# Notes on Differential Equations and Differential Inequalities 

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## 1 Real autonomous ODE

The following Lemma gives conditions for the existence of a solution of a differential equation which is bounded on the domain $\mathbb{R}$.

Lemma 1.1. Given real numbers $a<b$, if $f:(a, b) \rightarrow \mathbb{R}$ is a continuous, nonvanishing function, and there are some constants $C_{1}>0, C_{2}>0, \delta_{1} \in$ $(0, b-a), \delta_{2} \in(0, b-a)$ so that $|f(t)| \leq C_{1}(t-a)$ for $a<t<a+\delta_{1}$ and $|f(t)| \leq C_{2}(b-t)$ for $b-\delta_{2}<t<b$, then there exists a one-to-one, onto function $g: \mathbb{R} \rightarrow(a, b)$ so that $y=g(t)$ is a solution of the equation $\frac{d y}{d t}=f(y)$.
Proof. $\frac{1}{f(x)}$ is continuous on $(a, b)$, so the function

$$
\begin{equation*}
G(t)=\int_{\frac{a+b}{2}}^{t} \frac{1}{f(x)} d x \tag{1}
\end{equation*}
$$

is differentiable on $(a, b)$ with a nonvanishing, nonzero derivative, $\frac{d}{d t} G(t)=$ $\frac{1}{f(t)}$. Because $f$ and $\frac{1}{f}$ have constant sign, $G(t)$ is monotone on $(a, b)$. Suppose $f(t)>0$, so $G$ is increasing; the $f(t)<0$ case is similar.

For $t \in\left(b-\delta_{2}, b\right)$,

$$
G(t)=\int_{\frac{a+b}{2}}^{b-\delta_{2}} \frac{1}{f(x)} d x+\int_{b-\delta_{2}}^{t} \frac{1}{f(x)} d x \geq \int_{\frac{a+b}{2}}^{b-\delta_{2}} \frac{1}{f(x)} d x+\int_{b-\delta_{2}}^{t} \frac{1}{C_{2}(b-x)} d x
$$

[^0]which is unbounded. Similarly, $G$ is also unbounded at the other endpoint, so $G:(a, b) \rightarrow \mathbb{R}$ is onto and invertible. Let $T$ be any constant, and define
\[

$$
\begin{equation*}
g(t)=G^{-1}(t+T) \tag{2}
\end{equation*}
$$

\]

so $g: \mathbb{R} \rightarrow(a, b)$ is onto and increasing, and (by the Inverse Function Theorem, $[\mathrm{C}]), y=g(t)$ is differentiable with

$$
\frac{d y}{d t}=\frac{1}{G^{\prime}\left(G^{-1}(t+T)\right)}=\frac{1}{\frac{1}{f(y)}}=f(y)
$$

Such solutions with domain $\mathbb{R}$ are unique up to translation.
Lemma 1.2. Given an open (possibly infinite) interval $I$, if $f: I \rightarrow \mathbb{R}$ is a continuous, nonvanishing function, and $g_{1}: \mathbb{R} \rightarrow I$ and $g_{2}: \mathbb{R} \rightarrow I$ are solutions of the equation $\frac{d y}{d t}=f(y)$, then there exists a constant $T$ so that $g_{2}(t)=g_{1}(t+T)$.

Proof. Because $g_{1}^{\prime}(x)=f\left(g_{1}(x)\right)$ is continuous and nonzero, the Inverse Function Theorem applies. For $t \in \mathbb{R}$,

$$
\begin{align*}
\frac{d}{d t}\left(g_{1}^{-1}\left(g_{2}(t)\right)-t\right) & =\frac{1}{g_{1}^{\prime}\left(g_{1}^{-1}\left(g_{2}(t)\right)\right)} g_{2}^{\prime}(t)-1  \tag{3}\\
& =\frac{1}{f\left(g_{1}\left(g_{1}^{-1}\left(g_{2}(t)\right)\right)\right)} f\left(g_{2}(t)\right)-1 \equiv 0 .
\end{align*}
$$

There is also a local uniqueness theorem for solutions on an interval, with one initial condition.

Lemma 1.3. Given open (possibly infinite) intervals $I_{0}, I_{1}, I_{2}$, if $f: I_{0} \rightarrow \mathbb{R}$ is a continuous, nonvanishing function, and $g_{1}: I_{1} \rightarrow I_{0}$ and $g_{2}: I_{2} \rightarrow I_{0}$ are solutions of the equation $\frac{d y}{d t}=f(y)$, and there is a point $c \in I_{1} \cap I_{2}$ such that $g_{1}(c)=g_{2}(c)$, then there exists $\delta>0$ so that $g_{1}(t)=g_{2}(t)$ for all $t \in(c-\delta, c+\delta)$.

Proof. Because $g_{1}^{\prime}(x)=f\left(g_{1}(x)\right)$ is continuous and nonzero, the Inverse Function Theorem applies: there exists some $\delta_{1}>0$ so that $g_{1}$ is one-to-one on $\left(c-\delta_{1}, c+\delta_{1}\right)$. Suppose $f>0$, so $g_{1}$ is increasing; the $f<0$ case is similar. Let $\varepsilon=\min \left\{g_{1}\left(c+\frac{1}{2} \delta_{1}\right)-g_{1}(c), g_{1}(c)-g_{1}\left(c-\frac{1}{2} \delta_{1}\right)\right\}>0$. Because $g_{2}$ is continuous, there is some $\delta_{2}>0$ corresponding to $\varepsilon$, so that for all $t \in\left(c-\delta_{2}, c+\delta_{2}\right),\left|g_{2}(t)-g_{2}(c)\right|=\left|g_{2}(t)-g_{1}(c)\right|<\varepsilon$. Let $\delta=\min \left\{\delta_{1}, \delta_{2}\right\}>0$, then $(c-\delta, c+\delta) \subseteq I_{1} \cap I_{2}$, where both $g_{1}$ and $g_{2}$ are defined. Also, for any $t \in(c-\delta, c+\delta)$,

$$
g_{1}\left(c-\frac{1}{2} \delta_{1}\right) \leq g_{1}(c)-\varepsilon<g_{2}(t)<g_{1}(c)+\varepsilon \leq g_{1}\left(c+\frac{1}{2} \delta_{1}\right),
$$

and by the Intermediate Value Theorem, there is some $x \in\left(c-\frac{1}{2} \delta_{1}, c+\frac{1}{2} \delta_{1}\right)$ so that $g_{1}(x)=g_{2}(t)$; this shows that $x=g_{1}^{-1}\left(g_{2}(t)\right)$, so $g_{2}(t)$ is in the domain of $g_{1}^{-1}$. As in (3), for $c-\delta<t<c+\delta$,

$$
\frac{d}{d t}\left(g_{1}^{-1}\left(g_{2}(t)\right)\right) \equiv 1 \Longrightarrow g_{1}^{-1}\left(g_{2}(t)\right)=t+T
$$

for some constant $T$. Evaluating $g_{2}(t)=g_{1}(t+T)$ at $t=c$ gives $g_{2}(c)=$ $g_{1}(c+T)$, and $g_{2}(c)=g_{1}(c)$ by hypothesis, so $c+T=c$ because $g_{1}$ is one-to-one. It follows that $T=0$ and $g_{1}(t)=g_{2}(t)$ for all $t \in(c-\delta, c+\delta)$.

Lemma 1.4. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous, nonvanishing function, then there exist some open interval $I$ and a one-to-one, onto function $g: I \rightarrow \mathbb{R}$ so that $y=g(t)$ is a solution of the equation $\frac{d y}{d t}=f(y)$.

Proof. $\frac{1}{f(x)}$ is continuous on $\mathbb{R}$, so the function

$$
G(t)=\int_{0}^{t} \frac{1}{f(x)} d x
$$

is differentiable on $\mathbb{R}$ with a nonvanishing, nonzero derivative, $\frac{d}{d t} G(t)=\frac{1}{f(t)}$. Because $f$ and $\frac{1}{f}$ have constant sign, $G(t)$ is monotone on $\mathbb{R}$. Its image is some open interval $I$, so $G: \mathbb{R} \rightarrow I$ is invertible.

Let $T$ be any constant, and define $g(t)=G^{-1}(t+T)$ on the interval $I-T=\{x \in \mathbb{R}: x+T \in I\}$, so $g: I-T \rightarrow \mathbb{R}$ is invertible, and therefore not bounded. $g$ is a solution of the ODE as in Lemma 1.1.

Lemma 1.5. Given $b \in \mathbb{R}$, if $f:(-\infty, b) \rightarrow \mathbb{R}$ is a continuous, nonvanishing function, and there are some constants $C_{3}>0, \delta_{3} \in(0,1)$ so that $|f(t)| \leq$ $C_{3}(b-t)$ for $b-\delta_{3}<t<b$, then there exist an open interval I and a one-to-one, onto function $g: I \rightarrow(-\infty, b)$ so that $y=g(t)$ is a solution of the equation $\frac{d y}{d t}=f(y)$.
Proof. $\frac{1}{f(x)}$ is continuous on $(-\infty, b)$, so the function

$$
G(t)=\int_{b-1}^{t} \frac{1}{f(x)} d x
$$

is differentiable on $(-\infty, b)$ with a nonvanishing, nonzero derivative, $\frac{d}{d t} G(t)=$ $\frac{1}{f(t)}$. Because $f$ and $\frac{1}{f}$ have constant sign, $G(t)$ is monotone on $(-\infty, b)$. Suppose $f(t)>0$, so $G$ is increasing; the $f(t)<0$ case is similar.

For $t \in\left(b-\delta_{3}, b\right)$,

$$
G(t)=\int_{b-1}^{b-\delta_{3}} \frac{1}{f(x)} d x+\int_{b-\delta_{3}}^{t} \frac{1}{f(x)} d x \geq \int_{b-1}^{b-\delta_{3}} \frac{1}{f(x)} d x+\int_{b-\delta_{3}}^{t} \frac{1}{C_{3}(b-x)} d x
$$

which is unbounded. So, the image of $G$ is either $I=(L, \infty)$ or $I=\mathbb{R}$.
Let $T$ be any constant, and define $g(t)=G^{-1}(t+T)$ on the interval $I-T=\{x \in \mathbb{R}: x+T \in I\}$, so $g: I-T \rightarrow(-\infty, b)$ is invertible, and therefore not bounded. $g$ is a solution of the ODE as in Lemma 1.1.

Theorem 1.6. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a real analytic function. The following are equivalent:

1. there exists a non-constant, bounded, real analytic function $g: \mathbb{R} \rightarrow \mathbb{R}$ so that $y=g(t)$ is a solution of the equation $\frac{d y}{d t}=f(y)$;
2. there are at least two distinct points $a_{0}, b_{0}$ where $f\left(a_{0}\right)=f\left(b_{0}\right)=0$.

Proof. At any zero of $f$, say $c_{0}$, the real analyticity implies there is some constant $C_{0}$ so that $|f(t)| \leq C_{0}\left|x-c_{0}\right|$ for $x$ near $c_{0}$.

To show 2. $\Longrightarrow 1$., the property that $f$ is real analytic on $\mathbb{R}$ implies that $a_{0}$ is an isolated zero, and because $a_{0}$ is not the only zero of $f$, there is some open interval $(a, b)$ with either $a=a_{0}$ or $b=a_{0}$, satisfying the hypotheses of Lemma 1.1. The conclusion is that there exists a non-constant solution $g: \mathbb{R} \rightarrow(a, b)$, which by construction, (1) and (2), is real analytic.

To show 1. $\Longrightarrow$ 2., suppose, toward a contradiction, that there exists a solution $g_{1}$ as claimed, and that $f$ has fewer than two zeros.

Case 1. If $f$ is nonvanishing on $\mathbb{R}$ then Lemma 1.4 applies and there is some nonempty open interval $I$ and some onto function $g_{2}: I \rightarrow \mathbb{R}$ which is a solution of $\frac{d y}{d t}=f(y)$. By the construction of Lemma 1.4, $g_{2}$ is real analytic. On the interval $I, g_{2}-g_{1}$ is continuous, and because $g_{1}$ is bounded and $g_{2}$ is onto, $g_{2}-g_{1}$ attains some negative value and some positive value, so by the Intermediate Value Theorem, there is some $c$ so that $g_{1}(c)=g_{2}(c)$. By Lemma 1.3, $g_{1}(t) \equiv g_{2}(t)$ on some interval $(c-\delta, c+\delta)$, and because both functions are real analytic, $g_{1}(t) \equiv g_{2}(t)$ on $I$. The contradiction is that $g_{2}$ is unbounded on $I$ while $g_{1}$ is bounded.

Case 2. If $f$ has exactly one zero, $b \in \mathbb{R}$, then Lemma 1.5 applies to $f$ on $(-\infty, b)$ : there is some interval $I_{2}$ and some one-to-one, onto solution $g_{2}: I_{2} \rightarrow(-\infty, b)$. By an analogous existence result for $f$ on $(b, \infty)$, there is some interval $I_{3}$ and some one-to-one, onto solution $g_{3}: I_{3} \rightarrow(b, \infty)$. Because $g_{1}$ is non-constant, there is some $x_{0}$ where $g\left(x_{0}\right) \neq b$.

If $g_{1}\left(x_{0}\right)<b$, then there is some $T \in I_{2}$ with $g_{2}(T)=g_{1}\left(x_{0}\right)$, and the function $g_{4}(t)=g_{2}\left(t+T-x_{0}\right)$ is, by construction, a real analytic solution of $\frac{d y}{d t}=f(y)$ on an open interval $I_{2}-\left(T-x_{0}\right)$ satisfying $g_{4}\left(x_{0}\right)=g_{1}\left(x_{0}\right)$. As in Case 1., the uniqueness from Lemma 1.3 shows that $g_{4}=g_{1}$ on $I-\left(T-x_{0}\right)$, contradicting the boundedness of $g_{1}$.

The $g_{1}\left(x_{0}\right)>b$ case is similar, using $g_{3}$.

## 2 Ordinary differential inequalities

### 2.1 Linear differential inequalities

Lemma 2.1. If $u(t)$ is continuous on $[a, b]$ and $u(a)>0$, then either $u(t)>0$ for all $t \in[a, b]$, or there is some $t_{1} \in(a, b]$ so that $u(t)>0$ on $\left[a, t_{1}\right)$ and $u\left(t_{1}\right)=0$.

Proof. Consider the set $S=\{x \in[a, b]: u(t)>0$ for all $t \in[a, x)\}$; it is non-empty (by continuity of $u$ and $u(a)>0$ ), and bounded above, so it has a least upper bound $t_{1} \in(a, b]$. If there were some $t_{0}$ with $a<t_{0}<t_{1}$ and $u\left(t_{0}\right) \leq 0$, then there would be no $x \in S$ with $x>t_{0}$, and $t_{0}$ would be an upper bound, contradicting the least property of $t_{1}$. So, $u(t)>0$ for all $t \in\left[a, t_{1}\right)$. Case 1: If $u\left(t_{1}\right)=0$, then $u>0$ on $\left[a, t_{1}\right)$ as claimed. Case 2: If $u\left(t_{1}\right)<0$ then by continuity, there is some $t_{2}$ with $a<t_{2}<t_{1}$ and $u\left(t_{2}\right)<0$, contradicting the above property that $u(t)>0$ for all $t \in\left[a, t_{1}\right)$. Case 3: If $t_{1}=b$ and $u\left(t_{1}\right)>0$, then $u>0$ on $[a, b]$ as claimed. Case 4: If $t_{1}<b$ and $u\left(t_{1}\right)>0$, then by continuity, $u(t)$ would be positive on some interval $\left[a, t_{1}+\delta\right)$, contradicting the property that $t_{1}$ is an upper bound for $S$. Only Cases 1 and 3 do not lead to a contradiction.

Lemma 2.2. Suppose $a(t)$ is a real function on $[0,1)$ such that $a(t) \geq 0$ and $a$ is bounded on every subinterval $[0, x] \subseteq[0,1)$. If $y$ is continuous on $[0,1]$ with $y(0) \geq 0, \lim _{t \rightarrow 0^{+}} y^{\prime}(t) \geq 0$, and $y^{\prime \prime}(t) \geq a(t) y(t)$ for $0<t<1$, then $y(t) \geq 0$ for all $0 \leq t \leq 1$ and $y^{\prime}(t) \geq 0$ for all $0<t<1$.

Proof. Case 1: $y(0)>0$ and $\lim _{t \rightarrow 0^{+}} y^{\prime}(t)>0$. In this case, we can show that $y>0$ on $[0,1]$ and $y^{\prime}>0$ on $(0,1)$. By Lemma 2.1 applied to $y$ on $[0,1]$, either $y>0$ on $[0,1]$, or there is some $t_{1} \in(0,1]$ so that $y(t)>0$ for all $t \in\left[0, t_{1}\right)$ and $y\left(t_{1}\right)=0$. In the latter case, $y$ attains some positive maximum value on $\left[0, t_{1}\right]$. If $y(0)$ is the maximum, then by the Mean Value Theorem, for any $0<\delta<t_{1}$, there is some $t_{2}$ with $0<t_{2}<\delta$ and $y^{\prime}\left(t_{2}\right)=\frac{y(\delta)-y(0)}{\delta} \leq 0$, which contradicts $\lim _{t \rightarrow 0^{+}} y^{\prime}(t)>0$. If the maximum is at an interior point $t_{3}$ with $0<t_{3}<t_{1}$, then $y^{\prime}\left(t_{3}\right)=0$. From $\lim _{t \rightarrow 0^{+}} y^{\prime}(t)>0$, there is some $t_{4}$ with $0<t_{4}<t_{3}$ and $y^{\prime}\left(t_{4}\right)>0$. Applying the Mean Value Theorem to $y^{\prime}$ on $\left[t_{4}, t_{3}\right]$, there is some $t_{5}$ with $t_{4}<t_{5}<t_{3}$ and $y^{\prime \prime}\left(t_{5}\right)=\frac{y^{\prime}\left(t_{3}-y^{\prime}\left(t_{4}\right)\right.}{t_{3}-t_{4}}<0$. This contradicts $y^{\prime \prime}\left(t_{5}\right) \geq a\left(t_{5}\right) y\left(t_{5}\right) \geq 0$. We can conclude that $y\left(t_{1}\right)$ must be the
maximum value, and $y\left(t_{1}\right)>0$, which contradicts $y\left(t_{1}\right)=0$. This shows $y(t)>0$ for all $t \in[0,1]$.

For any $t_{7} \in(0,1)$, there is some $t_{8}$ with $0<t_{8}<t_{7}$ and $y^{\prime}\left(t_{8}\right)>0$. By the Mean Value Theorem, there is some $t_{9}$ with $\frac{y^{\prime}\left(t_{7}\right)-y^{\prime}\left(t_{8}\right)}{t_{7}-t_{8}}=y^{\prime \prime}\left(t_{9}\right) \geq$ $a\left(t_{9}\right) y\left(t_{9}\right) \geq 0$. It follows that $y^{\prime}\left(t_{7}\right) \geq y^{\prime}\left(t_{8}\right)>0$.

Case 1 did not use the boundedness of $a$, just $a \geq 0$.
Case 2: $y(0) \geq 0$ and $\lim _{t \rightarrow 0^{+}} y^{\prime}(t) \geq 0$. Suppose, toward a contradiction, that there is some $t_{0}$ with $0<t_{0}<1$ and $y\left(t_{0}\right)<0$. For $0 \leq t \leq t_{0}$, there is a bound $A>0$ with $0 \leq a(t) \leq A$. For $t \in\left[0, t_{0}\right]$, define $u(t)=$ $y(t)-\frac{1}{2} y\left(t_{0}\right) e^{\sqrt{A}\left(t-t_{0}\right)}$. Then, by construction, $u(0)>0$ and $u\left(t_{0}\right)<0$. For $0<t<t_{0}$,

$$
\begin{aligned}
u^{\prime}(t)= & y^{\prime}(t)-\frac{1}{2} y\left(t_{0}\right) \sqrt{A} e^{\sqrt{A}\left(t-t_{0}\right)} \\
\Longrightarrow \lim _{t \rightarrow 0^{+}} u^{\prime}(t)= & \left(\lim _{t \rightarrow 0^{+}} y^{\prime}(t)\right)-\frac{1}{2} y\left(t_{0}\right) \sqrt{A}>0, \\
u^{\prime \prime}(t)= & y^{\prime \prime}(t)-\frac{1}{2} y\left(t_{0}\right) A e^{\sqrt{A}\left(t-t_{0}\right)} \\
\geq & a(t) y(t)-\frac{1}{2} y\left(t_{0}\right) a(t) e^{\sqrt{A}\left(t-t_{0}\right)} \\
& +\frac{1}{2} y\left(t_{0}\right) a(t) e^{\sqrt{A}\left(t-t_{0}\right)}-\frac{1}{2} y\left(t_{0}\right) A e^{\sqrt{A}\left(t-t_{0}\right)} \\
= & a(t) u(t)+\frac{1}{2} y\left(t_{0}\right)(a(t)-A) e^{\sqrt{A}\left(t-t_{0}\right)} \\
\geq & a(t) u(t) .
\end{aligned}
$$

Let $w(t)=u\left(t_{0} t\right)$, just horizontally re-scaling $u$ to the domain $[0,1]$ so that $w(0)>0, w(1)<0, \lim _{t \rightarrow 0^{+}} w^{\prime}(t)>0$, and $w^{\prime \prime}(t) \geq t_{0}^{2} a\left(t_{0} t\right) w(t)$, so Case 1 applies to $w$, contradicting $w(1)<0$. The conclusion is that $y(t) \geq 0$ on $[0,1)$, and on $[0,1]$ by continuity.

To establish the inequality $y^{\prime} \geq 0$, for any $t_{1}$ with $0<t_{1}<1$ and any $\epsilon>0$, from $\lim _{t \rightarrow 0^{+}} y^{\prime}(t) \geq 0$, there is some $t_{2}$ with $0<t_{2}<t_{1}$ and $y^{\prime}\left(t_{2}\right)>$ $-\epsilon$. By the Mean Value Theorem, there is some $t_{3}$ with $t_{2}<t_{3}<t_{1}$ and $\frac{y^{\prime}\left(t_{1}\right)-y^{\prime}\left(t_{2}\right)}{t_{1}-t_{2}}=y^{\prime \prime}\left(t_{3}\right) \geq a\left(t_{3}\right) y\left(t_{3}\right) \geq 0$. It follows that $y^{\prime}\left(t_{1}\right) \geq y^{\prime}\left(t_{2}\right)>-\epsilon$.

Here's a higher order generalization, using only the Mean Value Theorem, not the maximum value.

Lemma 2.3. Let $k \geq 2$ be an integer. Suppose $a(t)$ is a real function on $[0,1)$ such that $a(t) \geq 0$ and $a$ is bounded on every subinterval $[0, x] \subseteq[0,1)$. If $y$ is continuous on $[0,1]$ with $y(0) \geq 0$, and $\lim _{t \rightarrow 0^{+}} y^{(j)} \geq 0$ for $j=1, \ldots, k-1$, and $y^{(k)}(t) \geq a(t) y(t)$ for $0<t<1$, then $y(t) \geq 0$ for all $0 \leq t \leq 1$ and $y^{(j)}(t) \geq 0$ for all $0<t<1, j=1, \ldots, k$.

Proof. Case 1: $y(0)>0$ and $\lim _{t \rightarrow 0^{+}} y^{(j)}(t)>0$. In this case, we can show that $y>0$ on $[0,1]$ and $y^{(j)}>0$ on $(0,1)$ for $j=1, \ldots, k-1$.

By Lemma 2.1 applied to $y$ on $[0,1]$, either $y>0$ on $[0,1]$, or there is some $t_{1} \in(0,1]$ so that $y(t)>0$ for all $t \in\left[0, t_{1}\right)$ and $y\left(t_{1}\right)=0$.

In the latter case, $y\left(t_{1}\right)=0<y(0)$, so by the Mean Value Theorem for $y$ on $\left[0, t_{1}\right]$, there is some $t_{2}$ with $0<t_{2}<t_{1}$ and $y^{\prime}\left(t_{2}\right)<0$. Then, the MVT applies to $y^{\prime}$ on $\left[t_{3}, t_{2}\right]$ for some $t_{3}>0$ where $y^{\prime}\left(t_{3}\right)>0$, using $\lim _{t \rightarrow 0^{+}} y^{\prime}(t)>0$, so there is some $t_{4}>0$ where $y^{\prime \prime}\left(t_{4}\right)=\frac{y^{\prime}\left(t_{2}\right)-y^{\prime}\left(t_{3}\right)}{t_{2}-t_{3}}<0$. Repeatedly applying this MVT argument to $y^{(j)}$ until $j=k$, gives some $t_{N}$ with $0<t_{N}<t_{1}$, $y^{(k)}\left(t_{N}\right)<0$, contradicting $y^{(k)}\left(t_{N}\right) \geq a\left(t_{N}\right) y\left(t_{N}\right) \geq 0$.

So, the only case not leading to a contradiction is that $y(t)>0$ on $[0,1]$.
The above MVT argument also shows that all $y^{(j)}$ are positive on $(0,1)$ for $j=1, \ldots, k-1$, since any point $t_{n}$ with $0<t_{n}<t_{1}=1$ and $y^{(j)}\left(t_{n}\right) \leq 0$ leads to another point $t_{m}$ with $0<t_{m}<t_{n}$ and $y^{(j+1)}\left(t_{m}\right)<0$, eventually contradicting $y^{(k)}\left(t_{N}\right) \geq a\left(t_{N}\right) y\left(t_{N}\right) \geq 0$.

Case 1 did not use the boundedness of $a$, just $a \geq 0$.
Case 2: $y(0) \geq 0$ and $\lim _{t \rightarrow 0^{+}} y^{(j)} \geq 0, j=1, \ldots, k-1$. Suppose, toward a contradiction, that there is some $t_{0}$ with $0<t_{0}<1$ and $y\left(t_{0}\right)<0$. For $0 \leq t \leq t_{0}$, there is a bound $A>0$ with $0 \leq a(t) \leq A$. For $t \in\left[0, t_{0}\right]$, define $u(t)=y(t)-\frac{1}{2} y\left(t_{0}\right) e^{A^{1 / k}\left(t-t_{0}\right)}$. Then, by construction, $u(0)>0$ and

$$
\begin{aligned}
& u\left(t_{0}\right)<0 . \text { For } 0<t<t_{0}, 1 \leq j \leq k-1, \\
& \qquad \begin{aligned}
u^{(j)}(t)= & y^{(j)}(t)-\frac{1}{2} y\left(t_{0}\right) A^{j / k} e^{A^{1 / k}\left(t-t_{0}\right)} \\
\Longrightarrow \lim _{t \rightarrow 0^{+}} u^{(j)}(t)= & \left(\lim _{t \rightarrow 0^{+}} y^{(j)}(t)\right)-\frac{1}{2} y\left(t_{0}\right) A^{j / k}>0, \\
u^{(k)}(t)= & y^{(k)}(t)-\frac{1}{2} y\left(t_{0}\right) A e^{A^{1 / k}\left(t-t_{0}\right)} \\
\geq & a(t) y(t)-\frac{1}{2} y\left(t_{0}\right) a(t) e^{A^{1 / k}\left(t-t_{0}\right)} \\
& +\frac{1}{2} y\left(t_{0}\right) a(t) e^{A^{1 / k}\left(t-t_{0}\right)}-\frac{1}{2} y\left(t_{0}\right) A e^{A^{1 / k}\left(t-t_{0}\right)} \\
= & a(t) u(t)+\frac{1}{2} y\left(t_{0}\right)(a(t)-A) e^{A^{1 / k}\left(t-t_{0}\right)} \\
\geq & a(t) u(t) .
\end{aligned}
\end{aligned}
$$

Let $w(t)=u\left(t_{0} t\right)$, just horizontally re-scaling $u$ to the domain $[0,1]$ so that $w(0)>0, w(1)<0, \lim _{t \rightarrow 0^{+}} w^{(j)}(t)>0$, and $w^{(k)}(t) \geq t_{0}^{k} a\left(t_{0} t\right) w(t)$, so Case 1 applies to $w$, contradicting $w(1)<0$. The conclusion is that $y(t) \geq 0$ on $[0,1)$, and on $[0,1]$ by continuity.

To establish the inequalities $y^{(j)} \geq 0$, start with $j=k-1$. Then, for any $t_{1}$ with $0<t_{1}<1$ and any $\epsilon>0$, from $\lim _{t \rightarrow 0^{+}} y^{(k-1)}(t) \geq 0$, there is some $t_{2}$ with $0<t_{2}<t_{1}$ and $y^{\prime}\left(t_{2}\right)>-\epsilon$. By the MVT, there is some $t_{3}$ with $t_{2}<t_{3}<t_{1}$ and $\frac{y^{(k-1)}\left(t_{1}\right)-y^{(k-1)}\left(t_{2}\right)}{t_{1}-t_{2}}=y^{(k)}\left(t_{3}\right) \geq a\left(t_{3}\right) y\left(t_{3}\right) \geq 0$. It follows that $y^{(k-1)}\left(t_{1}\right) \geq y^{(k-1)}\left(t_{2}\right)>-\epsilon$. A similar argument applies for $j$ decreasing from $k-1$ to 1 .

Lemma 2.4. If the left-side limit $\lim _{t \rightarrow b^{-}} f(t)=-\infty$, then there is no interval $(b-\delta, b)$ on which $f^{\prime}(t)$ is bounded below.

Proof. (See [C].)

Lemma 2.5. Suppose $a(t)$ is a real function on $[0,1)$ such that $a$ is bounded above on every subinterval $[0, x] \subseteq[0,1)$ and bounded below on every subinterval $\left[x_{1}, x_{2}\right] \subseteq(0,1)$. If $y$ is continuous on $[0,1]$ with $y(0) \geq 0$ and $y^{\prime}(t) \geq a(t) y(t)$ for $0<t<1$, then $y(t) \geq 0$ for all $0 \leq t \leq 1$.

Proof. Case 1: $y(0)>0$.
By Lemma 2.1 applied to $y$ on $[0,1]$, either $y>0$ on $[0,1]$, or there is some $t_{1} \in(0,1]$ so that $y(t)>0$ for all $t \in\left[0, t_{1}\right)$ and $y\left(t_{1}\right)=0$. If $t_{1}=1$, then $y \geq 0$ as claimed.

So, suppose toward a contradiction that $t_{1}<1$. On the interval $\left[\frac{1}{2} t_{1}, t_{1}\right]$, $a$ is bounded below: there is some $K<0$ so that $K \leq a(t)$. Define the function $f(t)=\ln (y(t))$ for $t$ in the interval $\left(\frac{1}{2} t_{1}, t_{1}\right)$. $f$ has left-side limit $\lim _{t \rightarrow t_{1}^{-}} f(t)=-\infty$. For all $t$ in $\left(\frac{1}{2} t_{1}, t_{1}\right)$, the derivative is bounded below: $f^{\prime}(t)=$ $\frac{1}{y(t)} y^{\prime}(t) \geq \frac{1}{y(t)} a(t) y(t)=a(t) \geq K$, but this contradicts Lemma 2.4.

Case 2: $y(0)=0$.
Suppose toward a contradiction that there is some $p \in[0,1]$ with $y(p)<0$. $p \neq 0$ by assumption, and if $p=1$, then by continuity of $y$, there is some nearby point $p-\delta_{1} / 2$ with $y\left(p-\delta_{1} / 2\right)<0$. So by re-labeling if necessary, we can assume $0<p<1$. On the interval $[0, p], a$ is bounded above: there is some $A>0$ so that $a(t) \leq A$.

Define $g(t)=-y(p-p t)$ on the domain $0 \leq t \leq 1$, so that $g(0)=-y(p)>$ $0, g(1)=-y(0)=0$, and $g$ is continuous on $[0,1]$. The derivative satisfies

$$
g^{\prime}(t)=p y^{\prime}(p-p t) \geq p a(p-p t) y(p-p t)=-p a(p-p t) g(t)
$$

and the coefficient $-p a(p-p t)$ is bounded below by $-p A$. Case 1 applies to $g$, so $g(t)>0$ on $[0,1)$ and $g(1)=0$. Define the function $f(t)=\ln (g(t))$ for $t$ in the interval $(0,1) . f$ has left-side limit $\lim _{t \rightarrow 1^{-}} f(t)=-\infty$. For all $t$ in $(0,1)$, the derivative is bounded below: $f^{\prime}(t)=\frac{1}{g(t)} g^{\prime}(t) \geq \frac{1}{g(t)}(-p a(p-p t)) g(t)=$ $-p a(p-p t) \geq-p A$, but this contradicts Lemma 2.4.

Lemma 2.6. Suppose $a(t)$ is a bounded real function on $[0, X]$. Then there is some $\delta$ with $0<\delta \leq X$, with the property that if $y$ is continuous on $[0, X]$ with $y(0) \geq 0, \lim _{t \rightarrow 0^{+}} y^{\prime}(t) \geq 0$, and $y^{\prime \prime}(t) \geq a(t) y(t)$ for $0<t<X$, then $y(t) \geq 0$ for all $0 \leq t \leq \delta$.

Proof. Step 1. Pick any $x$ in ( $0, X$ ], so that by hypothesis, there are some $A>$ 0 and $K<0$ so that $K \leq a(t) \leq A$ for $t \in[0, x]$. Let $\delta=\min \left\{x, \frac{1}{\sqrt{A}}, \frac{1}{\sqrt{-K}}\right\}>$ 0 . (Remark: depending on $a$, it may be possible to choose $x$ that optimizes $\delta$.) To show that this is a $\delta$ as claimed by the Lemma, suppose toward a contradiction that there is some $c$ with $0 \leq c \leq \delta$ and $y(c)<0 . c>0$ by hypothesis.

Step 2. Let $y(b)$ be the minimum value of $y$ on $[0, c]$, so $y(b) \leq y(c)<0$ and $0<b \leq c \leq \delta$. The MVT applies to $y$ on $[0, b]$ : there is some $t_{0}$ with $0<t_{0}<b$ and $y^{\prime}\left(t_{0}\right)=\frac{y(b)-y(0)}{b-0}$. The MVT applies to $y^{\prime}$ (extended to $\left.y^{\prime}(0)=\lim _{t \rightarrow 0^{+}} y(t) \geq 0\right)$ on $\left[0, t_{0}\right]$ : there is some $t_{1}$ with $0<t_{1}<t_{0}$ and

$$
y^{\prime \prime}\left(t_{1}\right)=\frac{y^{\prime}\left(t_{0}\right)-y^{\prime}(0)}{t_{0}-0}=\frac{\frac{y(b)-y(0)}{b}-y^{\prime}(0)}{t_{0}}
$$

By hypothesis,

$$
a\left(t_{1}\right) y\left(t_{1}\right) \leq y^{\prime \prime}\left(t_{1}\right)=\frac{y(b)-y(0)-b y^{\prime}(0)}{b t_{0}} \leq \frac{y(b)}{b t_{0}}<0 .
$$

If $a\left(t_{1}\right)>0$ and $y\left(t_{1}\right)<0$, then $y\left(t_{1}\right) \leq \frac{y(b)}{b t_{0} a\left(t_{1}\right)}<\frac{y(b)}{\delta^{2} A} \leq y(b)$, contradicting the minimum property of $y(b)$. So, $a\left(t_{1}\right)<0$ and $y\left(t_{1}\right)>0$.

Step 3. Lemma 2.1 applies to $y$ on the interval $\left[t_{1}, b\right]$, so there is some $t_{3}$ with $t_{1}<t_{3}<b, y\left(t_{3}\right)=0$, and $y(t)>0$ for all $t \in\left[t_{1}, t_{3}\right)$. A left-side version of Lemma 2.1 applies to $y$ on the interval $\left[0, t_{1}\right]$; there are two cases:

Case 1. There is some $t_{2}$ with $0 \leq t_{2}<t_{1}, y\left(t_{2}\right)=0$, and $y(t)>0$ for all $t \in\left(t_{2}, t_{1}\right]$.

Case 2. $y(t)>0$ for all $t \in\left[0, t_{1}\right]$. In this case denote $t_{2}=0$.
In either case, there is some interval $\left[t_{2}, t_{3}\right]$ where $0 \leq t_{2}<t_{1}<t_{3}<b$, $y\left(t_{3}\right)=0, y\left(t_{2}\right) \geq 0$, and $y(t)>0$ for all $t \in\left(t_{2}, t_{3}\right)$. Let $y\left(t_{4}\right)$ be the maximum value of $y$ on $\left[t_{2}, t_{3}\right]$. In Case $1, t_{2}<t_{4}<t_{3}$, so the maximum occurs at an interior point and $y^{\prime}\left(t_{4}\right)=0$. In Case 2, $t_{4}$ is either an interior point of $\left[t_{2}, t_{3}\right]$, or the maximum occurs at the endpoint $t_{4}=t_{2}=0$, where there is a right-side derivative $y^{\prime}(0) \geq 0$ as in Step 2 . In either case, $y^{\prime}\left(t_{4}\right) \geq 0$.

Step 4. The MVT applies to $y$ on $\left[t_{4}, t_{3}\right]$ : there is some $t_{5}$ with $t_{4}<t_{5}<t_{3}$ and $y^{\prime}\left(t_{5}\right)=\frac{y\left(t_{3}\right)-y\left(t_{4}\right)}{t_{3}-t_{4}}$. The MVT applies to $y^{\prime}$ on $\left[t_{4}, t_{5}\right]$ : there is some $t_{6}$ with $t_{4}<t_{6}<t_{5}$ and

$$
y^{\prime \prime}\left(t_{6}\right)=\frac{y^{\prime}\left(t_{5}\right)-y^{\prime}\left(t_{4}\right)}{t_{5}-t_{4}}=\frac{\frac{y\left(t_{3}\right)-y\left(t_{4}\right)}{t_{3}-t_{4}}-y^{\prime}\left(t_{4}\right)}{t_{5}-t_{4}}<\frac{-y\left(t_{4}\right)}{b^{2}} .
$$

Using the lower bound for $a$ and the property $y\left(t_{6}\right)>0$,

$$
\begin{aligned}
K y\left(t_{6}\right) & \leq a\left(t_{6}\right) y\left(t_{6}\right) \leq y^{\prime \prime}\left(t_{6}\right)<\frac{-y\left(t_{4}\right)}{b^{2}} \\
\Longrightarrow y\left(t_{6}\right) & >\frac{-y\left(t_{4}\right)}{b^{2} K} \geq \frac{-y\left(t_{4}\right)}{\delta^{2} K} \geq y\left(t_{4}\right)
\end{aligned}
$$

contradicting the maximum property of $y\left(t_{4}\right)$.
Theorem 2.7. Suppose $a(t)$ and $b(t)$ are real functions on $[0,1)$, and there is a point $X$ such that $0<X<1$ and $a, b$, and $b^{\prime}$ are bounded on $(0, X]$. Then there is some $\delta$ with $0<\delta \leq X$, with the property that if $y$ is continuous on $[0,1)$ with $y(0) \geq 0$, $\lim _{t \rightarrow 0^{+}} y^{\prime}(t) \geq 0$, and $y^{\prime \prime}(t) \geq a(t) y(t)+b(t) y^{\prime}(t)$ for $0<t<X$, then $y(t) \geq 0$ for all $0 \leq t \leq \delta$.

Proof. By hypothesis, there are some $A>0$ and $K<0$ so that $K \leq a(t) \leq A$ for $t \in[0, X]$, and there are some $B>0$ and $L<0$ so that $L \leq b(t) \leq B$ for $t \in[0, X]$. Let $x=\min \left\{X, \frac{1}{\sqrt{2 A}}, \frac{1}{4 B}\right\}>0$.

Recall the elementary calculus fact that if $b$ is continuous and bounded on ( $0, x$ ), then $b$ is (Riemann) integrable on $[0, x]$. Let $p(t)=\int_{0}^{t}-\frac{1}{2} b(x) d x$, so $p$ is continuous on $[0, x]$ and for $0<t<x, p^{\prime}(t)=-\frac{1}{2} b(t)$.

Let

$$
f(t)=e^{p(t)}\left[y(t)-K y(0) t^{2}-y(0)\right],
$$

so $f$ is continuous on $[0, x]$ and $f(0)=0$. For $0<t<x$,

$$
\begin{aligned}
f^{\prime}(t)= & e^{p(t)}\left[y^{\prime}(t)-2 K y(0) t\right] \\
& +e^{p(t)}\left(-\frac{1}{2} b(t)\right)\left[y(t)-K y(0) t^{2}-y(0)\right]
\end{aligned}
$$

and using $\lim _{t \rightarrow 0^{+}}(y(t)-y(0))=0$ and the boundedness of $b$, the limit exists:

$$
\lim _{t \rightarrow 0^{+}} f^{\prime}(t)=\lim _{t \rightarrow 0^{+}} y^{\prime}(t) \geq 0
$$

For $0<t<x$,

$$
\begin{align*}
f^{\prime \prime}(t)= & e^{p(t)}\left[y^{\prime \prime}(t)-2 K y(0)\right]+e^{p(t)}\left(-\frac{1}{2} b(t)\right)\left[y^{\prime}(t)-2 K y(0) t\right] \\
& +e^{p(t)}\left(-\frac{1}{2} b(t)\right)^{2}\left[y(t)-K y(0) t^{2}-y(0)\right] \\
& +e^{p(t)}\left(-\frac{1}{2} b^{\prime}(t)\right)\left[y(t)-K y(0) t^{2}-y(0)\right] \\
& +e^{p(t)}\left(-\frac{1}{2} b(t)\right)\left[y^{\prime}(t)-2 K y(0) t\right] \\
\geq & e^{p(t)}\left[a(t) y(t)+b(t) y^{\prime}(t)-2 K y(0)\right]-e^{p(t)} b(t)\left[y^{\prime}(t)-2 K y(0) t\right] \\
& +e^{p(t)}\left(-\frac{1}{2} b(t)\right)^{2}\left[y(t)-K y(0) t^{2}-y(0)\right] \\
& +e^{p(t)}\left(-\frac{1}{2} b^{\prime}(t)\right)\left[y(t)-K y(0) t^{2}-y(0)\right] \\
= & e^{p(t)}\left[y(t)-K y(0) t^{2}-y(0)\right]\left(a(t)+\frac{1}{4}(b(t))^{2}-\frac{1}{2} b^{\prime}(t)\right)  \tag{4}\\
& +e^{p(t)} y(0)\left(K\left(a(t) t^{2}+2 b(t) t-2\right)+a(t)\right) . \tag{5}
\end{align*}
$$

In the last step, the $y^{\prime}$ terms cancel by construction. Term (4) is equal to $\tilde{a}(t) f(t)$, where $\tilde{a}(t)=\left(a(t)+\frac{1}{4}(b(t))^{2}-\frac{1}{2} b^{\prime}(t)\right)$ is bounded by hypothesis. The upper bounds $a(t) \leq A$ and $b(t) \leq B$ and the initial choice of $x$ imply, for $0<t<x$,

$$
\begin{aligned}
a(t) t^{2}+2 b(t) t-2 & \leq A t^{2}+2 B t-2 \\
& \leq A\left(\frac{1}{2 A}\right)+2 B\left(\frac{1}{4 B}\right)-2=-1 \\
\Longrightarrow K\left(a(t) t^{2}+2 b(t) t-2\right)+a(t) & \geq-K+a(t) \geq 0,
\end{aligned}
$$

so the entire term (5) is non-negative, and for $0<t<x, f^{\prime \prime}(t) \geq \tilde{a}(t) f(t)$. Lemma 2.6 applies to $f$, so there is some $\delta_{1}$ depending on $a, b, b^{\prime}, X$, but not on $y$, with $f \geq 0$ on $\left[0, \delta_{1}\right]$. The factor $\left[y(t)-K y(0) t^{2}-y(0)\right]$ is non-negative on the same interval, where

$$
y(t)-K y(0) t^{2}-y(0) \geq 0 \Longrightarrow y(t) \geq y(0)\left(1+K t^{2}\right)
$$

so $y(t) \geq 0$ for $0 \leq t \leq \delta=\min \left\{\delta_{1}, \frac{1}{\sqrt{-K}}\right\}$.

### 2.2 A nonlinear differential inequality

Theorem 2.8. Given a function $f$ that satisfies $f^{\prime \prime} f-\left(f^{\prime}\right)^{2} \geq 0$ on $(a, b)$, at every critical point $c$ with $f(c) \neq 0$, there is either a positive local min. or a negative local max.

Proof. Suppose $c$ is a critical point, meaning $f^{\prime}(c)=0$. Suppose also that $f(c) \neq 0$, so that the function $g(x)=f^{\prime}(x) / f(x)$ is defined on a neighborhood $N=(c-\delta, c+\delta) \subseteq(a, b)$. By the quotient rule,

$$
g^{\prime}(x)=\frac{f(x) f^{\prime \prime}(x)-\left(f^{\prime}(x)\right)^{2}}{(f(x))^{2}}
$$

which is $\geq 0$ on $N$ by hypothesis. It follows that $g(x)$ is weakly increasing on $N$. For $c<x<c+\delta, f^{\prime}(x) / f(x)=g(x) \geq g(c)=0$, and for $c-\delta<x<c$, $f^{\prime}(x) / f(x)=g(x) \leq g(c)=0$.

If $f(c)>0$, then $f(x)>0$ on $N$ so $f^{\prime}(x) \geq 0$ on the right and $f^{\prime}(x) \leq 0$ on the left. $f(c)$ is a local min. by the first derivative test.

If $f(c)<0$, then $f(x)<0$ on $N$ so $f^{\prime}(x) \leq 0$ on the right and $f^{\prime}(x) \geq 0$ on the left. $f(c)$ is a local max.

Note that $\mathcal{C}^{2}$ is not used, just the existence of $f^{\prime \prime}$. Constant functions trivially satisfy both the hypotheses and conclusions.

Lemma 2.9. If $p(x)$ satisfies $p^{\prime \prime}(x) \geq 0$ on $(a, b)$ then for any $c \in(a, b), p$ satisfies $p(x) \geq p(c)+p^{\prime}(c)(x-c)$ for all $x \in(a, b)$.

Proof. (See [C].)
Theorem 2.10. Given a function $f$ that satisfies $f^{\prime \prime} f-\left(f^{\prime}\right)^{2} \geq 0$ on $(a, b)$, if there is a point $c$ in $(a, b)$ with $f(c)>0$, then $f$ satisfies

$$
f(x) \geq f(c) \cdot \exp \left(\frac{f^{\prime}(c)}{f(c)}(x-c)\right)
$$

for all $x \in(a, b)$.
Proof. By continuity, there is some neighborhood $(s, t) \subseteq(a, b)$ so that $s<$ $c<t$ and $f(x)>0$ on $(s, t)$. Suppose $f(z)=0$ for some $z \in(c, b)$. Then, the set $\{t: f(x)>0$ on $(c, t)\}$ is non-empty and has sup $=T \leq z<b$. By construction and using continuity again, $f(T)=0$ and $f(x)>0$ on $(s, T)$.

Consider $h(x)=\ln (f(x))$, which is well-defined on $(s, T) . h^{\prime}=f^{\prime} / f=g$, from Theorem 2.8, so $h^{\prime \prime}=g^{\prime} \geq 0$ on $(s, T)$. By Lemma 2.9, $h(x) \geq h(c)+$ $h^{\prime}(c) \cdot(x-c)$ on $(s, T)$ :

$$
\begin{aligned}
\ln (f(x)) & \geq \ln (f(c))+\frac{f^{\prime}(c)}{f(c)} \cdot(x-c) \\
f(x) & \geq f(c) \cdot \exp \left(\frac{f^{\prime}(c)}{f(c)} \cdot(x-c)\right)
\end{aligned}
$$

for all $x$ in $(s, T)$. This implies $\lim _{x \rightarrow T^{-}} f(x)=f(T)>0$, which contradicts the construction of $T$. We can conclude that $f$ is never zero on $(c, b)$, and always positive there, so the inequality holds on $(s, b)$. The inequality on the other side of $c$ follows from an analogous inf argument.

It follows that if $c$ is a critical point with $f(c)>0$, then $f(c)$ is a global minimum. It further follows that either $f$ is constant or there is at most one point $c$ where $f^{\prime}(c)=0$ and $f(c)>0$.

## References

[C] A. Coffman, Notes on first semester calculus. (unpublished course notes)
http://users.pfw.edu/CoffmanA/


[^0]:    *Supported in part by National Science Foundation DMS-1265330.

