z)) \cdot (\partial + \overline{\partial}) f \\ 0 &= (J_B(f(z)) + J_{std}) \overline{\partial} f + (J_B(f(z)) - J_{std}) \partial f \\ 0 &= \overline{\partial} f + (J_B(f(z)) + J_{std})^{-1} \cdot (J_B(f(z)) - J_{std}) \cdot \partial f \\ \Longrightarrow \overline{\partial} f &= Q(f(z)) \cdot \partial f, \end{aligned} \tag{43}$$

where $Q : B \to \mathrm{Hom}_a(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ is a map whose definition does not depend on f: for $\vec{x} \in B$,

$$Q(\vec{x}) = (J_B(\vec{x}) + J_{std})^{-1} \cdot (J_{std} - J_B(\vec{x})).$$

For each point $\vec{x} \in B$, the matrix $Q(\vec{x})$ is well-defined by our earlier assumption that B is chosen small enough so that $J_B(\vec{x}) + J_{std}$ is invertible for all $\vec{x} \in B$, and then the matrix $Q(\vec{x})$ is a-linear with respect to J_{std}, J_{std}, as in Equation (2) and Lemma 1.6. By construction, $Q(\vec{0})$ is the zero matrix, and Q is a $\mathcal{C}^s$ map (same regularity as J_B).

$Q(\vec{x})$ is zero if and only if $J_B(\vec{x}) = J_{std}$, and Q is identically zero if and only if $J_B(\vec{x})$ is the constant CSO J_{std}, in which case the condition for J-holomorphic becomes just $\overline{\partial} f = 0$, so f is holomorphic in the usual sense. (Comment: The a-linear operator Q is denoted $\overline{Q}$ by [R], but otherwise our sign conventions are the same.)

5.2 Complex diagonalization and the Cauchy-Riemann equations

The eigenvalues of J_{std} are $\pm i$, and for the 2×2 case, the eigenvectors in $\mathbb{C}^2$ are $\begin{bmatrix} 1 \\ i \end{bmatrix}$ with eigenvalue $-i$, and $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ with eigenvalue i. Let $J_{2\times 2} = J(x,y)$ be a variable CSO, near J_{std}. The eigenvalues are the same (Lemma 1.11), but the eigenvectors may depend on the position, so suppose there are complex valued functions $v_1(x,y) \approx 1$, $v_2(x,y) \approx 0$ so that the $-i$ eigenspace of J is the complex line spanned by

$$v_1(x,y)\begin{bmatrix} 1 \\ i \end{bmatrix} + v_2(x,y)\begin{bmatrix} 1 \\ -i \end{bmatrix}. \tag{44}$$

Because J is real, the i eigenspace is spanned by the conjugate vector,

$$\overline{v_2(x,y)}\begin{bmatrix} 1 \\ i \end{bmatrix} + \overline{v_1(x,y)}\begin{bmatrix} 1 \\ -i \end{bmatrix}.$$

This diagonalizes J over $\mathbb{C}$: let $P_{2\times 2}(x,y) = \begin{bmatrix} v_1 + v_2 & \bar{v}_1 + \bar{v}_2 \\ iv_1 - iv_2 & i\bar{v}_2 - i\bar{v}_1 \end{bmatrix}$, so

$$\begin{aligned} P^{-1} &= \frac{1}{2(v_1\bar{v}_1 - v_2\bar{v}_2)}\begin{bmatrix} \bar{v}_1 - \bar{v}_2 & -i\bar{v}_1 - i\bar{v}_2 \\ v_1 - v_2 & i(v_1 + v_2) \end{bmatrix}, \\ J \cdot P &= P \cdot \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}. \end{aligned} \tag{45}$$

Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2 : (x,y) \mapsto (f^1(x,y), f^2(x,y))$ is pseudoholomorphic with respect to J_{std} and J. Then combining (42) with (45) gives:

$$\begin{aligned}
J(f(x,y)) \cdot \begin{bmatrix} f^1_x \\ f^2_x \end{bmatrix} &= \begin{bmatrix} f^1_y \\ f^2_y \end{bmatrix} \\
P \cdot \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \cdot P^{-1} \begin{bmatrix} f^1_x \\ f^2_x \end{bmatrix} &= \begin{bmatrix} f^1_y \\ f^2_y \end{bmatrix} \\
\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \bar{v}_1 - \bar{v}_2 & -i\bar{v}_1 - i\bar{v}_2 \\ v_1 - v_2 & i(v_1 + v_2) \end{bmatrix} \begin{bmatrix} f^1_x \\ f^2_x \end{bmatrix} &= \begin{bmatrix} \bar{v}_1 - \bar{v}_2 & -i\bar{v}_1 - i\bar{v}_2 \\ v_1 - v_2 & i(v_1 + v_2) \end{bmatrix} \begin{bmatrix} f^1_y \\ f^2_y \end{bmatrix} \\
\begin{bmatrix} -i(\bar{v}_1 - \bar{v}_2) f^1_x - (\bar{v}_1 + \bar{v}_2) f^2_x \\ i(v_1 - v_2) f^1_x - (v_1 + v_2) f^2_x \end{bmatrix} &= \begin{bmatrix} (\bar{v}_1 - \bar{v}_2) f^1_y - i(\bar{v}_1 + \bar{v}_2) f^2_y \\ (v_1 - v_2) f^1_y + i(v_1 + v_2) f^2_y \end{bmatrix}.
\end{aligned}$$

The first and second entries are complex conjugate, so the above vector equality is equivalent to setting the second entries equal and dividing by i:

$$\begin{aligned}
(v_1 - v_2) f^1_x + i(v_1 + v_2) f^2_x &= -i(v_1 - v_2) f^1_y + (v_1 + v_2) f^2_y \qquad (46) \\
v_1 \cdot ((f^1_x - f^2_y) + i(f^1_y + f^2_x)) &= v_2 \cdot ((f^1_x + f^2_y) - i(f^2_x - f^1_y)) \\
v_1(f(x,y)) \frac{\partial}{\partial \bar{z}}(f^1 + if^2) &= v_2(f(x,y)) \overline{\frac{\partial}{\partial z}(f^1 + if^2)} \qquad (47)
\end{aligned}$$

so (46) is equivalent to (47), a perturbation of the classical Cauchy-Riemann equation $\frac{\partial f}{\partial \bar{z}} = 0$. The complex conjugation on the RHS is analogous to the anti-linearity of the operator Q from Section 5.1. Equation (47) and the subspace (44) both depend only on the ratio $\frac{v_2}{v_1}$.

5.3 The effect of re-scaling

Some results in analysis require an *a priori* estimate that $J_B - J_{std}$ is small (possibly in some norm sense involving its derivatives) on the whole unit ball B. The following construction will start with a given $J_B(\vec{x})$ on B as in the previous Subsection 5.1, and modify it by "re-scaling" to get a new almost complex structure on the same set B.

It is convenient to use some previously established notation and return to the global setting of the almost complex manifold M (although $M = B$ is a suitable example). Recall the coordinate chart $\phi_j : U_j \to B$, and consider any number $0 < t \leq 1$. Let $B_t \subseteq B$ denote the ball centered at $\vec{0}$ with radius t, and let $\frac{1}{t} \cdot Id$ be the scalar multiplication (or "dilatation") operator on $\mathbb{R}^{2n}$, which maps B_t onto B. Let $U_k = \phi_j^{-1}(B_t) \subseteq U_j \subseteq M$, and define $\phi_k : U_k \to B$ by $(\frac{1}{t} \cdot Id) \circ \phi_j$.

By the transformation rule (30), the local representation J_k of the almost complex structure on the chart U_k is related to J_j by a similarity transformation, but the conjugating matrix is $\frac{1}{t} \cdot Id$ which commutes with J_j, so $J_k = J_j$.

So, in the new k coordinate system, the original almost complex structure on B, $J_B(\vec{x}) = J_j(\phi_j^{-1}(\vec{x}))$, is replaced by

$$J_{B,t}(\vec{x}) = J_k(\phi_k^{-1}(\vec{x})) = J_j(\phi_j^{-1}(t \cdot \vec{x})) = J_B(t \cdot \vec{x}),$$

that is, the new almost complex structure is related to the old one by re-scaling the input vector $\vec{x} \in B$ by t. Since all this is just a matter of different local coordinate systems on the same almost complex manifold M, for the local analysis there is no loss of generality in replacing $J_B(\vec{x})$ with $J_{B,t}(\vec{x}) = J_B(t \cdot \vec{x})$, and no change in the $\mathcal{C}^s$ regularity. The normalization condition $J_{B,t}(\vec{0}) = J_{std}$ still holds, and also the condition that $J_{B,t} - J_{std}$ is invertible still holds.

We can think of $J_{B,t}$ as a parametrized family of almost complex structures on B, where $J_{B,1} = J_B$, and using the continuity of J_B, there is a pointwise limit: for all $\vec{x} \in B$,

$$\lim_{t \to 0^+} J_{B,t}(\vec{x}) = \lim_{t \to 0^+} J_B(t \cdot \vec{x}) = J_B(\vec{0}) = J_{std},$$

so $J_{B,t}$ approaches the constant complex structure on B as $t \to 0^+$, and we can define $J_{B,0} = J_{std}$, even though the above coordinate system construction does not apply when $t = 0$.

Suppose there is some norm $\| \ \|$ on the space of $\mathcal{C}^s$ maps $B \to \mathrm{Hom}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ that has the property that if $H(\vec{0}) = 0$, then $\|H \circ (t \cdot Id)\| \le c \cdot t \cdot \|H\|$ for some c and all t such that $0 < t < t_0$, c and t_0 depending on H. For example, when $s = 2$, the usual $\mathcal{C}^2$ norm has this property. Then, $J_B - J_{std}$ has the property $J_B(\vec{0}) - J_{std} = 0$ and if $\|J_B - J_{std}\|$ is finite, then given any $\epsilon > 0$, there is some t_1 so that $\|J_{B,t} - J_{std}\| = \|J_B \circ (t \cdot Id) - J_{std}\| < \epsilon$ for all $0 < t < t_1$.

The composite $Q(\vec{x}) = (J_B(\vec{x}) + J_{std})^{-1} \cdot (J_{std} - J_B(\vec{x}))$ is also just re-scaled:

$$\begin{aligned} Q_t(\vec{x}) &= (J_{B,t}(\vec{x}) + J_{std})^{-1} \cdot (J_{std} - J_{B,t}(\vec{x})) \\ &= (J_B(t \cdot \vec{x}) + J_{std})^{-1} \cdot (J_{std} - J_B(t \cdot \vec{x})) = Q(t \cdot \vec{x}). \end{aligned} \quad (48)$$

If $\|Q\| < \infty$ and $\epsilon > 0$ is given, then there is some t_1 so that $\|Q \circ (t \cdot Id)\| < \epsilon$ for all $0 < t < t_1$.

The conclusion is that if an estimate of the form $\|Q\| < \epsilon$ is ever required, then the above construction shows that a "re-scaling" exists so that Q can be replaced by some Q_t which satisfies the estimate. If there is also a map $f : D_1 \to B$ under consideration, then the domain coordinates may also have to be transformed, just by starting over at step (39) in the above construction of local coordinates.

Lemma 5.4. *For $0 \le t \le 1$, if $f : D_1 \to B$ is $J_{B,t}$-holomorphic, then $t \cdot f : D_1 \to B$ is J_B-holomorphic.*

Proof. The $t = 0$ case is trivial. Otherwise, there are two approaches to the proof. The first is to use the notion that the property of being pseudoholomorphic is coordinate invariant; the composite $\phi_k^{-1} \circ f : D_1 \to U_k \subseteq M$ is J-holomorphic (J being the global structure on M), so $\phi_j|_{U_k \subseteq U_j} \circ \phi_k^{-1} \circ f : D_1 \to B$ is J_B-holomorphic, and this composite equals $t \cdot f$.

Alternatively, we can just check the differential equation (42):

$$\begin{aligned} \frac{df}{dy} &= J_{B,t}(f(z)) \cdot \frac{df}{dx} \\ \Longrightarrow \frac{df}{dy} &= J_B(t \cdot f(z)) \cdot \frac{df}{dx} \\ \Longrightarrow \frac{d(t \cdot f)}{dy} &= J_B(t \cdot f(z)) \cdot \frac{d(t \cdot f)}{dx}, \end{aligned}$$

where the second line is multiplied by t to get the last line, which is the definition of $t \cdot f$ being J_B-holomorphic. ∎

5.4 Local Existence

We recall from [Z] a basic version of the Implicit Function Theorem.

Proposition 5.5. *Given Banach spaces X, Y, and Z, a neighborhood $U \subseteq X$ of u_0, a neighborhood $V \subseteq Y$ of v_0, and a $\mathcal{C}^r$ map $F : U \times V \to Z$, $r \ge 1$, if $F(u_0, v_0) = \vec{0}$ and $\mathrm{D}_v F(u_0, v_0) : Y \to Z$ is invertible, then there exist $\epsilon_1 > 0$, $\epsilon_2 > 0$, and a $\mathcal{C}^r$ function $\psi : B(u_0, \epsilon_1) \to B(v_0, \epsilon_2)$ such that $F(u, \psi(u)) = \vec{0}$.* ∎

The notation D_vF refers to a partial derivative, the (possibly infinite-dimensional) Jacobian linearization of $F(u_0, \underline{\cdot}) : (\{u = u_0\} \times V) \to Z$.

Corollary 5.6. *For X, Y, Z, $(u_0, v_0) \in U \times V$, r, and F as in Proposition 5.5, if $F(u_0, v_0) = z_0$ and $\mathrm{D}_vF(u_0, v_0) : Y \to Z$ is invertible, then there is some $\epsilon_2 > 0$, some $\epsilon_3 > 0$, and some $\epsilon_4 > 0$ so that $B(z_0, \epsilon_3) \subseteq F(\{u_1\} \times B(v_0, \epsilon_2))$ for each $u_1 \in B(u_0, \epsilon_4)$.*

Proof. Consider the function

$$G : Z \times U \times V \to Z : (z, u, v) \mapsto F(u, v) - z.$$

It satisfies $G(z_0, u_0, v_0) = \vec{0}$, it has the same $\mathcal{C}^r$ regularity as F, and $\mathrm{D}_vG(z_0, u_0, v_0) : Y \to Z$ (where z_0 and u_0 are both fixed) is equal to the invertible map $\mathrm{D}_vF(u_0, v_0) : Y \to Z$, so Proposition 5.5 applies to G. There exists some $\psi : B((z_0, u_0), \epsilon_1) \to B(v_0, \epsilon_2)$ such that $G(z, u, \psi(z, u)) = \vec{0}$. There is some product of balls, $B(z_0, \epsilon_3) \times B(u_0, \epsilon_4) \subseteq B((z_0, u_0), \epsilon_1)$, and for (z, u) in this set, $F(u, \psi(z, u)) = z$. ∎

The next result proves the local existence theorem of Nijenhuis and Woolf, following the sketch appearing in [S]. Some of the technical details are omitted, as described in the remarks.

Theorem 5.7. *Given $r > 1$, a $\mathcal{C}^{r+1}$ manifold M, and a $\mathcal{C}^r$ almost complex structure J, for any $v \in M$ there is some neighborhood U of $\vec{0} \in T_vM$ such that for all $\vec{X} \in U$, there exists a J-holomorphic map $f : D_1 \to M$ such that $f(0) = v$ and $df(0) \cdot \frac{d}{dx} = \vec{X}$.*

Proof. This being a local result, we can replace M with the unit ball B, and point v with $\vec{0}$, and then the $\mathcal{C}^r$ structure is represented on B as J_B, normalized and scaled as previously, so that $J_B(\vec{0}) = J_{std}$ and the $\mathcal{C}^r$ norm $\|Q\|$ is less than some sufficiently small $\epsilon_1 > 0$.

Define the following map:

$$\begin{aligned} \Phi : (-1, 1] \times \mathcal{C}^{r+1}(D_1, B) &\to \mathcal{C}^{r+1}(D_1, \mathbb{R}^{2n}) \\ (t, f) &\mapsto f - T((Q \circ (t \cdot f)) \cdot \partial f), \end{aligned}$$

where T is the Cauchy-Green operator satisfying $\overline{\partial} \circ T = Id$.

(* Remark: The regularity of both the input and the output of T should be checked. This may be where we need the *a priori* norm on Q? *)

Φ satisfies $\Phi(0, f) = f$, so $\Phi(0, \underline{\cdot})$ is the canonical embedding. The function Φ is a $\mathcal{C}^r$ map of Banach spaces in a neighborhood of the origin, and this is enough for the Implicit Function Theorem to apply.

(* Remark: the $\mathcal{C}^r$ property should be checked. This is one place where $r > 1$ is used — see [IR]. *)

The first conclusion from Corollary 5.6 is that there is some ϵ_3-neighborhood of the origin, $W \subseteq \mathcal{C}^{r+1}(D_1, \mathbb{R}^{2n})$ and some $\epsilon_4 > 0$ so that for all $t \in [0, \epsilon_4]$, the image of

$$\Phi(t, \underline{\cdot}) : \mathcal{C}^{r+1}(D_1, B) \to \mathcal{C}^{r+1}(D_1, \mathbb{R}^{2n})$$

contains W.

Let $h : \mathbb{R}^{2n} \to \mathcal{C}^{r+1}(D_1, \mathbb{R}^{2n})$ denote the linear map $\vec{v} \mapsto h_{\vec{v}}$, where

$$h_{\vec{v}} : z = (x, y) \mapsto z \cdot \vec{v} = x \cdot \vec{v} + y \cdot J_{std} \cdot \vec{v}. \tag{49}$$

There is some ball $B_{\epsilon_5} \subseteq \mathbb{R}^{2n}$ so that $\vec{v} \in B_{\epsilon_5} \implies h_{\vec{v}} \in W$. In particular, for any $|t| < \epsilon_4$ and $\vec{v} \in B_{\epsilon_5}$, there exists $f_{t,\vec{v}} = \psi(t, h_{\vec{v}}) \in \mathcal{C}^{r+1}(D_1, B)$ such that $h_{\vec{v}} = \Phi(t, f_{t,\vec{v}})$. The second conclusion from Corollary 5.6 is that the map ψ is $\mathcal{C}^r$.

Applying $\overline{\partial}$ to both sides of $h_{\vec{v}} = \Phi(t, f_{t,\vec{v}})$ gives

$$\vec{0} = \overline{\partial} f_{t,\vec{v}} - (Q \circ (t \cdot f_{t,\vec{v}})) \cdot \partial f_{t,\vec{v}}.$$

By Equations (43) and (48), this means $f_{t,\vec{v}}$ is pseudoholomorphic with respect to $J_{B,t}$.

(* Remark: This is where it should be checked that the $\overline{\partial} \circ T = Id$ identity applies as claimed. *)

Define

$$\begin{aligned} \varphi : (-\epsilon_4, \epsilon_4) \times B_{\epsilon_5} &\to \mathbb{R}^{2n} \\ (t, \vec{v}) &\mapsto df_{t,\vec{v}}(0) \cdot \frac{d}{dx}. \end{aligned}$$

In the case $t = 0$, $\Phi(0, h_{\vec{v}}) = h_{\vec{v}} \implies f_{0,\vec{v}} = h_{\vec{v}}$, so $\varphi(0, \vec{v}) = df_{0,\vec{v}}(0) \cdot \frac{d}{dx} = \vec{v}$.

$\varphi(0, \underline{\cdot})$ is the identity map on B_{ϵ_5}, and φ is $\mathcal{C}^r$, being the composite of a $\mathcal{C}^r$ map with two linear maps: $\varphi = E \circ \psi \circ (Id \times h)$, where E is the linear map evaluating the derivative, $g \mapsto dg(0) \cdot \frac{d}{dx} = \frac{dg}{dx}(0)$. Corollary 5.6 applies again.

The conclusion is that there is some $0 < t_0 < \epsilon_4$, some $0 < \epsilon_6 < \epsilon_5$, and some $0 < \epsilon_7$ so that the image of $\varphi(t_0, \underline{\cdot}) : B_{\epsilon_6} \to \mathbb{R}^{2n}$ contains B_{ϵ_7}: for any $\vec{Y} \in B_{\epsilon_7}$, there is a $\vec{v} \in B_{\epsilon_6}$ so that $df_{t_0,\vec{v}}(0) \cdot \frac{d}{dx} = \vec{Y}$. Let $U = t_0 \cdot B_{\epsilon_7} = B_{t_0 \cdot \epsilon_7}$, so that then for any $\vec{X}$ in U, $\vec{X} = t_0 \cdot \vec{Y}$ for some $\vec{Y} \in \varphi(t_0, B_{\epsilon_6})$, and

$$\vec{X} = t_0 \cdot \vec{Y} = t_0 \cdot df_{t_0,\vec{v}}(0) \cdot \frac{d}{dx} = d(t_0 \cdot f_{t_0,\vec{v}})(0) \cdot \frac{d}{dx}.$$

The map $t_0 \cdot f_{t_0,\vec{v}}$ is pseudoholomorphic with respect to J_B by Lemma 5.4. ∎

Another local existence theorem is for a curve connecting two points. This proof follows [D].

Theorem 5.8. *Given $r > 1$, a $\mathcal{C}^{r+1}$ manifold M, and a $\mathcal{C}^r$ almost complex structure J, for any $v \in M$ there is some neighborhood U of v such that for all points $p, q \in U$, there exists a J-holomorphic map $f : D_1 \to M$ such that $f(0) = p$ and $f(\frac{1}{2}) = q$.*

Proof. Again it is enough to work locally, and show that there is some neighborhood U of $\vec{0} \in B$ so that for $\vec{p}, \vec{q} \in U$, there is a map $f : D_1 \to B$ with $f(0) = \vec{p}$ and $f(\frac{1}{2}) = \vec{q}$.

The first part of the Proof proceeds exactly as in the Proof of Theorem 5.7, including the construction of the same Φ, ψ, and the same neighborhood W, just before Equation (49).

This time, define $h : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathcal{C}^{r+1}(D_1, \mathbb{R}^{2n}) : (\vec{p}, \vec{q}) \mapsto h_{\vec{p},\vec{q}}$, where

$$h_{\vec{p},\vec{q}} : z = (x, y) \mapsto \vec{p} + 2z \cdot (\vec{q} - \vec{p}) = \vec{p} + 2x \cdot (\vec{q} - \vec{p}) + 2y \cdot J_{std} \cdot (\vec{q} - \vec{p}).$$

There is some ball $B_{\epsilon_5} \subseteq \mathbb{R}^{2n}$ so that $\vec{p}, \vec{q} \in B_{\epsilon_5} \implies h_{\vec{p},\vec{q}} \in W$. In particular, for any $|t| < \epsilon_4$ and $\vec{p}, \vec{q} \in B_{\epsilon_5}$, there exists $f_{t,\vec{p},\vec{q}} = \psi(t, h_{\vec{p},\vec{q}}) \in \mathcal{C}^{r+1}(D_1, B)$ such that $h_{\vec{p},\vec{q}} = \Phi(t, f_{t,\vec{p},\vec{q}})$.

Again, $h_{\vec{p},\vec{q}}$ being holomorphic implies $f_{t,\vec{p},\vec{q}}$ is $J_{B,t}$-holomorphic.

Define

$$\begin{aligned} \varphi : (-\epsilon_4, \epsilon_4) \times B_{\epsilon_5} \times B_{\epsilon_5} &\to \mathbb{R}^{2n} \times \mathbb{R}^{2n} \\ (t, \vec{p}, \vec{q}) &\mapsto (f_{t,\vec{p},\vec{q}}(0), f_{t,\vec{p},\vec{q}}(\frac{1}{2})). \end{aligned}$$

In the case $t = 0$, $\Phi(0, h_{\vec{p},\vec{q}}) = h_{\vec{p},\vec{q}} \implies f_{0,\vec{p},\vec{q}} = h_{\vec{p},\vec{q}}$, so $\varphi(0, \vec{p}, \vec{q}) = (f_{0,\vec{p},\vec{q}}(0), f_{0,\vec{p},\vec{q}}(\frac{1}{2})) = (\vec{p}, \vec{q})$.

So, $\varphi(0, \_, \_)$ is the identity map on $B_{\epsilon_5} \times B_{\epsilon_5}$, and φ is $\mathcal{C}^r$, being the composite of a $\mathcal{C}^r$ map with two linear maps: $\varphi = E \circ \psi \circ (Id \times h)$, where E is the linear map evaluating at a pair of points, $g \mapsto (g(0), g(\frac{1}{2}))$. Corollary 5.6 applies again.

The conclusion is that there is some $0 < t_0 < \epsilon_4$, some $0 < \epsilon_6 < \epsilon_5$, and some $0 < \epsilon_7$ so that the image of $\varphi(t_0, \_, \_) : B_{\epsilon_6} \times B_{\epsilon_6} \to \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ contains $B_{\epsilon_7} \times B_{\epsilon_7}$: for any $\vec{p}_0, \vec{q}_0 \in B_{\epsilon_7}$, there are $\vec{p}_1, \vec{q}_1 \in B_{\epsilon_6}$ so that $f_{t_0,\vec{p}_1,\vec{q}_1}(0) = \vec{p}_0$ and $f_{t_0,\vec{p},\vec{q}}(\frac{1}{2}) = \vec{q}_0$. Let $U = t_0 \cdot B_{\epsilon_7} = B_{t_0 \cdot \epsilon_7}$, so that then for any $\vec{p}, \vec{q}$ in U, $(\vec{p}, \vec{q}) = t_0 \cdot (\vec{p}_0, \vec{q}_0)$ for some $(\vec{p}_0, \vec{q}_0) = (\frac{1}{t_0}\vec{p}, \frac{1}{t_0}\vec{q}) \in \varphi(t_0, B_{\epsilon_6}, B_{\epsilon_6})$, and

$$\begin{aligned}(\vec{p}, \vec{q}) &= t_0 \cdot (\vec{p}_0, \vec{q}_0) = t_0 \cdot (f_{t_0,\vec{p}_1,\vec{q}_1}(0), f_{t_0,\vec{p}_1,\vec{q}_1}(\frac{1}{2})) \\ &= ((t_0 \cdot f_{t_0,\vec{p}_1,\vec{q}_1})(0), (t_0 \cdot f_{t_0,\vec{p}_1,\vec{q}_1})(\frac{1}{2})).\end{aligned}$$

The map $t_0 \cdot f_{t_0,\vec{p}_1,\vec{q}_1}$ is pseudoholomorphic with respect to J_B by Lemma 5.4. ∎

Yet another local existence theorem is for a curve with specified higher-order derivatives. This proof follows [IR] Prop. 1.1, which claims further that the regularity hypothesis on J can be improved to $\mathcal{C}^{r-1}$, by a different proof.

Theorem 5.9. *Given $1 \le k < r$, and a $\mathcal{C}^r$ almost complex structure J on the ball $B \subseteq \mathbb{R}^{2n}$, for any $\vec{v} \in B$ there is some neighborhood U of $\vec{v}$ and some $\epsilon > 0$ such that for all points $\vec{p} \in U$, and all $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in B_\epsilon$, there exists a J-holomorphic map $f : D_1 \to M$ such that $f(0) = \vec{p}$ and $(\frac{d}{dx})^\ell f(0) = \vec{v}_\ell$.*

Proof. Again since it is enough to work locally, we can assume $\vec{v} = \vec{0} \in B$, and show that there is some neighborhood U of $\vec{0} \in B$ so that for $\vec{p}, \vec{v}_1, \ldots, \vec{v}_k \in U$, there is a map $f : D_1 \to B$ with $f(0) = \vec{p}$ and $(\frac{d}{dx})^\ell f(0) = \vec{v}_\ell$.

The first part of the Proof proceeds exactly as in the Proof of Theorem 5.7, including the construction of the same Φ, ψ, and the same neighborhood W, just before Equation (49).

This time, define $h : \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k \to \mathcal{C}^{r+1}(D_1, \mathbb{R}^{2n}) : (\vec{p}, V) \mapsto h_{\vec{p},V}$, where $V = (\vec{v}_1, \ldots, \vec{v}_k)$ and

$$h_{\vec{p},V} : z = (x, y) \mapsto \vec{p} + \sum_{\ell=1}^{k} \frac{1}{\ell!} z^\ell \vec{v}_\ell.$$

There is some ball $B_{\epsilon_5} \subseteq \mathbb{R}^{2n}$ so that $\vec{p}, \vec{v}_1, \ldots, \vec{v}_k \in B_{\epsilon_5} \implies h_{\vec{p},V} \in W$. In particular, for any $|t| < \epsilon_4$ and $\vec{p}, \vec{v}_1, \ldots, \vec{v}_k \in B_{\epsilon_5}$, there exists $f_{t,\vec{p},V} = \psi(t, h_{\vec{p},V}) \in \mathcal{C}^{r+1}(D_1, B)$ such that $h_{\vec{p},V} = \Phi(t, f_{t,\vec{p},V})$.

Again, $h_{\vec{p},V}$ being holomorphic implies $f_{t,\vec{p},V}$ is $J_{B,t}$-holomorphic.

Define

$$\begin{aligned} \varphi : (-\epsilon_4, \epsilon_4) \times B_{\epsilon_5} \times (B_{\epsilon_5})^k &\to \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^k \\ (t, \vec{p}, V) &\mapsto (f_{t,\vec{p},V}(0), \frac{d}{dx} f_{t,\vec{p},V}(0), \ldots, (\frac{d}{dx})^k f_{t,\vec{p},V}(0)). \end{aligned}$$

In the case $t = 0$, $\Phi(0, h_{\vec{p},V}) = h_{\vec{p},V} \implies f_{0,\vec{p},V} = h_{\vec{p},V}$, so $\varphi(0, \vec{p}, V) = (f_{0,\vec{p},V}(0), \frac{d}{dx} f_{0,\vec{p},V}(0), \ldots, (\frac{d}{dx})^k f_{0,\vec{p},V}(0)) = (\vec{p}, V)$.

So, $\varphi(0, \underline{\cdot}, \underline{\cdot})$ is the identity map on $B_{\epsilon_5} \times (B_{\epsilon_5})^k$, and φ is $\mathcal{C}^r$, being the composite of a $\mathcal{C}^r$ map with two linear maps: $\varphi = E \circ \psi \circ (Id \times h)$, where E is the linear map evaluating the map and its x derivatives at 0. Corollary 5.6 applies again.

The conclusion is that there is some $0 < t_0 < \epsilon_4$, some $0 < \epsilon_6 < \epsilon_5$, and some $0 < \epsilon_7$ so that the image of $\varphi(t_0, \underline{\cdot}, \underline{\cdot}) : B_{\epsilon_6} \times (B_{\epsilon_6})^k \to \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^{2k}$ contains $B_{\epsilon_7} \times (B_{\epsilon_7})^k$: for any $\vec{p}^{\,0}, \vec{v}_1^{\,0}, \ldots, \vec{v}_k^{\,0} \in B_{\epsilon_7}$, there are $\vec{p}^1, \vec{v}_1^1, \ldots, \vec{v}_k^1 \in B_{\epsilon_6}$ so that $f_{t_0,\vec{p}^1,V^1}(0) = \vec{p}^{\,0}$ and $(\frac{d}{dx})^\ell f_{t_0,\vec{p}^1,V^1}(0) = \vec{v}_\ell^{\,0}$. Let $U = t_0 \cdot B_{\epsilon_7} = B_{t_0 \cdot \epsilon_7}$, so that then for any $\vec{p}, \vec{v}_1, \ldots, \vec{v}_k$ in U, $(\vec{p}, V) = t_0 \cdot (\vec{p}^{\,0}, \vec{v}_1^{\,0}, \ldots, \vec{v}_k^{\,0})$ for some $(\vec{p}^{\,0}, \vec{v}_1^{\,0}, \ldots, \vec{v}_k^{\,0}) = (\frac{1}{t_0}\vec{p}, \frac{1}{t_0}\vec{v}_1, \ldots, \frac{1}{t_0}\vec{v}_k) \in \varphi(t_0, B_{\epsilon_6}, B_{\epsilon_6}, \ldots, B_{\epsilon_6})$, and

$$\begin{aligned} & (\vec{p}, V) \\ = \;& t_0 \cdot (\vec{p}_0, \vec{v}_1^{\,0}, \ldots, \vec{v}_k^{\,0}) \\ = \;& t_0 \cdot (f_{t_0,\vec{p}^1,V^1}(0), \frac{d}{dx} f_{t_0,\vec{p}^1,V^1}(0), \ldots, (\frac{d}{dx})^k f_{t_0,\vec{p}^1,V^1}(0)) \\ = \;& ((t_0 \cdot f_{t_0,\vec{p}^1,V^1})(0), \frac{d}{dx}(t_0 \cdot f_{t_0,\vec{p}^1,V^1})(0), \ldots, (\frac{d}{dx})^k (t_0 \cdot f_{t_0,\vec{p}^1,V^1})(0)). \end{aligned}$$

The map $t_0 \cdot f_{t_0,\vec{p}^1,V^1}$ is pseudoholomorphic with respect to J_B by Lemma 5.4. ∎

6 Normal form for coordinates near a disk

Recall D_1 is the unit disk in $\mathbb{C}$, with a $\mathcal{C}^\infty$ differentiable structure and the constant, 2×2, $\mathcal{C}^\infty$ almost complex structure J_{std}. In this Section, an important property of D_1 is that it is a contractible topological space; by the

Riemann Mapping Theorem, any contractible open subset of $\mathbb{C}$ is either $\mathbb{C}$ or holomorphically equivalent to D_1, so such a set could replace D_1 without changing the results.

Let M be a $\mathcal{C}^r$ manifold with $r \geq 1$ and $\dim M = 2n$, and let J be a $\mathcal{C}^s$ almost complex structure on M as in Section 3 with $0 \leq s \leq r - 1$.

We will be interested in J-holomorphic maps $u : D_1 \to M$, and our goal in this Section is to follow a construction of [IR] (Proof of Theorem A1) and [MS_1] (Lemma 2.2.2), to find a convenient chart for a neighborhood of the whole image $u(D_1)$ and a simple form for J in that chart. So, this is not the local problem as in Section 5, this is a global construction for "big" disks. See also [R], [ST_2].

6.1 Differential topology: real coordinate charts

To start, we assume only that u is a $\mathcal{C}^\rho$ map $D_1 \to M$, which is also a (global) embedding, so $1 \leq \rho \leq r$. (For maps which are not embeddings, one could restrict the domain to avoid singularities or self-intersections, but once u is an embedding of a disk, we do not want to shrink the domain any further.)

Theorem 6.1. *Given an embedding $u : D_1 \to M$ as above, there exists a $\mathcal{C}^\rho$ differentiable structure on M containing (U, ϕ), where U is a neighborhood of the image $u(D_1)$, and $\phi : U \to D_1 \times \mathbb{R}^{2n-2} \subseteq \mathbb{R}^{2n}$ is an onto chart such that $(\phi \circ u)(x, y) = (x, y, 0, 0, \ldots, 0)$ for all $(x, y) \in D_1$.*

Proof. As a notational convenience, the map $u : D_1 \to M$ factors as a composite $\iota \circ u_0$, where $\iota : u(D_1) \to M$ is the inclusion, and $u_0 : D_1 \to u(D_1)$ is a homeomorphism of the disk onto its image.

By Proposition 2.14, there is a $\mathcal{C}^\rho$ differentiable structure on M so that the image $u(D_1)$ is a $\mathcal{C}^\rho$ 2-submanifold of M. ι is a $\mathcal{C}^\rho$ inclusion, and u_0 is a $\mathcal{C}^\rho$ homeomorphism, which has a $\mathcal{C}^\rho$ inverse by Proposition 2.15.

By Proposition 2.17, there exists a "tubular neighborhood" of $u(D_1)$ in M, given by the following: there is a $\mathcal{C}^\rho$ $(2n-2)$-bundle $p : E \to u(D_1)$, with zero section $\theta_E : u(D_1) \to E : x \mapsto [x, k, \vec{0}]$, and a $\mathcal{C}^\rho$ embedding $f : E \to M$ such that $U = f(E)$ is a neighborhood of $u(D_1)$ in M, and $f \circ \theta_E = \iota$.

The bundle $E \to u(D_1)$ pulls back (as in Definition 2.6) to $u_0^* E \to D_1$ so that the canonical bimorphism $\varepsilon : u_0^* E \to E$ is a $\mathcal{C}^\rho$ homeomorphism. Since D_1 is contractible, there exists a trivial vector bundle $p_D : D_1 \times \mathbb{R}^{2n-2}$ and an isomorphism of vector bundles $\tau : D_1 \times \mathbb{R}^{2n-2} \to u_0^* E$ which is a $\mathcal{C}^\rho$ homeomorphism ([H] Cor. 4.2.5.) and is the identity on the base D_1. Denote the zero section of the trivial bundle $\theta_D : D_1 \to D_1 \times \mathbb{R}^{2n-2} : (x, y) \mapsto (x, y, 0, 0, \ldots, 0)$.

$$\begin{tikzcd}
D_1 \times \mathbb{R}^{2n-2} \arrow[r, "\tau"] \arrow[dr, shift left, "p_D"] & u_0^*E \arrow[r, "\varepsilon"] \arrow[d] & E \arrow[d, shift left, "p"] \arrow[dr, "f"] & \\
& D_1 \arrow[r, "u_0"] \arrow[ul, shift left, "\theta_D"] & u(D_1) \arrow[r, "\iota"] \arrow[u, shift left, "\theta_E"] & M
\end{tikzcd}$$

For $U = f(E)$, let $\phi = (f \circ \varepsilon \circ \tau)^{-1}$, then $\phi : U \to D_1 \times \mathbb{R}^{2n-2}$ is the claimed coordinate chart:

$$\begin{aligned}
\phi \circ u &= \tau^{-1} \circ \varepsilon^{-1} \circ f^{-1} \circ \iota \circ u_0 \\
&= \tau^{-1} \circ \varepsilon^{-1} \circ \theta_E \circ u_0 \\
&= \theta_D.
\end{aligned}$$

∎

In preparation for another change of coordinates on $\mathbb{R}^{2n}$, which fixes the disk $D_1 \times \{\vec{0}\}$, we will need the following consequence of the Inverse Function Theorem, a special case of Exercise 1.8.14. of [GP].

Theorem 6.2. *For $\sigma \geq 1$, let $H : D_1 \times \mathbb{R}^{2n-2} \to \mathbb{R}^{2n}$ be a $\mathcal{C}^\sigma$ map such that at every point $(x, y, \vec{0})$, $H(x, y, \vec{0}) = (x, y, \vec{0})$ and $\mathrm{D}_{(x,y,\vec{0})}H$ is nonsingular. Then there is an open neighborhood U of $D_1 \times \{\vec{0}\}$ such that $H|_U$ is invertible with a $\mathcal{C}^\sigma$ inverse.*

Proof. Let $(x_0, y_0, \vec{0})$ be any element of $D_1 \times \{\vec{0}\}$. By the Inverse Function Theorem, there is some neighborhood $U_{(x_0,y_0)}$ of $(x_0, y_0, \vec{0})$ in $D_1 \times \mathbb{R}^{2n-2}$ so that $H(U_{(x_0,y_0)})$ is open in $\mathbb{R}^{2n}$ and $H|_{U_{(x_0,y_0)}} : U_{(x_0,y_0)} \to H(U_{(x_0,y_0)})$ is invertible with a $\mathcal{C}^\sigma$ inverse $H(U_{(x_0,y_0)}) \to U_{(x_0,y_0)}$.

Because $H(x_0, y_0, \vec{0}) = (x_0, y_0, \vec{0})$, $U_{(x_0,y_0)} \cap H(U_{(x_0,y_0)})$ is an open neighborhood of $(x_0, y_0, \vec{0})$ in $D_1 \times \mathbb{R}^{2n-2}$, and there is an open set $V_{(x_0,y_0)}$ such that $(x_0, y_0, \vec{0}) \in V_{(x_0,y_0)} \subseteq \overline{V_{(x_0,y_0)}} \subseteq U_{(x_0,y_0)} \cap H(U_{(x_0,y_0)})$ (where the bar denotes closure in $\mathbb{R}^{2n}$). Denote

$$h_{(x_0,y_0)} = \left(\left(H|_{U_{(x_0,y_0)}} \right)^{-1} \right) \Big|_{\overline{V_{(x_0,y_0)}}},$$

so for $\vec{v} \in \overline{V_{(x_0,y_0)}}$, $h_{(x_0,y_0)}(\vec{v}) \in U_{(x_0,y_0)}$ and $H(h_{(x_0,y_0)}(\vec{v})) = \vec{v}$. For $\vec{v} \in \overline{V_{(x_0,y_0)}}$ of the form $\vec{v} = (x, y, \vec{0})$, $H(h_{(x_0,y_0)}(\vec{v})) = \vec{v} = H(\vec{v})$, and because H is one-to-one on $U_{(x_0,y_0)}$, $h_{(x_0,y_0)}(\vec{v}) = \vec{v}$.

The collection of all open subsets $V_{(x_0,y_0)}$ for every point $(x_0, y_0) \in D_1$ is an open cover of $D_1 \times \{\vec{0}\}$. By the paracompact property of $D_1 \times \{\vec{0}\}$, this cover has a locally finite open refinement: a collection of open sets V_k indexed by k, that covers $D_1 \times \{\vec{0}\}$, where each V_k is contained in some $V_{(x_0,y_0)}$, and every point $(x_0, y_0, \vec{0})$ has some neighborhood Q so that $Q \cap V_j$ is non-empty for only finitely many j. For each k, we choose some (x_0, y_0) so that $V_k \subseteq V_{(x_0,y_0)}$ and $\overline{V_k} \subseteq \overline{V_{(x_0,y_0)}} \subseteq U_{(x_0,y_0)}$, and denote this $U_{(x_0,y_0)}$ by U_k. Define $h_k = h_{(x_0,y_0)}|_{\overline{V_k}} : \overline{V_k} \to U_k$, so for $\vec{v} \in \overline{V_k}$, $h_k(\vec{v}) = h_{(x_0,y_0)}(\vec{v}) \in U_{(x_0,y_0)} = U_k$ and $H(h_k(\vec{v})) = \vec{v}$.

For indices j and k, define the following closed set:

$$W_{jk} = \overline{\{\vec{x} \in V_j \cap V_k : h_k(\vec{x}) \neq h_j(\vec{x})\}},$$

so $W_{jk} \subseteq \overline{V_j \cap V_k} \subseteq \overline{V_j}$. Consider $(x, y, \vec{0}) \in V_k$ and the following two cases.

Case 1. If $(x, y, \vec{0}) \in V_k \setminus \overline{V_j}$, then $V_k \setminus \overline{V_j}$ is an open neighborhood of $(x, y, \vec{0})$ disjoint from W_{jk}.

Case 2. If $(x, y, \vec{0}) \in V_k \cap \overline{V_j}$, then $H(x, y, \vec{0}) = (x, y, \vec{0}) \in V_k$ and $H(x, y, \vec{0}) = (x, y, \vec{0}) \in \overline{V_j} \subseteq U_j$, so $(x, y, \vec{0}) \in V_k \cap H(V_k \cap U_j)$. To show that $V_k \cap H(V_k \cap U_j)$ is disjoint from W_{jk}, suppose, toward a contradiction, that there is some $\vec{v} \in (V_k \cap H(V_k \cap U_j)) \cap W_{jk}$. From $\vec{v} \in W_{jk}$, any open set containing $\vec{v}$ must also contain some element $\vec{x} \in V_j \cap V_k$ with $h_k(\vec{x}) \neq h_j(\vec{x})$. Since $V_k \cap H(V_k \cap U_j)$ is an open set containing $\vec{v}$, there is some such $\vec{x} \in (V_k \cap H(V_k \cap U_j)) \cap (V_j \cap V_k)$. So, $\vec{x} = H(\vec{w})$ for $\vec{w} \in V_k \cap U_j$. $h_k(\vec{x}) \in U_k$, and $H(h_k(\vec{x})) = \vec{x} = H(\vec{w})$, and since H is one-to-one on U_k, $h_k(\vec{x}) = \vec{w}$. $h_j(\vec{x}) \in U_j$, and $H(h_j(\vec{x})) = \vec{x} = H(\vec{w})$, and since H is one-to-one on U_j, $h_j(\vec{x}) = \vec{w}$; however, this contradicts $h_k(\vec{x}) \neq h_j(\vec{x})$.

From Cases 1. and 2., we can conclude that every point $(x, y, \vec{0}) \in V_k$ is in either the open set $V_k \setminus \overline{V_j}$ or the open set $V_k \cap H(V_k \cap U_j)$, and the union

$$N_{jk} = \left(V_k \setminus \overline{V_j}\right) \cup (V_k \cap H(V_k \cap U_j))$$

is an open neighborhood of the set $\{(x, y, \vec{0}) \in V_k\}$, disjoint from W_{jk}.

Consider a point $(x, y, \vec{0}) \in D_1 \times \{\vec{0}\}$. The local finiteness property of the cover $\{V_j\}$ is that there exists some neighborhood Q of $(x, y, \vec{0})$ that has a non-empty intersection with only finitely many V_j. For each of the (finitely many) k such that $(x, y, \vec{0}) \in V_k$, $Q \cap N_{jk}$ is an open neighborhood of $(x, y, \vec{0})$ in V_k, disjoint from W_{jk}. If $Q \cap V_j = \emptyset$, then $Q \cap V_k \subseteq V_k \setminus \overline{V_j} \subseteq N_{jk}$, so $Q \cap N_{jk} = Q \cap V_k$ and the intersection over all j, $P_k = \bigcap_j Q \cap N_{jk}$ is the same as a finite intersection, and it is an open neighborhood of $(x,y,\vec{0})$ in V_k which is disjoint from W_{jk} for all j. Let $P_{(x,y)}$ be the intersection of the finitely many P_k, so $P_{(x,y)}$ is an open neighborhood of $(x,y,\vec{0})$ contained in every V_k neighborhood of $(x,y,\vec{0})$.

Let P be the union of all open sets $P_{(x,y)}$ for $(x,y)\in D_1$; we will define $h:P\to D_1\times\mathbb{R}^{2n-2}$. Given $\vec{p}\in P$, there is some (x,y) and some k so that $\vec{p}\in P_{(x,y)}\subseteq P_k\subseteq V_k$, and P_k is contained in N_{jk} for all j. Define $h(\vec{p})=h_k(\vec{p})$; by construction, there is no j such that $h_j(\vec{p})$ is defined but not equal to $h_k(\vec{p})$. If there is some other (x',y') and j with $\vec{p}\in P_{(x',y')}\subseteq P_j\subseteq V_j$, then $h(\vec{p})=h_j(\vec{p})$ is equal to the previously calculated $h_k(\vec{p})$. For any $\vec{p}\in P$, there is some (x,y) and some k so that $\vec{p}\in P_{(x,y)}\subseteq P_k$, so $H(h(\vec{p}))=H(h_k(\vec{p}))=\vec{p}$.

Given $(x,y)\in D_1$, there is some k so that $P_{(x,y)}\subseteq P_k\subseteq V_k$, and $h(P_{(x,y)})=h_k(P_{(x,y)})$, so $h(P_{(x,y)})$ is an open neighborhood of $(x,y,\vec{0})$ in $U_k\subseteq D_1\times\mathbb{R}^{2n-2}$. Let U be the union of the open sets $h(P_{(x,y)})$, so $D_1\times\{\vec{0}\}\subseteq U=h(P)\subseteq D_1\times\mathbb{R}^{2n-2}$. For any $\vec{x}\in U$, there is some (x,y) so that $\vec{x}=h(\vec{p})$ for $\vec{p}\in P_{(x,y)}$, and there is some k so that $\vec{x}=h_k(\vec{p})$ for $\vec{p}\in P_{(x,y)}\subseteq P_k$. $h(H(\vec{x}))=h(H(h_k(\vec{p})))=h(\vec{p})=h_k(\vec{p})=\vec{x}$. The conclusion is that $h:P\to U$ is the inverse of $H|_U:U\to P$. ∎

6.2 Linear algebra: normalizing the complex structure operator

Now consider M with $\mathcal{C}^s$ almost complex structure J as in Section 3, and a map $u:D_1\to M$ which is a J-holomorphic, $\mathcal{C}^\rho$ embedding, and such that $J\circ u:D_1\to\mathrm{Hom}(TM,TM)$ is $\mathcal{C}^t$.

Initially, M has some $\mathcal{C}^r$ structure, $s\le r-1$, $1\le\rho\le r$, and $t\le r-1$. Let $\phi_k:U_k\to\mathbb{R}^{2n}$ be a coordinate chart on M, where J has matrix representation $J_k:U_k\to\mathrm{Hom}(\mathbb{R}^{2n},\mathbb{R}^{2n})$, so $J_k\circ\phi_k^{-1}$ is $\mathcal{C}^s$. The map u restricts to $u:u^{-1}(U_k)\to U_k$, so that $\phi_k\circ u:u^{-1}(U_k)\to\mathbb{R}^{2n}$ is $\mathcal{C}^\rho$. The local coordinate representation of $J\circ u:u^{-1}(U_k)\to\mathrm{Hom}(\mathbb{R}^{2n},\mathbb{R}^{2n})$ is $(J_k\circ\phi_k^{-1})\circ(\phi_k\circ u)=J_k\circ u$, which is $\mathcal{C}^t$, $t\ge\min\{\rho,s\}$.

Let U^1 be the neighborhood of $u(D_1)$ from Theorem 6.1, and let $\phi:U^1\to D_1\times\mathbb{R}^{2n-2}$ be the $\mathcal{C}^\rho$ chart with $\phi\circ u=\theta_D$. As in Example 3.3, the matrix representation of J on this chart is $J_D:D_1\times\mathbb{R}^{2n-2}\to\mathrm{Hom}(\mathbb{R}^{2n},\mathbb{R}^{2n})$, defined by (32) for $\vec{x}\in D_1\times\mathbb{R}^{2n-2}$ by picking any k such that $\phi^{-1}(\vec{x})\in U^1\cap U_k$;

then

$$J_D(\vec{x}) = (\mathrm{D}_{\vec{x}}(\phi_k \circ \phi^{-1}))^{-1} \cdot J_k(\phi^{-1}(\vec{x})) \cdot \mathrm{D}_{\vec{x}}(\phi_k \circ \phi^{-1})$$

does not depend on k. J_D has regularity $\mathcal{C}^{\min\{s,\rho-1\}}$ on $D_1 \times \mathbb{R}^{2n-2}$, but for $\vec{x}$ of the form $(x, y, \vec{0})$,

$$\begin{aligned} J_D(x,y,\vec{0}) &= J_D(\theta_D(x,y)) = J_D(\phi(u(x,y))) \\ &= (\mathrm{D}_{(x,y,\vec{0})}(\phi_k \circ \phi^{-1}))^{-1} \cdot J_k(u(x,y)) \cdot \mathrm{D}_{(x,y,\vec{0})}(\phi_k \circ \phi^{-1}), \end{aligned}$$

which has regularity $\mathcal{C}^{\min\{t,\rho-1\}}$.

Now we use the J-holomorphic property of u. It follows from general principles that its local representation $\phi \circ u = \theta_D$ is pseudoholomorphic, but it is worth checking the specifics in this case. To check θ_D is pseudoholomorphic with respect to J_{std} on D_1 and $J_D = d\phi \circ J \circ d(\phi^{-1})$,

$$\begin{aligned} J_D \circ d\theta_D &= d\phi \circ J \circ d(f \circ \varepsilon \circ \tau) \circ d\theta_D = d\phi \circ J \circ d(f \circ \varepsilon \circ \tau \circ \theta_D) \\ &= d\phi \circ J \circ du = d\phi \circ du \circ J_{std} = d(\phi \circ u) \circ J_{std} = d\theta_D \circ J_{std}. \end{aligned}$$

In the $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$ coordinate system of $D_1 \times \mathbb{R}^{2n-2}$, the differential of θ_D is given by

$$d\theta_D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}_{2n \times 2}. \tag{50}$$

The above equation then becomes

$$J_D \circ d\theta_D = d\theta_D \circ J_{std} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}_{2n \times 2}.$$

One can conclude that for points on the disk, $\vec{x} = (x, y, 0, \ldots 0) = \theta_D(x, y)$, the matrix representation of $J_D(\vec{x})$ is

$$J_D(x,y,0,\ldots,0) = \begin{pmatrix} 0 & -1 & * & \ldots & * \\ 1 & 0 & * & \ldots & * \\ 0 & 0 & * & \ldots & * \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & * & \ldots & * \end{pmatrix}_{2n\times 2n} = \begin{pmatrix} J_{std} & B \\ 0 & J_2 \end{pmatrix}. \qquad (51)$$

The lower right $(2n-2)\times(2n-2)$ block J_2 is a CSO on the tangent space to the fiber $\mathbb{R}^{2n-2} = p_D^{-1}(x,y)$ at $\vec{x}$. Both $B_{2\times(2n-2)}$ and J_2 are functions of (x,y). On the whole space $D_1\times\mathbb{R}^{2n-2}$, $J_D(\vec{v})$ has regularity $\mathcal{C}^{\min\{s,\rho-1\}}$ and may not have the above block form; however, the restriction $J_D(x,y,\vec{0})$ and the blocks $B(x,y)$ and $J_2(x,y)$ may have some higher order of smoothness, $\mathcal{C}^t$ for $t\geq\min\{s,\rho-1\}$.

We now want to find matrices G so that $G\cdot J_D\cdot G^{-1} = J_{std}$, at all points $(x,y,\vec{0})$ on the disk. There are two methods; Method 1 gives a canonical formula, which only applies under a certain condition, while Method 2 works for any J_D but involves making some arbitrary choices.

Method 1. If $J_D(x,y,\vec{0})$ has the property that $J_2(x,y,\vec{0})+J_{std}$ is invertible, then from (6) in Example 1.9, there exists $G(x,y)$ such that:

$$\begin{aligned} G(x,y) &= \begin{pmatrix} Id & -\frac{1}{2}B(x,y)\cdot J_2(x,y) \\ 0 & (Id+A(x,y))^{-1} \end{pmatrix}, & (52)\\ A(x,y) &= (J_2(x,y)+J_{std})^{-1}\cdot(J_{std}-J_2(x,y)), \\ J_D(x,y,\vec{0}) &\mapsto G(x,y)\cdot J_D(x,y,\vec{0})\cdot G(x,y)^{-1} = \begin{pmatrix} J_{std} & 0 \\ 0 & J_{std} \end{pmatrix} = J_{std}. & (53) \end{aligned}$$

$G(x,y)$ has the same $\mathcal{C}^t$ regularity as $J_D(x,y,\vec{0})$.

The invertibility of $J_2(x,y,\vec{0})+J_{std}$ on the whole disk D_1 is a significant assumption. J_D can be normalized to J_{std} at one point by a linear transformation of $\mathbb{R}^{2n}$, as in Section 3.2, and then $J_D\approx J_{std}$ near that point, but the formula (52) may still not be applicable globally.

Method 2. The Proof of Lemma 1.11 can be modified to construct $G(x,y)$, depending on $J_D(x,y,\vec{0})$. At each point $(x,y,\vec{0})\in D_1\times\mathbb{R}^{2n-2}$, we want to find a basis of $T_{(x,y,\vec{0})}(D_1\times\mathbb{R}^{2n-2}) = \mathbb{R}^{2n}$. In particular, we will construct vector fields $\vec{v}_k : (D_1\times\{\vec{0}\})\to\mathbb{R}^{2n}$. Let $\vec{v}_1(x,y,\vec{0}) = \vec{e}_1$, the constant vector in the x_1 direction; then $J_D(x,y,\vec{0})\cdot\vec{v}_1(x,y,\vec{0}) = \vec{e}_2$ is also a constant vector field (by the form of (51)). Let $\vec{v}_2(x,y,\vec{0}) = \vec{e}_3$ be a third constant vector field; then $J_D(x,y,\vec{0})\cdot\vec{e}_3$ is a $\mathcal{C}^t$ vector expression, using entries of B and J_2 from (51), and the list $\left(\vec{e}_1,\vec{e}_2,\vec{e}_3,J_D(x,y,\vec{0})\cdot\vec{e}_3\right)$ is independent at every point by Lemma 1.10. If $n=2$, we have a basis of $\mathbb{R}^4$. For $n>2$, that list gives four independent sections of the trivial bundle $D_1\times\mathbb{R}^{2n}\to D_1$, spanning a $\mathcal{C}^t$ sub-bundle. There exists a $\mathcal{C}^t$ complementary sub-bundle ([H] Theorem 4.2.2. — we can think of it as the normal bundle), which is trivial ([H] Cor. 4.2.5.), so there exists a non-vanishing $\mathcal{C}^t$ section $\vec{v}_3(x,y,\vec{0})$, such that the five element list $(\vec{v}_1,\ldots,\vec{v}_3)$ is independent at every point. (By an approximation, $\vec{v}_3$ can be chosen to be a $\mathcal{C}^\infty$ section of $D_1\times\mathbb{R}^{2n}$, but this is not a significant improvement.) Then $J_D(x,y,\vec{0})\cdot\vec{v}_3(x,y,\vec{0})$, a $\mathcal{C}^t$ vector field, so that the six element list $\left(\vec{v}_1,\ldots,\vec{v}_3(x,y,\vec{0}),J_D(x,y,\vec{0})\cdot\vec{v}_3(x,y,\vec{0})\right)$ is independent at every point by Lemma 1.10. This can be repeated — choosing another independent vector field $\vec{v}_k$ and then adding $J_D\cdot\vec{v}_k$, until there are $2n$ vector fields forming a basis at every point. The construction of Lemma 1.11 still works: let

$$G(x,y)=\left[\vec{v}_1,J(\vec{v}_1),\vec{v}_2,J(\vec{v}_2),\ldots,\vec{v}_{n-1},J(\vec{v}_{n-1}),\vec{v}_n,J(\vec{v}_n)\right]^{-1}.$$

Then $G(x,y)$ has a block form as in (52) with $\mathcal{C}^t$ entries, and satisfies (53).

Using $G(x,y)$ defined by either Method 1 or Method 2, define:

$$H:D_1\times\mathbb{R}^{2n-2}\to\mathbb{R}^{2n}:\vec{x}=\begin{bmatrix}x_1\\y_1\\\vdots\\x_n\\y_n\end{bmatrix}\mapsto G(x_1,y_1)\cdot\vec{x},\tag{54}$$

a $\mathcal{C}^t$ mapping which, by the form (52) of $G(x,y)$, fixes $D_1\times\{\vec{0}\}$ pointwise.

If $t\ge 1$ (a significant new assumption), then the Jacobian of H is $\mathrm{D}_{\vec{x}}H=$

$$\begin{pmatrix}1+\frac{\partial G_{13}}{\partial x_1}x_2+\cdots+\frac{\partial G_{1,2n}}{\partial x_1}y_n & 0+\frac{\partial G_{13}}{\partial y_1}x_2+\cdots+\frac{\partial G_{1,2n}}{\partial y_1}y_n & G_{13}\ldots\\ 0+\frac{\partial G_{23}}{\partial x_1}x_2+\cdots+\frac{\partial G_{2,2n}}{\partial x_1}y_n & 1+\frac{\partial G_{23}}{\partial y_1}x_2+\cdots+\frac{\partial G_{2,2n}}{\partial y_1}y_n & G_{23}\ldots\\ \vdots & \vdots & G_{33}\ldots\\ 0+\frac{\partial G_{2n,3}}{\partial x_1}x_2+\cdots+\frac{\partial G_{2n,2n}}{\partial x_1}y_n & & \ldots G_{2n,2n}\end{pmatrix}.$$

In particular, $D_{(x_1,y_1,\vec{0})}H=G(x_1,y_1)$, which is invertible, so Theorem 6.2 applies: there is an open neighborhood U^2 of $D_1\times\{\vec{0}\}$ such that $H|_{U^2}$:

$U^2 \to H(U^2) \subseteq \mathbb{R}^{2n}$ is invertible with a $\mathcal{C}^t$ inverse. If $B \equiv 0$, H is a vector bundle isomorphism of $D_1 \times \mathbb{R}^{2n-2}$, so U^2 can be taken to be $D_1 \times \mathbb{R}^{2n-2}$, instead of using Theorem 6.2.

As in Example 3.1, the almost complex structure J_D restricts to U^2, and induces an almost complex structure $dH \circ J_D \circ d(H^{-1})$ on $H(U^2)$, with regularity $\mathcal{C}^{\min\{s,\rho-1,t-1\}}$. By construction, the matrix representation at points $(x, y, \vec{0})$ is $G(x, y) \cdot J_D(x, y, \vec{0}) \cdot (G(x, y))^{-1} = J_{std}$.

Let $U = \phi^{-1}(U^2)$, a neighborhood of $u(D_1)$ in U^1. The composite $H \circ \phi : U \to \mathbb{R}^{2n}$ has local coordinate representation $H \circ \phi \circ \phi_k^{-1}$, which is $\mathcal{C}^{\min\{\rho,t\}}$. The matrix representation of J in the $H \circ \phi$ chart is as in Example 3.3, formula (32) for $\vec{x} \in H(U^2) \subseteq \mathbb{R}^{2n}$:

$$\begin{aligned}
J'(\vec{x}) &= (\mathrm{D}_{\vec{x}}(\phi_k \circ \phi^{-1} \circ H^{-1}))^{-1} \cdot J_k((H \circ \phi)^{-1}(\vec{x})) \cdot \mathrm{D}_{\vec{x}}(\phi_k \circ \phi^{-1} \circ H^{-1}) \\
&= \mathrm{D}_{H^{-1}(\vec{x})} H \cdot (\mathrm{D}_{H^{-1}(\vec{x})}(\phi_k \circ \phi^{-1}))^{-1} \cdot J_k(\phi^{-1}(H^{-1}(\vec{x}))) \qquad (55) \\
&\quad \cdot \mathrm{D}_{H^{-1}(\vec{x})}(\phi_k \circ \phi^{-1}) \cdot (\mathrm{D}_{H^{-1}(\vec{x})} H)^{-1} \\
&= \mathrm{D}_{H^{-1}(\vec{x})} H \cdot J_D(H^{-1}(\vec{x})) \cdot (\mathrm{D}_{H^{-1}(\vec{x})} H)^{-1},
\end{aligned}$$

and (55) is a $\mathcal{C}^{\min\{s,\rho-1,t-1\}}$ expression.

For a point on the image $u(D_1)$, $\vec{x} = H(\phi(u(x, y))) = H(\theta_D(x, y)) = (x, y, \vec{0})$,

$$\begin{aligned}
J'(x, y, \vec{0}) &= \mathrm{D}_{(x,y,\vec{0})} H \cdot (\mathrm{D}_{(x,y,\vec{0})}(\phi_k \circ \phi^{-1}))^{-1} \cdot J_k(\phi^{-1}((x, y, \vec{0}))) \\
&\quad \cdot \mathrm{D}_{(x,y,\vec{0})}(\phi_k \circ \phi^{-1}) \cdot (\mathrm{D}_{(x,y,\vec{0})} H)^{-1} \\
&= \mathrm{D}_{(x,y,\vec{0})} H \cdot (\mathrm{D}_{(x,y,\vec{0})}(\phi_k \circ \phi^{-1}))^{-1} \cdot J_k(u(x, y)) \\
&\quad \cdot \mathrm{D}_{(x,y,\vec{0})}(\phi_k \circ \phi^{-1}) \cdot (\mathrm{D}_{(x,y,\vec{0})} H)^{-1} \\
&= G(x, y) \cdot J_D(x, y, \vec{0}) \cdot (G(x, y))^{-1} = J_{std}.
\end{aligned}$$

7 Normal coordinates in 4 dimensions

The goal of this Section is to find a coordinate chart where the matrix representation has a normal form at every point, not just on the disk. In general, this can only be achieved locally. The notion of "normal coordinates" is considered by [S], [ST_1], [T] — we work out some of the linear algebra details, but do not prove the main analytical step (Proposition 7.1).

7.1 The construction

We continue with the construction from Section 6, but in the special case where $n = \dim M = 4$ and everything is smooth: $r = s = t = \rho = \infty$. There is a J-holomorphic curve $u : D_1 \to M$ with an open neighborhood $u(D_1) \subseteq U$, and a coordinate chart $H \circ \phi : U \to \mathbb{R}^4$ so that $H \circ \phi \circ u = \theta_D : (x, y) \mapsto (x, y, 0, 0)$. The matrix representation J' of J in this chart satisfies, for $(x, y) \in D_1$,

$$J'(x, y, 0, 0) = J_{std} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $u(z_0) \in M$ be any point on the given J-holomorphic curve. Without loss of generality (by re-parameterizing u), we can assume z_0 is the center 0 of the disk D_1.

The idea is that given a J-holomorphic curve, an implicit function theorem argument, similar to the local existence results in Section 5, shows that there exists a (complex) one-parameter family of nearby curves. The curves and the parameter can be used to define a chart with two complex coordinates ζ and w. For $c \in D_\rho$, denote

$$\theta_c : D_\rho \to D_\rho \times D_\rho : \zeta \mapsto (\zeta, c).$$

The following Proposition is adapted from Lemmas 5.4 and 5.5 of [T].

Proposition 7.1. *Given J' on a neighborhood of $\vec{0}$ in $\mathbb{R}^4$ as above, there exists some $\rho > 0$ and a diffeomorphism $\Theta : D_\rho \times D_\rho \to \mathbb{R}^4$ of the form*

$$\Theta(\zeta, w) \mapsto (\zeta, w + \tau(\zeta, w))$$

such that:

- $\Theta : (\zeta, 0) \mapsto (\zeta, 0)$*;*
- $\Theta : (0, w) \mapsto (0, w)$*;*
- *for each constant $w = c$, the composite*

$$\Theta \circ \theta_c : \zeta \mapsto (\zeta, c + \tau(\zeta, c)) \tag{56}$$

is pseudoholomorphic with respect to J_{std} on D_ρ and J'.

$\blacksquare$

For $(\zeta_0, w_0) \in D_\rho \times D_\rho$, the Jacobian of Θ is

$$\mathrm{D}_{(\zeta_0,w_0)}\Theta = \begin{bmatrix} I & 0 \\ T_1 & I+T_2 \end{bmatrix}$$

for 2×2 blocks including the identity matrix I and T_1 and T_2 depending on τ. The Jacobian of θ_c is as in (50), so writing $J'(\Theta(\zeta_0, c))$ in terms of 2×2 blocks, the J'-holomorphic property from (56) gives

$$\begin{aligned} \mathrm{D}_{\zeta_0}(\Theta \circ \theta_c) \cdot J_{std} &= J' \cdot \mathrm{D}_{\zeta_0}(\Theta \circ \theta_c) \\ \begin{bmatrix} I \\ T_1 \end{bmatrix} \cdot J_{std} &= \begin{bmatrix} J_{std} + B_1 & B_2 \\ B_3 & J_{std} + B_4 \end{bmatrix} \cdot \begin{bmatrix} I \\ T_1 \end{bmatrix} \\ \begin{bmatrix} J_{std} \\ T_1 \cdot J_{std} \end{bmatrix} &= \begin{bmatrix} J_{std} + B_1 + B_2 \cdot T_1 \\ B_3 + J_{std} \cdot T_1 + B_4 \cdot T_1 \end{bmatrix}. \qquad (57) \end{aligned}$$

The matrix representation of J' using Θ^{-1} as a chart, as in Example 3.3, formula (32), is the following CSO at $(\zeta_0, c) \in D_\rho \times D_\rho$:

$$\begin{aligned} J_0(\zeta_0, c) &= (\mathrm{D}_{(\zeta_0,c)}\Theta)^{-1} \cdot J'(\Theta(\zeta_0, c)) \cdot (\mathrm{D}_{(\zeta_0,c)}\Theta) \\ &= \begin{bmatrix} I & 0 \\ T_1 & I+T_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} J_{std} + B_1 & B_2 \\ B_3 & J_{std} + B_4 \end{bmatrix} \cdot \begin{bmatrix} I & 0 \\ T_1 & I+T_2 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -(I+T_2)^{-1} \cdot T_1 & (I+T_2)^{-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} J_{std} + B_1 + B_2 \cdot T_1 & B_2 \cdot (I+T_2) \\ B_3 + J_{std} \cdot T_1 + B_4 \cdot T_1 & (J_{std} + B_4) \cdot (I+T_2) \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -(I+T_2)^{-1} \cdot T_1 & (I+T_2)^{-1} \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} J_{std} & B_2 \cdot (I+T_2) \\ T_1 \cdot J_{std} & (J_{std} + B_4) \cdot (I+T_2) \end{bmatrix} \qquad (58) \\ &= \begin{bmatrix} J_{std} & B_2 \cdot (I+T_2) \\ 0 & (I+T_2)^{-1} \cdot (J_{std} + B_4 - T_1 \cdot B_2) \cdot (I+T_2) \end{bmatrix} \qquad (59) \end{aligned}$$

where step (58) used (57). Expression (59) is also the matrix representation of the original CSO J, using the chart $\Theta^{-1} \circ H \circ \phi$ on some small neighborhood of $u(z_0)$. When $c = 0$, $J'(\Theta(\zeta_0, 0)) = J_{std}$ and all the B_k blocks are 0, so

$$J_0(\zeta_0, 0) = \begin{bmatrix} J_{std} & 0 \\ 0 & (I+T_2)^{-1} \cdot J_{std} \cdot (I+T_2) \end{bmatrix}.$$

From $\Theta(0,w)=w$, $T_2=0$ at $(\zeta_0,w_0)=(0,0)$, so $J_0(0,0)=J_{std}$.

Formula (59) can be re-written

$$J_0(\zeta,w)=\left[\begin{array}{cc} J_{std} & B_5 \\ 0 & J_{std}+B_6 \end{array}\right], \tag{60}$$

where $B_5(\zeta,0)=0$ and $B_6(0,0)=0$.

Remark 7.2. A alternative normalization as in [S], [ST$_2$] §4, using similar methods, results in a block normal form

$$\left[\begin{array}{cc} J_{std}+B_7 & 0 \\ 0 & J_{std}+B_8 \end{array}\right]_{4\times 4},$$

where $B_7(\zeta,0)=0$ and $B_8(0,0)=0$.

7.2 Entries in the matrix representation

Formula (60) can be written in terms of real entries, (depending on ζ, w):

$$J_0(\zeta,w)=\left[\begin{array}{cc} J_{std} & B_5 \\ 0 & J_{std}+B_6 \end{array}\right]=\left[\begin{array}{cccc} 0 & -1 & a_1 & a_2 \\ 1 & 0 & a_3 & a_4 \\ 0 & 0 & b_1 & -1+b_2 \\ 0 & 0 & 1+b_3 & b_4 \end{array}\right].$$

The property $J^2=-Id_{\mathbb{R}^4}$ constrains the entries:

$$\begin{array}{rcl} (b_2-1)a_3 & = & a_1b_1b_2-a_2b_1^2-a_1b_1-a_2 \\ a_4 & = & a_1b_2-a_2b_1-a_1 \\ (1-b_2)b_3 & = & b_1^2+b_2 \\ b_4 & = & -b_1. \end{array}$$

For (ζ,w) near the origin, J_0 is close to J_{std}, so the fractions in the following expression are well-defined, with $|b_2|<1$.

$$J_0(\zeta,w)=\left[\begin{array}{cccc} 0 & -1 & a_1 & a_2 \\ 1 & 0 & \frac{a_1b_1b_2-a_2b_1^2-a_1b_1-a_2}{b_2-1} & a_1b_2-a_2b_1-a_1 \\ 0 & 0 & b_1 & -1+b_2 \\ 0 & 0 & 1+\frac{b_1^2+b_2}{1-b_2} & -b_1 \end{array}\right]. \tag{61}$$

This real matrix acts by matrix multiplication on column vectors; considering column vectors in $\mathbb{C}^4$, the eigenvalues are $\pm i$, and the $-i$ eigenspace is spanned by:

$$\begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} + \frac{b_2 - ib_1}{b_2 - 2 + ib_1} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -i \end{bmatrix} + \frac{a_1 + i(a_1 b_2 - a_2 b_1 - a_1)}{b_2 - 2 + ib_1} \begin{bmatrix} 1 \\ -i \\ 0 \\ 0 \end{bmatrix}.$$

The $+i$ eigenspace is spanned by the complex conjugates of these vectors. The above set of $-i$ eigenvectors can be re-written with complex coefficients β_1, β_2:

$$\begin{aligned} T^{0,1} &= \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{\zeta}}, \frac{\partial}{\partial \bar{w}} + \beta_1 \frac{\partial}{\partial w} + \beta_2 \frac{\partial}{\partial \zeta} \right\} \\ \beta_1(\zeta, w) &= \frac{b_2 - ib_1}{b_2 - 2 + ib_1} \\ \beta_2(\zeta, w) &= \frac{a_2 + i(a_1 b_2 - a_2 b_1 - a_1)}{b_2 - 2 + ib_1}. \end{aligned} \tag{62}$$

Conversely, given complex coefficients β_1, β_2 in an expression of the form (62) with $|\beta_1| < 1$, the real entries a_1, a_2, b_1, b_2 in a CSO of the form (61) are uniquely determined by:

$$\begin{aligned} a_1 + ia_2 &= \frac{2i(\beta_1 \overline{\beta_2} + \beta_2)}{\beta_1 \overline{\beta_1} - 1} \\ b_1 + ib_2 &= \frac{2i\beta_1(\overline{\beta_1} + 1)}{\beta_1 \overline{\beta_1} - 1}. \end{aligned}$$

In terms of β_1, β_2, the matrix (61) for $J_0(\zeta, w)$ is:

$$\begin{bmatrix} 0 & -1 & \frac{2(\mathrm{Im}(\beta_2)\mathrm{Re}(\beta_1) - \mathrm{Im}(\beta_1)\mathrm{Re}(\beta_2) - \mathrm{Im}(\beta_2))}{|\beta_1|^2 - 1} & \frac{2(\mathrm{Im}(\beta_2)\mathrm{Im}(\beta_1) + \mathrm{Re}(\beta_2)\mathrm{Re}(\beta_1) + \mathrm{Re}(\beta_2))}{|\beta_1|^2 - 1} \\ 1 & 0 & -\frac{2(\mathrm{Im}(\beta_2)\mathrm{Im}(\beta_1) + \mathrm{Re}(\beta_2)\mathrm{Re}(\beta_1) - \mathrm{Re}(\beta_2))}{|\beta_1|^2 - 1} & \frac{2(\mathrm{Im}(\beta_2)\mathrm{Re}(\beta_1) - \mathrm{Im}(\beta_1)\mathrm{Re}(\beta_2) + \mathrm{Im}(\beta_2))}{|\beta_1|^2 - 1} \\ 0 & 0 & -\frac{2\mathrm{Im}(\beta_1)}{|\beta_1|^2 - 1} & -1 + \frac{2(|\beta_1|^2 + \mathrm{Re}(\beta_1))}{|\beta_1|^2 - 1} \\ 0 & 0 & 1 - \frac{2(|\beta_1|^2 - \mathrm{Re}(\beta_1))}{|\beta_1|^2 - 1} & \frac{2\mathrm{Im}(\beta_1)}{|\beta_1|^2 - 1} \end{bmatrix}.$$

As in Section 5.2, the eigenvectors of the matrix J_0 can be used to find the nonlinear Cauchy-Riemann equations satisfied by J-holomorphic curves.

The following calculations are analogous to (44)–(47). The diagonalizing matrix of eigenvectors, its inverse, and the diagonalization of J_0 are:

$$
\begin{aligned}
P &= \begin{bmatrix} 1 & 1 & \beta_2 & \bar{\beta}_2 \\ i & -i & -i\beta_2 & i\bar{\beta}_2 \\ 0 & 0 & 1+\beta_1 & 1+\bar{\beta}_1 \\ 0 & 0 & i-i\beta_1 & -i+i\bar{\beta}_1 \end{bmatrix}, \\
P^{-1} &= \frac{1}{2}\begin{bmatrix} 1 & -i & \frac{\bar{\beta}_2(1-\beta_1)}{\beta_1\bar{\beta}_1-1} & \frac{i\bar{\beta}_2(1+\beta_1)}{\beta_1\bar{\beta}_1-1} \\ 1 & i & \frac{\beta_2(1-\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1} & \frac{-i\beta_2(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1} \\ 0 & 0 & \frac{\bar{\beta}_1-1}{\beta_1\bar{\beta}_1-1} & \frac{i(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1} \\ 0 & 0 & \frac{\beta_1-1}{\beta_1\bar{\beta}_1-1} & \frac{-i(1+\beta_1)}{\beta_1\bar{\beta}_1-1} \end{bmatrix}, \\
D &= \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.
\end{aligned}
$$

If $f : D_1 \to \mathbb{R}^4$, $f(x,y) = (f^1, f^2, f^3, f^4)$, is the coordinate representation as in (40) of a J-holomorphic curve in a neighborhood of $\vec{0} \in \mathbb{R}^4$ where J_0 has the form (61), (62), then from

$$\frac{df}{dy} = J_0(f(x,y))\frac{df}{dx} = PDP^{-1}\frac{df}{dx},$$

this equality of vectors follows:

$$
\begin{aligned}
P \cdot D \cdot P^{-1}\begin{bmatrix} f_x^1 \\ f_x^2 \\ f_x^3 \\ f_x^4 \end{bmatrix} &= \begin{bmatrix} f_y^1 \\ f_y^2 \\ f_y^3 \\ f_y^4 \end{bmatrix} \\
\begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix} P^{-1}\begin{bmatrix} f_x^1 \\ f_x^2 \\ f_x^3 \\ f_x^4 \end{bmatrix} &= P^{-1}\begin{bmatrix} f_y^1 \\ f_y^2 \\ f_y^3 \\ f_y^4 \end{bmatrix} \\
\begin{bmatrix} -if_x^1 - f_x^2 - i\frac{\bar{\beta}_2(1-\beta_1)}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{\bar{\beta}_2(1+\beta_1)}{\beta_1\bar{\beta}_1-1}f_x^4 \\ if_x^1 - f_x^2 + i\frac{\beta_2(1-\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{\beta_2(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_x^4 \\ -i\frac{\bar{\beta}_1-1}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{1+\bar{\beta}_1}{\beta_1\bar{\beta}_1-1}f_x^4 \\ i\frac{\beta_1-1}{\beta_1\bar{\beta}_1-1}f_x^3 + \frac{1+\beta_1}{\beta_1\bar{\beta}_1-1}f_x^4 \end{bmatrix} &= \begin{bmatrix} f_y^1 - if_y^2 + \frac{\bar{\beta}_2(1-\beta_1)}{\beta_1\bar{\beta}_1-1}f_y^3 + i\frac{\bar{\beta}_2(1+\beta_1)}{\beta_1\bar{\beta}_1-1}f_y^4 \\ f_y^1 + if_y^2 + \frac{\beta_2(1-\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_y^3 - i\frac{\beta_2(1+\bar{\beta}_1)}{\beta_1\bar{\beta}_1-1}f_y^4 \\ \frac{\bar{\beta}_1-1}{\beta_1\bar{\beta}_1-1}f_y^3 + i\frac{1+\bar{\beta}_1}{\beta_1\bar{\beta}_1-1}f_y^4 \\ \frac{\beta_1-1}{\beta_1\bar{\beta}_1-1}f_y^3 - i\frac{1+\beta_1}{\beta_1\bar{\beta}_1-1}f_y^4 \end{bmatrix}.
\end{aligned}
$$

The first and second entries on either side are complex conjugate, and the third and fourth entries are also conjugate, so for $|\beta_1| \neq 1$, the above vector equality is equivalent to a system of two complex equations (63), (64). In analogy with (46), setting the fourth entries equal and multiplying by $|\beta_1|^2 - 1$:

$$\begin{aligned} i(\beta_1 - 1)f_x^3 + (1+\beta_1)f_x^4 &= (\beta_1 - 1)f_y^3 - i(1+\beta_1)f_y^4 \qquad (63) \\ \Longrightarrow \frac{\partial}{\partial \bar{z}}(f^3 + if^4) &= \beta_1(f(x,y)) \cdot \overline{\frac{\partial}{\partial z}(f^3 + if^4)}. \end{aligned}$$

Setting the second entries equal and multiplying by $|\beta_1|^2 - 1$:

$$\begin{aligned} & (\beta_1\bar{\beta}_1 - 1)(if_x^1 - f_x^2) - i\beta_2(\bar{\beta}_1 - 1)f_x^3 + \beta_2(1+\bar{\beta}_1)f_x^4 \\ = \; & (\beta_1\bar{\beta}_1 - 1)(f_y^1 + if_y^2) - \beta_2(\bar{\beta}_1 - 1)f_y^3 - i\beta_2(1+\bar{\beta}_1)f_y^4 \qquad (64) \end{aligned}$$

$$\begin{aligned} \Longrightarrow \frac{\partial}{\partial \bar{z}}(f^1 + if^2) &= \frac{1}{1 - \beta_1\bar{\beta}_1}\left(-\beta_2\bar{\beta}_1\frac{\partial}{\partial \bar{z}}(f^3 + if^4) + \beta_2\overline{\frac{\partial}{\partial z}(f^3 + if^4)}\right) \\ &= \beta_2(f(x,y)) \cdot \overline{\frac{\partial}{\partial z}(f^3 + if^4)}. \end{aligned}$$

Equation (64) looks more complicated than (46) or (63), but there is a significant simplification using (63) in the last step.

If a local parametric equation for a pseudoholomorphic curve is written in complex form as

$$(\zeta, w) = (u^1(z), u^2(z)) = (f^1 + if^2, f^3 + if^4),$$

then u^2 satisfies a Beltrami equation

$$u_{\bar{z}}^2 = \beta_1(u^1(z), u^2(z))\overline{u_z^2},$$

and u^1 satisfies a nonlinear inhomogeneous Cauchy-Riemann equation

$$u_{\bar{z}}^1 = \beta_2(u^1(z), u^2(z))\overline{u_z^2}.$$

Example 7.3. By construction, $(\zeta, w) = (u^1(z), c)$ is J-holomorphic for any holomorphic u^1 and constant c.

Example 7.4. In the special case $\beta_1 \equiv 0$, the matrix (61) for $J_0(\zeta, w)$ is:

$$J_0(\zeta, w) = \begin{bmatrix} 0 & -1 & a_1 & a_2 \\ 1 & 0 & a_2 & -a_1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 2\mathrm{Im}(\beta_2) & -2\mathrm{Re}(\beta_2) \\ 1 & 0 & -2\mathrm{Re}(\beta_2) & -2\mathrm{Im}(\beta_2) \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The projection $(\zeta, w) \mapsto w$ is a pseudoholomorphic map $D_\rho \times D_\rho \to D_\rho$; the fibers are the J-holomorphic curves (z, c) — ([ST$_1$] §3) calls this the "pseudo-holomorphically fibered" case. A curve of the form $(\zeta, w) = (u^1(z), u^2(z))$ is J-holomorphic if u^2 is holomorphic, and $u^1_{\bar{z}} = \beta_2(u^1(z), u^2(z))\overline{u^2_z}$. In particular, a curve in the form of a graph over the w-axis, $(\zeta, w) = (u^1(z), z)$, is J-holomorphic if $u^1_{\bar{z}} = \beta_2(u^1(z), z)$.

8 Pointwise properties and relation to CR singularities

Given a $\mathcal{C}^\rho$, $\rho \geq 1$, map $f : D_1 \to B$ as in Section 5.1, but not necessarily J-holomorphic, we consider just the differential of f at the origin. The Jacobian matrix $df(0)$ is a real $2n \times 2$ matrix representation of a real linear map from $T_0D_1 \to T_{\vec{0}}B$; both these tangent spaces have the standard CSOs J_{std}.

Lemma 8.1. *If $df(0)$ is c-linear and f is singular at 0, then $df(0)$ is the zero matrix.*

Proof. The definition of "f singular at 0" is that $df(0)$ has rank < 2, so Lemma 1.7 applies. ∎

Definition 8.2. Given an almost complex $\mathcal{C}^r$ manifold M, $r \geq 1$, with arbitrary ($\mathcal{C}^0$) almost complex structure J, and an embedded two-dimensional submanifold $S \subseteq M$ containing $p \in S$ (so that the tangent plane T_pS is well-defined), the point $p \in S$ is a "CR singular" point if the tangent plane T_pS is a $J(p)$-invariant subspace of T_pM.

Lemma 8.3. *If $df(0)$ is c-linear and f is not singular at 0, then there is a neighborhood U of 0 in D_1 so that the image $f(U)$ is an embedded real surface in B with a CR singularity at $\vec{0}$.*

Proof. The definition of "not singular at 0" is that $df(0)$ has maximum rank, 2, so there is some neighborhood U of 0 in the domain so that the restriction $f : U \to B$ is an embedding. The image of $df(0)$ is a two-dimensional subspace of $T_{\vec{0}}B$, equal to the tangent space of the image of the embedding at $\vec{0}$. If $df(0)$ is c-linear, then the image subspace is invariant under J_{std} in $T_{\vec{0}}B$: for $df(0) : \vec{u} \mapsto \vec{v}$ in the image of $df(0)$, $J_{std} \cdot \vec{v} = J_{std} \cdot df(0) \cdot \vec{u} = df(0) \cdot J_{std} \cdot \vec{u}$ is also in the image of $df(0)$. ∎

The product space $D_1 \times B$ has an almost complex structure. The tangent space at $(z, \vec{x})$ is a direct sum $T_zD_1 \oplus T_{\vec{x}}B$, and the map $(\vec{a}, \vec{b}) : (J_{std} \cdot \vec{a}, J_B(\vec{x}) \cdot \vec{b})$ is a CSO. In matrix form, the product CSO is a $(2 + 2n) \times (2 + 2n)$ block matrix, where J_{std} and $J_B(\vec{x})$ are the upper left and lower right blocks. At $(0, \vec{0})$, the CSO is exactly the $(2 + 2n) \times (2 + 2n)$ standard CSO J_{std}.

The following Lemma applies to both singular and non-singular maps f.

Lemma 8.4. *If $df(0)$ is c-linear, then the "graph" map*

$$g : D_1 \to D_1 \times B : z \mapsto (z, f(z))$$

has the property that its image $g(D_1)$ is an embedded real surface with a CR singularity at $(0, \vec{0})$.

Proof. The map g has the property that $dg(0) = Id \oplus df(0)$, that is, it is a $(2 + 2n) \times 2$ matrix with a 2×2 identity block stacked on top of a $2n \times 2$ $df(0)$ block. It has rank 2 (from the Id block), and is c-linear with respect to the 2×2 and $(2 + 2n) \times (2 + 2n)$ standard CSOs, so Lemma 8.3 applies. ∎

So, g is an embedding of the whole disk D_1, not just a neighborhood near 0.

There is nothing special about the single point 0 in the above Lemmas. If f is J-holomorphic (so that df is c-linear at every point), then the rank of df will be 0 at every singular point of f, and centered at every non-singular point, there is a small disk whose image under f is an embedded surface which is CR singular at every point, so it could be called an embedded J-holomorphic disk. The image of a graph g of a J-holomorphic map f is an embedded J-holomorphic disk $g(D_1)$ in $D_1 \times B$, even if f is singular.

Returning to the general case of Lemma 8.3, where f is not necessarily J-holomorphic, but at the single point 0, the differential $df(0)$ is c-linear and f is non-singular at 0, we can put f into a "standard position" by a linear coordinate change. In fact, corresponding to any J_{std}-invariant real

2-dimensional subspace S of $T_{\vec{0}}B$, there is some c-linear transformation H such that H maps S to the subspace

$$S_1 = \{(x_1, y_1, 0, \ldots, 0)^T\} \subseteq \mathbb{R}^{2n}.$$

In the case where S is the image of the c-linear map $df(0)$, the composite map $H \circ f$ has the property that $d(H \circ f) = H \cdot df$, and the image of $d(H \circ f)(0)$ is the subspace S_1. If B is a ball centered at $\vec{0}$, then H can be chosen to be unitary, so the target space does not change: $H(B) = B$. H can even be chosen so that $d(H \circ f) : \frac{d}{dx} \mapsto \frac{d}{dx_1}$.

Putting a non-singular map f with c-linear differential $df(0)$ into standard position by a c-linear transformation H can also be thought of as just choosing a different coordinate chart for the target M. Returning to the global set-up (40), where $f = \phi_j \circ u \circ \psi_{j'}^{-1}$, let $\phi_k = H \circ \phi_j$. This will be a coordinate chart for M in which the local representation $\phi_k \circ u \circ \psi_{j'}^{-1}$ of the map u is in standard position, since $H \circ f = H \circ (\phi_j \circ u \circ \psi_{j'}^{-1}) = \phi_k \circ u \circ \psi_{j'}^{-1}$. The local representation of the almost complex structure transforms from the j chart to the k chart by (30), $J_k = H \cdot J_j \cdot H^{-1}$, where $H = \phi_k \circ \phi_j^{-1}$. Since H is c-linear, the almost complex structure still satisfies the normalization conditions: at the origin, $H \cdot J_B(0) \cdot H^{-1} = H \cdot J_{std} \cdot H^{-1} = J_{std}$, and at every point, $H \cdot J_B \cdot H^{-1} + J_{std} = H \cdot (J_B + J_{std}) \cdot H^{-1}$ is invertible.

Continuing to consider a non-singular map f with c-linear differential $df(0)$, Lemma 8.3 applies, and we further may suppose f is in standard position, so that the image of some small disk in the domain is an embedded real surface whose tangent plane at $\vec{0}$ is the J_{std}-invariant subspace S_1. So, there is some even smaller neighborhood of $\vec{0}$ in B, in which this surface patch can be described as the graph of $2n-2$ real functions of class $\mathcal{C}^\rho$ over the tangent space. If $\mathbb{R}^{2n}$ has coordinates $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, then the equations of the surface are, for x_1, y_1 near $(0, 0)$:

$$\begin{aligned} x_2 &= H_1(x_1, y_1) \\ &\vdots \\ y_n &= H_{2n-2}(x_1, y_1), \end{aligned}$$

where at the origin, the H_ℓ functions have value 0 and first derivatives 0.

If f is not just c-linear and non-singular at 0 but also J-holomorphic in a neighborhood of 0, then there is a non-linear change of coordinates ([MS$_1$] Lemma 2.2.2) so that the image is just the z_1-axis; in the above notation, the graphing functions H_ℓ are all identically zero.

In the case where $df(0)$ is the zero matrix, the graph map $g(z) = (z, f(z))$ is already in standard position since it is non-singular and $dg(0)$ has image equal to T_0D_1 inside $T_0D_1 \oplus T_{\vec{0}}B$. In terms of the above construction, the defining equations of the image of g are exactly the components of f:

$$\begin{aligned} x_1 &= f_1(x, y) \\ &\vdots \\ y_n &= f_n(x, y), \end{aligned}$$

and again at the origin, the f_ℓ functions have value 0 and first derivatives 0.

Example 8.5. Consider a target space $\mathbb{C}^3$, which is $\mathbb{R}^6$ with the standard almost complex structure J_{std}. Let $f : D_1 \to \mathbb{C}^3$ be given by

$$f(z) = (z, \bar{z}^2, z\bar{z})^T,$$

or in terms of the real coordinates,

$$f(x, y) = (x, y, x^2 - y^2, -2xy, x^2 + y^2, 0)^T.$$

Then

$$df = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & -2y \\ -2y & -2x \\ 2x & 2y \\ 0 & 0 \end{pmatrix},$$

so $df(x, y)$ is c-linear only at the origin. The map f is an embedding in standard position, and the image is totally real except for the CR singular point at $\vec{0}$ where the surface is tangent to the z_1-axis. This surface, the algebraic normal form for non-degenerate CR singular surfaces in $\mathbb{C}^3$ as in [C$_1$], already happens to be given in the form of a graph and could be written in terms of the target coordinates (z_1, z_2, z_3) only: $\{z_2 = \bar{z}_1^2, z_3 = z_1\bar{z}_1\}$.

Example 8.6. The map $f : D_1 \to \mathbb{C}^2$ given by $f(z) = (\bar{z}^2, z\bar{z})$ is singular at the origin, where its differential is the 4×2 zero matrix, and non-singular and not c-linear at every other point. The map is two-to-one, branched at the origin, since $f(z) = f(-z)$. The image of $f(x, y) = (x^2 - y^2, -2xy, x^2 + y^2, 0)$ in the (x_1, y_1, x_2, y_2) coordinate system is exactly the circular cone $\{x_1^2 + y_1^2 =$

$x_2^2, x_2 \geq 0\}$ in the three-dimensional subspace $\{y_2 = 0\}$, so the vertex of the cone is the image of the singular point, and the smooth points of the cone are totally real. The image of the graph $g(z) = (z, f(z))$ is exactly the embedded surface with an isolated CR singularity in $\mathbb{C}^3$ from the previous Example.

Example 8.7. For $f(z) = (\bar{z}^2, z\bar{z})$ as in Example 8.6, consider the composite $F(z) = f(z^2)$, so the map $F : D_1 \to \mathbb{C}^2$ given by $F(z) = (\bar{z}^4, z^2\bar{z}^2)$ is singular at the origin, where its differential is the 4×2 zero matrix, and non-singular and not c-linear at every other point. The map is four-to-one, branched at the origin, since $F(z) = F(-z) = F(iz) = F(-iz)$. The image of $F(x, y) = ((x^2-y^2)^2-4x^2y^2, -4xy(x^2-y^2), (x^2+y^2)^2, 0)$ in the (x_1, y_1, x_2, y_2) coordinate system is exactly the same circular cone $\{x_1^2 + y_1^2 = x_2^2, x_2 \geq 0, y_2 = 0\}$ as the image of f. The image of the graph $G(z) = (z, F(z))$ is the embedded surface with an isolated CR singularity:

$$\{(z_1, z_2, z_3) : z_2 = \bar{z}_1^4, z_3 = z_1^2\bar{z}_1^2\},$$

which has a higher order of contact with its complex tangent plane in $\mathbb{C}^3$ than the surface from Example 8.5.

The following result generalizes Lemma 8.1.

Lemma 8.8. *Given a $\mathcal{C}^s$ almost complex structure J_B on B and a $\mathcal{C}^{s+1}$ map $f : D_1 \to B$, if there is an integer $k \leq s+1$ such that*

$$\overline{\partial}_J f = o(|z|^{k-1})$$

and, for all ℓ such that $0 \leq \ell \leq k$,

$$\left(\frac{d}{dx}\right)^\ell f(0) = \vec{0},$$

then, for all (j, m) such that $j + m \leq k$,

$$\left(\frac{d}{dx}\right)^j \left(\frac{d}{dy}\right)^m f(0) = \vec{0}.$$

Proof. From the definition (38) of $\overline{\partial}_J$ and the calculation of (41), the first hypothesis implies

$$\begin{aligned} \overline{\partial}_J f = df + J_B \cdot df \cdot J_{std} &= o(|z|^{k-1}) \\ \implies J_B \cdot df &= df \cdot J_{std} + o(|z|^{k-1}) \\ \implies \frac{df}{dy}(z) &= J_B(f(z)) \cdot \frac{df}{dx}(z) + o(|z|^{k-1}). \end{aligned}$$

The proof of the claim is by induction on m. The $m=0$ case is exactly the second hypothesis.

For the inductive step establishing the claim for $m>0$, assume

$$\left(\frac{d}{dx}\right)^j\left(\frac{d}{dy}\right)^\ell f(0)=\vec{0}$$

for all (j,ℓ) such that $\ell<m$ and $j+\ell\leq k$. Then, for any j such that $j+m\leq k$,

$$\begin{aligned}\left(\frac{d}{dx}\right)^j\left(\frac{d}{dy}\right)^m f &= \left(\frac{d}{dx}\right)^j\left(\frac{d}{dy}\right)^{m-1}\frac{df}{dy}\\ &= \left(\frac{d}{dx}\right)^j\left(\frac{d}{dy}\right)^{m-1}\left(J_B(f(z))\cdot\frac{df}{dx}+o(|z|^{k-1})\right).\end{aligned}$$

The derivative of the second term is $o(1)$, and the derivative of the first term, when evaluated at 0, is $\vec{0}$ by the rules for derivatives, the existence of $k-1$ derivatives of J_B, and the inductive hypothesis. ∎

Example 8.9. Consider, for $k\geq 1$, the smooth map

$$f(x,y)=(y^k,xy^{k-1},y^{2k},0).$$

Then $f:D_1\to B$, where $B\subseteq\mathbb{C}^2$ has the standard complex structure.

The image of f in the (x_1,y_1,x_2,y_2) coordinate system is contained in (but not equal to) the parabolic cylinder $\{x_2=x_1^2,y_2=0\}$ (not depending on k).

Since f maps the real axis to the single point $\vec{0}$, $\left(\frac{d}{dx}\right)^\ell f(0)=\vec{0}$ for all $\ell>0$, and the second hypothesis of Lemma 8.8 is satisfied for any k.

By construction, $(\frac{d}{dy})^k f(0)\neq\vec{0}$ and $(\frac{d}{dx})(\frac{d}{dy})^{k-1}f(0)\neq\vec{0}$. The conclusion from Lemma 8.8 is that $\overline{\partial}f=o(|z|^{k-1})$ must be false.

The same conclusion can be drawn more directly, from expanding

$$f(z,\bar{z})=\left(\left(\frac{z-\bar{z}}{2i}\right)^k+i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)^{k-1},\left(\frac{z-\bar{z}}{2i}\right)^{2k}\right),$$

so $\frac{d}{d\bar{z}}f(z,\bar{z})$ involves terms of the form $\bar{z}^{k-1}$.

The graph g with image in $\mathbb{C}^3$, $g(z)=(z,f(z,\bar{z}))$ maps the real axis to $(x,0,0)$, so the image of g coincides with the z_1-axis along a real line, which is the CR singular locus of the image.

The Example shows that $\overline{\partial} f$ can vanish to arbitrarily high order, so f is smooth but not holomorphic, f has high order of contact with a holomorphic map, and f is constant on an entire segment in the domain, but f is not constant.

The following result, used in [IR], generalizes Lemma 8.8.

Lemma 8.10. *Given a $\mathcal{C}^s$ almost complex structure J_B on B and $\mathcal{C}^{s+1}$ maps $u, v : D_1 \to B$, if there is an integer $k \le s+1$ such that*

$$\overline{\partial}_J u = o(|z|^{k-1}), \qquad \overline{\partial}_J v = o(|z|^{k-1})$$

and, for all ℓ such that $0 \le \ell \le k$,

$$\left(\frac{d}{dx}\right)^{\ell} u(0) = \left(\frac{d}{dx}\right)^{\ell} v(0),$$

then, for all (j, m) such that $j + m \le k$,

$$\left(\frac{d}{dx}\right)^{j} \left(\frac{d}{dy}\right)^{m} u(0) = \left(\frac{d}{dx}\right)^{j} \left(\frac{d}{dy}\right)^{m} v(0).$$

Proof. As in the Proof of Lemma 8.8,

$$\begin{aligned}
\overline{\partial}_J u &= o(|z|^{k-1}) \implies \frac{du}{dy}(z) = J_B(u(z)) \cdot \frac{du}{dx}(z) + o(|z|^{k-1}), \\
\overline{\partial}_J v &= o(|z|^{k-1}) \implies \frac{dv}{dy}(z) = J_B(v(z)) \cdot \frac{dv}{dx}(z) + o(|z|^{k-1}).
\end{aligned}$$

The proof of the claim is by induction on m. The $m = 0$ case is exactly the second hypothesis.

For the inductive step establishing the claim for $m > 0$, assume

$$\left(\frac{d}{dx}\right)^{j} \left(\frac{d}{dy}\right)^{\ell} u(0) = \left(\frac{d}{dx}\right)^{j} \left(\frac{d}{dy}\right)^{\ell} v(0)$$

for all (j, ℓ) such that $\ell < m$ and $j + \ell \le k$. Then, for any j such that $j + m \le k$,

$$\begin{aligned}
\left(\frac{d}{dx}\right)^{j} \left(\frac{d}{dy}\right)^{m} u &= \left(\frac{d}{dx}\right)^{j} \left(\frac{d}{dy}\right)^{m-1} \frac{du}{dy} \\
&= \left(\frac{d}{dx}\right)^{j} \left(\frac{d}{dy}\right)^{m-1} \left(J_B(u(z)) \cdot \frac{du}{dx} + o(|z|^{k-1})\right).
\end{aligned}$$

The derivative of the second term is $o(1)$. The derivative of the first term is, by the chain rule and the product rule, assuming the existence of $k-1$ derivatives of J_B, a sum of (possibly repeated) terms of the form $(J_B)_{x^\alpha y^\beta}(u(z)) \cdot u_{x^\gamma y^\delta} \cdots u_{x^\eta y^\theta}$, with $\alpha + \beta \le k-1$, and few enough derivatives $u_{x^\gamma y^\delta}$, etc., to satisfy the inductive hypothesis, so that when evaluated at 0, the derivative is $(J_B)_{x^\alpha y^\beta}(u(0)) \cdot u_{x^\gamma y^\delta}(0) \cdots u_{x^\eta y^\theta}(0) = (J_B)_{x^\alpha y^\beta}(v(0)) \cdot v_{x^\gamma y^\delta}(0) \cdots v_{x^\eta y^\theta}(0)$. ∎

Example 8.11. Consider, for $k \ge 1$, the smooth maps

$$u(x,y) = (x + y^k, y + xy^{k-1}, x + y^{2k}, y), \quad v(x,y) = (x,y,x,y).$$

Then $u, v : D_1 \to \mathbb{C}^2$ and $u = v + f$ where f is the smooth map from Example 8.9. The map v is just a holomorphic embedding of the disk into the line $z_1 = z_2$, $v(z) = (z,z)$, and u is a smooth but not holomorphic map, which is an embedding near 0, satisfying the hypotheses of Lemma 8.10.

Note $u(x,0) = v(x,0)$ for all x, so the maps coincide along the x-axis.

By construction, $\left(\frac{d}{dx}\right)^\ell u(0) = \left(\frac{d}{dx}\right)^\ell v(0)$ for all ℓ, but $(\frac{d}{dy})^k u(0) \ne \vec{0} = (\frac{d}{dy})^k v(0)$ and $(\frac{d}{dx})(\frac{d}{dy})^{k-1} u(0) \ne \vec{0} = (\frac{d}{dx})(\frac{d}{dy})^{k-1} v(0)$. The conclusion from Lemma 8.10 is that $\bar{\partial} u = o(|z|^{k-1})$ must be false.

The same conclusion can be drawn more directly, from expanding

$$u(z,\bar{z}) = \left(z + \left(\frac{z-\bar{z}}{2i}\right)^k + i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)^{k-1}, z + \left(\frac{z-\bar{z}}{2i}\right)^{2k} \right),$$

so $\frac{d}{d\bar{z}} u(z,\bar{z})$ involves terms of the form $\bar{z}^{k-1}$.

9 Finding J so that a surface is J-holomorphic

For J_B as in Subsection 5.1, recall $f : D_1 \to (B, J_B)$ is J_B-holomorphic if and only if it satisfies Equation (43): $\bar{\partial} f = Q(f(z)) \cdot \partial f$, where Q is an a-linear operator depending on the position $\vec{x} \in B$: $Q(\vec{x}) = (J_B(\vec{x}) + J_{std})^{-1} \cdot (J_{std} - J_B(\vec{x}))$.

We consider the following problem: Consider $\mathbf{u} : D_1 \to \mathbb{C}^N$ — is there a continuous CSO J defined near $\vec{0} \in \mathbb{C}^N$ so that $\mathbf{u}$ is J-holomorphic? A necessary condition is that $\mathbf{u}$ satisfies $\bar{\partial}\mathbf{u}(z) = \mathbf{Q}(z)\partial\mathbf{u}$ for $z \in D_1$, where $\mathbf{Q}$ is some continuous function from D_1 to the space of a-linear operators on $\mathbb{C}^N$, with $\mathbf{Q}(0) = \mathbf{0}$.

From the linear algebra in Section 1, the a-linearity of $\mathbf{Q}(z)$ is equivalent to complex $N \times N$ matrix representations $B_1(z)$ or $B_2(z)$, where

$$\overline{\partial}\mathbf{u}(z) = B_1(z)\overline{\partial\mathbf{u}} = \overline{B_2(z)\partial\mathbf{u}}.$$

It is convenient to assume $\mathbf{u}(0) = \mathbf{0}$, and that $\mathbf{u}$ is continuous on the closed disk $\overline{D_1}$ and continuously differentiable on D_1. It is key to the construction to assume that $\mathbf{u}$ is one-to-one on $\overline{D_1}$. Then the image of $\mathbf{u}$ is the compact, continuously embedded disk $\mathbf{u}(\overline{D_1}) \subseteq \mathbb{C}^N$, and the inverse map $\mathbf{u}^{-1} : \mathbf{u}(\overline{D_1}) \to \overline{D_1}$ is automatically continuous.

Given $\mathbf{u}$, one (not necessarily unique) way to find an a-linear operator $\mathbf{Q}(z)$ with $\overline{\partial}\mathbf{u}(z) = \mathbf{Q}(z)\partial\mathbf{u}(z)$ is to construct the following $N \times N$ complex matrix, $\tilde{Q}(z)$:

$$\begin{aligned}
\overline{\partial}\mathbf{u} &= \begin{bmatrix} u^1_{\bar{z}} \\ \vdots \\ u^N_{\bar{z}} \end{bmatrix}_{N\times 1} = \begin{bmatrix} u^1_{\bar{z}} \\ \vdots \\ u^N_{\bar{z}} \end{bmatrix}_{N\times 1} \frac{\begin{bmatrix} u^1_z \dots & u^N_z \end{bmatrix}_{1\times N}}{|u^1_z|^2 + \cdots + |u^N_z|^2} \begin{bmatrix} \overline{u^1_z} \\ \vdots \\ \overline{u^N_z} \end{bmatrix}_{N\times 1} \\
&= \frac{1}{|u^1_z|^2 + \cdots + |u^N_z|^2} \begin{bmatrix} u^1_{\bar{z}}u^1_z & \cdots & \\ \vdots & & \\ & \cdots & u^N_{\bar{z}}u^N_z \end{bmatrix}_{N\times N} \begin{bmatrix} \overline{u^1_z} \\ \vdots \\ \overline{u^N_z} \end{bmatrix}_{N\times 1} \\
&= (\tilde{Q}(z) \circ C)\partial\mathbf{u}.
\end{aligned}$$

Multiplying by $\tilde{Q}$ is c-linear; C is the a-linear complex conjugation as in Example 1.12, Section 1. We now make some more assumptions — that $\mathbf{u}$ has the properties that $\partial\mathbf{u} \neq \mathbf{0}$ for $z \neq 0$, and that $\tilde{Q}$ as defined above extends continuously to $\overline{D_1}$, including the origin, where $\tilde{Q}(0) = \mathbf{0}$. The complex entries of $\tilde{Q}_{N\times N}$ (and also the real entries of the real $2N \times 2N$ representation) are all bounded by $\|\overline{\partial}\mathbf{u}\|/\|\partial\mathbf{u}\|$. For example, if $\mathbf{u}$ is J_{std}-holomorphic and non-constant, then $\tilde{Q} \equiv \mathbf{0}$.

Using the assumption that $\mathbf{u}$ is one-to-one,

$$\overline{\partial}\mathbf{u} \;=\; (\tilde{Q}(\mathbf{u}^{-1}(\mathbf{u}(z))) \circ C)\partial\mathbf{u}$$

will match Equation (43) if, for $\vec{x} = \mathbf{u}(z)$,

$$\tilde{Q}(\mathbf{u}^{-1}(\vec{x})) \circ C = (J_B(\vec{x}) + J_{std})^{-1} \cdot (J_{std} - J_B(\vec{x})).$$

Using the inverse formula (2), if $A = \tilde{Q}(\mathbf{u}^{-1}(\vec{x})) \circ C$ is small enough so that $Id+A$ is invertible (which holds for $\vec{x} = \mathbf{u}(z)$ sufficiently close to $\vec{0}$, and which can be assumed to hold for all $z \in \overline{D_1}$ by re-scaling exactly as in Section 5.3), then, for each $\vec{x}$, the real $2N \times 2N$ operator $J(\vec{x})$ is a CSO so that $\mathbf{u}$ and J satisfy (43):

$$J(\vec{x}) = \Big(Id + \tilde{Q}(\mathbf{u}^{-1}(\vec{x})) \circ C\Big) \circ J_{std} \circ \Big(Id + \tilde{Q}(\mathbf{u}^{-1}(\vec{x})) \circ C\Big)^{-1}.$$

As a function of $\vec{x} \in \mathbf{u}(\overline{D_1})$, J is continuous, and $J(\vec{0}) = J_{std}$. By point-set topology, the continuous function $\tilde{Q} \circ \mathbf{u}^{-1}$ extends from the closed set $\mathbf{u}(\overline{D_1}) \subseteq \mathbb{C}^N$ to a continuous function $\mathbb{C}^N \to \mathrm{Hom}_a(\mathbb{C}^N, \mathbb{C}^N)$, and $J(\vec{x})$ also extends from the closed set $\mathbf{u}(\overline{D_1}) \subseteq \mathbb{C}^N$ to a continuous almost complex structure on all of $\mathbb{C}^N$. The conclusion is that $\mathbf{u}$ is J-holomorphic with respect to this continuous extension.

Pointwise estimates for $\tilde{Q}$ and J are related to re-scaling as in Section 5.3. However, any stronger estimate for J, for example, of the form $\|J(\vec{x}) - J_{std}\| \leq C_1\|\vec{x}\|^\alpha$, for $\alpha > 0$, would require an estimate of a similar form on $\tilde{Q} \circ \mathbf{u}^{-1}$,

$$\|(\tilde{Q} \circ \mathbf{u}^{-1})(\vec{x})\| \leq C_2\|\vec{x}\|^\alpha, \tag{65}$$

measured at points $\vec{x} = \mathbf{u}(z) \in \mathbf{u}(\overline{D_1})$, or equivalently, as a function of z:

$$\|\tilde{Q}(z)\| = \|\tilde{Q}(\mathbf{u}^{-1}(\mathbf{u}(z)))\| \leq C_2\|\mathbf{u}(z)\|^\alpha.$$

The composition with the inverse $\mathbf{u}^{-1}$ is a big loss in (65); even if $\mathbf{u}$ is smooth, $\mathbf{u}^{-1}$ is continuous but fast-growing, and not necessarily differentiable. The continuous extension of the composite $\tilde{Q} \circ \mathbf{u}^{-1}$ might extend to something α-Hölder continuous, or differentiable, at the origin, but would be hard to estimate without an explicit formula for $\mathbf{u}$.

Example 9.1. In the Example from [CP$_1$], $\mathbf{u}$ is smooth and vanishing to infinite order, and $\tilde{Q}(z)$ also vanishes to infinite order, where $\|\tilde{Q}(z)\|$ is comparable to $\|\overline{\partial}\mathbf{u}\|/\|\partial\mathbf{u}\|$, but

$$\frac{\|\tilde{Q}(z)\|}{\|\mathbf{u}(z)\|^\alpha} \approx \frac{\|\overline{\partial}\mathbf{u}\|}{\|\mathbf{u}(z)\|^\alpha\|\partial\mathbf{u}\|}$$

is unbounded as $z \to 0$, for any $\alpha > 0$. (We did not find any other examples of smooth functions $\mathbf{u}$ with bounded $\frac{\|\overline{\partial}\mathbf{u}\|}{\|\mathbf{u}(z)\|^\alpha\|\partial\mathbf{u}\|}$.)

The Example $(u^1(z), u^2(z))$ from [CP$_1$] is also not one-to-one $\mathbb{C} \to \mathbb{C}^2$, but it is non-zero except at $z = 0$, so it can be modified to

$$\mathbf{u}(z) = (u^1, u^2, z \cdot u^1, z \cdot u^2)$$

to get a smooth, one-to-one map $\mathbb{C} \to \mathbb{C}^4$.

$$\partial\mathbf{u} = (u^1_1, u^2_z, u^1 + zu^1_z, u^2 + zu^2_z)$$

is still non-vanishing as a vector, and $\|\partial\mathbf{u}\| \geq |u^1_z| \geq F(n)p(n)|z|^{p(n)-1}$ on even annuli A_n.

$$\overline{\partial}\mathbf{u} = (u^1_{\bar{z}}, u^2_{\bar{z}}, zu^1_{\bar{z}}, zu^2_{\bar{z}})$$

satisfies $\|\overline{\partial}\mathbf{u}\| = (1 + |z|^2)^{1/2}\|\overline{\partial}(u^1, u^2)\|$, so the estimates for $\|\overline{\partial}\mathbf{u}\|/\|\partial\mathbf{u}\|$ on D_1 are comparable to the estimates in [CP$_1$], and the complex entries of $\tilde{Q}_{4\times 4}$ are bounded by $\|\overline{\partial}\mathbf{u}\|/\|\partial\mathbf{u}\|$. As described above, this $\mathbf{u}$ is J-holomorphic with respect to a continuous almost complex structure on $\mathbb{C}^4$, and still has the property of having an isolated zero of infinite order.

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