Handout for Pi Day at Science Central
by Professor Adam Coffman, IPFW.
`http://www.ipfw.edu/math/`

3.14159265358979323846264338327950288419716939937510582097494459230781640628
6208998628034825342117067982148086513282306647093844609550582231725359408128
481117450284102701938521105559644622948954930381964428810975665933446128475648233
. . .

Some links on the internet about π:

At the Math Forum: `http://mathforum.org/library/topics/pi/`

David H. Bailey's page: `http://crd.lbl.gov/~dhbailey/pi/`

David Blatner's page: `http://www.joyofpi.com/`

St. Andrew's History of Mathematics Archive: `http://www-groups.dcs.st-and.ac.uk/~history/`

Did the Indiana state legislature really pass a law in 1897 declaring π to be equal to 3.2? No, it almost did, but a Purdue math professor happened to be in the capitol at the time and stopped it:
`http://www.agecon.purdue.edu/crd/Localgov/topics.htm`
`http://en.wikipedia.org/wiki/Indiana_Pi_Bill`

Some links on the internet about Pi Day (March 14):

At the Math Forum: `http://mathforum.org/t2t/faq/faq.pi.html`

`http://www.teachpi.org/`

Some books at IPFW's Helmke Library about π and other numbers:
P. BECKMANN, *A History of Pi*. QA 484.B4 1971.
J. H. CONWAY and R. GUY, *The Book of Numbers*. QA 241.C6897 1996.
D. WELLS, *Curious and Interesting Numbers*. QA 241.W36 1997.
E. ZEBROWSKI, *A History of the Circle*. Q 176.Z42 1999.
(these two are at a more advanced level:)
I. NIVEN, *Irrational Numbers*. QA 247.5.N57.
S. LANG, *Introduction to Transcendental Numbers*. QA 247.5.L3.

How do we know π is an irrational number? The fact that π is irrational can be <u>proved</u> so we call it a <u>Theorem</u>. The first proof was given by J. H. Lambert in 1768. The proof on the next page is based on one by I. Niven from 1947. It uses only single-variable calculus.

Definition. A real number x is a "rational" number if there are integers p and q, with $q \neq 0$, so that $x = p/q$. If a real number x is not rational, then x is called "irrational."

Theorem. π *is irrational.*

Proof. In fact, we will show something even better: π^2 is irrational. Then π itself can't be rational, since if it were, $\pi = p/q$ would imply $\pi^2 = p^2/q^2$ is rational, which is about to be proved wrong.

First, for an integer n (which we'll pick later and leave unknown for now), define the function $f(x) = x^n(1-x)^n/n!$. For all x, $f(x) = f(1-x)$, and on the interval $0 \le x \le 1$, the function values satisfy $0 \le f(x) \le \frac{1}{n!}$. Expanding $f(x)$ as a polynomial,

$$f(x) = \frac{1}{n!}(c_0 x^n + c_1 x^{n+1} + c_2 x^{n+2} + \cdots + c_n x^{2n}),$$

where the coefficients $c_0, \ldots, c_n$ are integers. For any non-negative integer j, the jth derivative of f, denoted $\left(\frac{d}{dx}\right)^j f = f^{(j)}$, has the property that $f^{(j)}(0)$ is an integer: for $j < n$ or $j > 2n$, $f^{(j)}$ is a polynomial with constant term 0, so $f^{(j)}(0) = 0$; for $n \le j \le 2n$, applying the Power Rule j times to x^j gives the constant $j!$, and it follows that $f^{(j)}(0) = j! c_{j-n}/n!$, which is an integer since all the factors cancel from the denominator. Since $\left(\frac{d}{dx}\right)^j (f(x)) = \left(\frac{d}{dx}\right)^j (f(1-x)) = (-1)^j f^{(j)}(1-x)$ by the Chain Rule, $f^{(j)}(1) = (-1)^j f^{(j)}(1-1) = (-1)^j f^{(j)}(0)$ is also an integer for any j.

Now, assume that π^2 is a rational number, so $\pi^2 = a/b$, where a and b are integers. We'll see that making this assumption leads to a contradiction, so π^2 must be irrational. Define another polynomial function

$$
\begin{aligned}
F(x) &= b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f^{(2k)}(x) \\
&= b^n \cdot \left(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) - \cdots + (-1)^n \pi^0 f^{(2n)}(x) \right).
\end{aligned}
$$

The coefficients $b^n \pi^{2(n-k)} = b^n(a/b)^{n-k} = a^{n-k} b^k$ are all integers, and we showed earlier that $f^{(j)}(0)$ and $f^{(j)}(1)$ are also integers, so $F(0)$ and $F(1)$ are integers.

The first step of this calculation is taking the derivative using the Product Rule and Chain Rule, the second step is a cancellation of all but one term from the sum formula for F:

$$
\begin{aligned}
\frac{d}{dx}\left(F'(x)\sin(\pi x) - \pi F(x)\cos(\pi x)\right) &= \left(F''(x) + \pi^2 F(x)\right)\sin(\pi x) \\
&= b^n \pi^{2n+2} f(x)\sin(\pi x) = \pi^2 a^n f(x)\sin(\pi x).
\end{aligned}
$$

By the Fundamental Theorem of Calculus,

$$\pi a^n \int_0^1 f(x)\sin(\pi x)dx = \left[\frac{1}{\pi}F'(x)\sin(\pi x) - F(x)\cos(\pi x)\right]_0^1 = F(1) + F(0),$$

which is an integer. Since $0 < f(x)\sin(\pi x) < \frac{1}{n!}\sin(\pi x)$ for $0 < x < 1$, we also get the estimate

$$0 < \pi a^n \int_0^1 f(x)\sin(\pi x)dx < \pi a^n \int_0^1 \frac{1}{n!}\sin(\pi x)dx = \pi a^n \cdot \frac{1}{n!} \cdot \frac{2}{\pi} = \frac{2a^n}{n!}.$$

We can now pick n big enough so that $n > a$ and $n > \frac{2a^{a+1}}{a!}$. Then $\frac{2a^n}{n!} = \frac{a}{n} \cdot \frac{a}{n-1} \cdots \frac{a}{a+1} \cdot \frac{2a^a}{a!} \le \frac{1}{n} \cdot \frac{2a^{a+1}}{a!} < 1$ (the middle fractions $\frac{a}{n-1} \cdots \frac{a}{a+1}$ are each < 1). However, this results in the inequality

$$0 < F(0) + F(1) = \pi a^n \int_0^1 f(x)\sin(\pi x)dx < 1,$$

which contradicts the previous observation that $F(0) + F(1)$ is an integer. ∎

What's the connection between rational numbers and decimal expansions? It turns out that rational numbers have repeating decimal expansions, and irrational numbers have non-repeating decimal expansions. I'll use a bar notation for repeating decimal blocks, so if the block 246 repeats, we can write $0.246246246246246\ldots = 0.\overline{246}$. To be precise about the connection between fractions and decimals, there are some more theorems.

Theorem. *If the real number x has a repeating decimal expansion, then x is a rational number.*

Proof. There are three cases. First, x could be a "terminating" decimal, which we count as repeating since we could add repeating 0 digits at the end, like $\frac{3}{8} = 0.375 = 0.375000000000\ldots$. In the terminating case, if $x = N + 0.DIGITS\overline{000}$, so N is the integer part, then $x = N + \frac{DIGITS}{1000000}$. Use as many 0 digits in the denominator as there are digits in the terminating block.

If x is repeating a block of digits, of the form $x = 0.\overline{BLOCK}$, then $x = \frac{BLOCK}{99999}$. Use as many 9 digits in the denominator as there are digits in the repeating block. (I am skipping the part of the proof that shows this always works, but this is something you can try on your calculator!)

If x is a combination, so it is a finite list of digits followed by a repeating block, of the form $x = N + 0.DIGITS\overline{BLOCK}$, then we can write it as

$$x = N + 0.DIGITS + \frac{1}{1000000} \cdot 0.\overline{BLOCK} = N + \frac{DIGITS}{1000000} + \frac{BLOCK}{99999000000}.$$

Then we can add the fractions to get the rational number expression $x = p/q$. ∎

For example, if $x = 6.7154343434343\overline{43}$, then $x = 6 + \frac{715}{1000} + \frac{43}{99000} = \frac{664828}{99000}$.

It also follows that π does not have a repeating decimal expansion, since a repeating decimal expansion would imply that π is rational, contradicting our earlier Theorem.

Theorem. *If x is a rational number, then x has a repeating decimal expansion.*

Proof. The main part of the proof is the following statement about integers: if N is a non-negative integer and D is a positive integer, then there are integers Q and R so that $N = QD + R$ and $0 \leq R < D$. The idea is that the fraction $\frac{N}{D}$ can always be simplified into an integer quotient Q, with some integer remainder R, so that $\frac{N}{D} = Q + \frac{R}{D}$, where $0 \leq \frac{R}{D} < 1$. Multiplying both sides by D gives the $N = QD + R$ expression.

So, we start with $x = p/q$ from the definition of rational, and simplify (as in the above division statement) to get $x = p/q = Q + \frac{r_0}{q}$, where Q is the integer part, and r_0 is the integer remainder with $0 \leq r_0 < q$.

What we want to find is the digit sequence for the fractional part: $\frac{r_0}{q} = 0.d_1 d_2 d_3 d_4 \ldots d_n \ldots$, and we also want to show that the decimal expansion is repeating.

Consider $10r_0$ and divide q into it to get a quotient and remainder, $\frac{10r_0}{q} = d_1 + \frac{r_1}{q}$, or equivalently, $10r_0 = d_1 \cdot q + r_1$, where d_1 and r_1 are integers so that $0 \leq r_1 < q$ and $0 \leq d_1 \leq \frac{10r_0}{q} < 10$. The single digit d_1 is in fact the first decimal place we were looking for, since dividing both sides of $\frac{10r_0}{q} = d_1 + \frac{r_1}{q}$ by 10 gives $\frac{r_0}{q} = \frac{d_1}{10} + \frac{1}{10} \cdot \frac{r_1}{q}$, where $0 \leq \frac{1}{10} \cdot \frac{r_1}{q} < \frac{1}{10}$.

We can repeat the above step with r_1, so that $\frac{10r_1}{q} = d_2 + \frac{r_2}{q}$ for some single digit d_2 and remainder $0 \leq r_2 < q$. Then $\frac{r_1}{q} = \frac{d_2}{10} + \frac{1}{10} \cdot \frac{r_2}{q}$, and we can plug this into the above expression for $\frac{r_0}{q}$ to get

$$\frac{r_0}{q} = \frac{d_1}{10} + \frac{1}{10} \cdot \frac{r_1}{q} = \frac{d_1}{10} + \frac{1}{10} \cdot \left(\frac{d_2}{10} + \frac{1}{10} \cdot \frac{r_2}{q} \right) = \frac{d_1}{10} + \frac{d_2}{100} + \frac{1}{100} \frac{r_2}{q}.$$

This shows that d_2 is the second digit.

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Continuing to repeat this process, we see that every time we get a remainder r_k, we can divide $10r_k$ by q to get the next digit d_{k+1} and a new remainder r_{k+1}, both of which depend only on r_k and q. Since the only possible r_k remainder values we can get are integer values between 0 and $q-1$, after q steps, the list of remainders $r_0, r_1, \ldots, r_{q-1}$ is going to either contain a 0 (in which case the decimal expansion terminates) or the list of q numbers between 1 and $q-1$ will have at least one repeated value: $r_k = r_n$ for some $0 \le k < n \le q-1$. Since r_k and r_n are equal, they determine the same digit $d_{k+1} = d_{n+1}$ and the same remainders $r_{k+1} = r_{n+1}$. Since the digit sequence depends only on the remainders, the same outputs will occur in a block of $n-k$ digits: $(d_{k+1}d_{k+2}\ldots d_n) = (d_{n+1}d_{n+2}\ldots d_{n+n-k})$, and this is the block that repeats, so

$$x = \frac{p}{q} = Q + 0.d_1\ldots d_{k-1}d_k\overline{d_{k+1}\ldots d_n}.$$

∎

We see in the above proof that a rational number of the form $\frac{p}{q}$ not only has a repeating decimal expansion, but the block that repeats has size less than q. For example, we can try $x = \frac{2}{7}$, and it should have a repeating block with length less than 7. The above procedure (essentially the "long division" algorithm) says starting with $r_0 = 2$, the first digit is the integer part of $\frac{20}{7}$, which is $d_1 = 2$, with remainder $r_1 = 6$. The next digit is the integer part of $\frac{60}{7}$, which is 8 with remainder $r_2 = 4$. The next digit is the integer part of $\frac{40}{7}$, which is 5 with remainder $r_3 = 5$. The next digit is the integer part of $\frac{50}{7}$, which is 7 with remainder $r_4 = 1$. The next digit is the integer part of $\frac{10}{7}$, which is 1 with remainder $r_5 = 3$. The next digit is the integer part of $\frac{30}{7}$, which is 4 with remainder $r_6 = 2$. But that remainder, $r_6 = 2$, is a repeat of the numerator $r_0 = 2$ that started the process! The sequence of remainders was $(2, 6, 4, 5, 1, 3, 2)$, and we got six out of the seven possible remainders $0, 1, 2, 3, 4, 5, 6$ before getting a repeated 2 that tells us that the decimal places also start repeating. The conclusion is that $\frac{2}{7} = 0.\overline{285714}$.