J-holomorphic curves in rough almost complex structures

Adam Coffman

based on joint work with Yifei Pan and Yuan Zhang

Indiana University - Purdue University Fort Wayne

MAA MathFest in Chicago

http://ipfw.edu/math http://google.com/+AdamCoffman

July 2017

Holomorphic functions

$\mathbb{C} = (\mathbb{R}^2, i)$ has coordinates $z = x + iy$, $\bar{z} = x - iy$.

Let f be a continuous function on a connected open set $\Omega \subseteq \mathbb{C}$, $f : \Omega \to \mathbb{C}$, with real/imaginary parts:

$$f(z) = u(x, y) + iv(x, y)$$

Notation for (classical, pointwise) partial derivatives:

$$\frac{\partial f}{\partial z} = f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right), \qquad \frac{\partial f}{\partial \bar{z}} = f_{\bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

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f is <u>holomorphic</u> means it satisfies the Cauchy-Riemann Equations:

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \quad \underset{\textit{lin. alg.}}{\iff} \quad \frac{\partial f}{\partial \bar{z}} \equiv 0.$$

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- SUCP: f has the Strong Unique Continuation Property: If all the derivatives vanish at some point p: for all a, b, c,

$$\left.\frac{\partial^a f}{\partial z^a}\right|_p = 0 \iff \left.\frac{\partial^{b+c} f}{\partial x^b \partial y^c}\right|_p = 0,$$

  then $f \equiv 0$.

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1. small changes to the "coefficients" of the differential equations;
2. increase the target dimension to get vector valued $f : \Omega \to \mathbb{C}^n$

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- Matrix version of generalized C-R equation:

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = J(f(z)) \cdot \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

  for a 2×2 real matrix J with:
  - $J \cdot J = -Id$,
  - entries depending continuously on the coordinates in the target space
  - $J(x, y) \approx J_{std} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

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Nice properties of solutions of $\frac{\partial f}{\partial \bar{z}} = \mu(z) \cdot \overline{\frac{\partial f}{\partial z}}$
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- SUCP $\implies$ WUCP (again assuming derivatives exist).

Holomorphic curves

A holomorphic curve is a map $\Omega \to \mathbb{C}^n$,

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$$\left[d\vec{f}(x,y)\right]_{2n\times 2} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{bmatrix}_{2n\times 2n} \cdot \left[d\vec{f}(x,y)\right]_{2n\times 2}.$$

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Holomorphic curves have the same nice properties as holomorphic functions.

Pseudoholomorphic curves — or J-holomorphic curves

Modify the $2n \times 2n$ coefficient matrix to get an "Almost Complex Structure" $\ldots J_{2n\times 2n}$ with real entries depending continuously on the coordinates in $\mathbb{C}^n$, satisfying $J \cdot J = -Id$.

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A J-holomorphic curve $\vec{f}(z) = [f_1(z), \ldots, f_n(z)]$ is a differentiable map $\Omega \to \mathbb{C}^n = \mathbb{R}^{2n}$ satisfying:

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For J close to J_{std}, some linear algebra $\implies$

$$\begin{bmatrix} \frac{\partial f_1}{\partial \bar{z}} \\ \vdots \\ \frac{\partial f_n}{\partial \bar{z}} \end{bmatrix} = [\mathbf{Q}(f(z))]_{n\times n} \cdot \begin{bmatrix} \overline{\frac{\partial f_1}{\partial z}} \\ \vdots \\ \overline{\frac{\partial f_n}{\partial z}} \end{bmatrix}.$$

for some matrix $\mathbf{Q}$ with complex entries, derived from J with the same "regularity", $\mathbf{Q} = \mathbf{0}$ when $J = J_{std}$.

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Reference: [Ivashkovich-Shevchishin$_{2011}$]

Counterexamples for uniqueness: $\mathcal{C}^{0,\alpha}$ structure

For $0 < \alpha < 1$, an almost complex structure on $\mathbb{C}^2 = \mathbb{R}^4$:

$$J(z_1, z_2) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -2|z_2|^\alpha & 0 & -1 \\ -2|z_2|^\alpha & 0 & 1 & 0 \end{bmatrix}.$$

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$\mathcal{C}^1$ maps $\mathbb{R}^2 \to \mathbb{R}^4$:

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This ODE doesn't have unique solutions for initial conditions $u(0) = u'(0) = 0$:

$$u(x) = \begin{cases} 0 & x \leq c \\ ((2-2\alpha)(x-c))^{1/(1-\alpha)} & x > c \end{cases}$$

Conclude: $\vec{f} \equiv \vec{g}$ on an open set but $\vec{f} \not\equiv \vec{g}$,

so for (non-Lipschitz) J with Hölder continuity $\mathcal{C}^{0,\alpha}$, the Weak Unique Continuation Property need not hold.

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For $0 < \alpha \leq \frac{2}{3}$, this phenomenon was used by [Ivashkovich-Pinchuk-Rosay$_{2005}$] to construct an example of an almost complex manifold where J is $\mathcal{C}^{0,\alpha}$ and the Kobayashi-Royden pseudo-norm is not upper semicontinuous.

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(still open for Lipschitz or $\mathcal{C}^1$ cases)

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Proposition

[Rosay$_{2010}$] *There exist:*

- *a complex* 2×2 *matrix* $\mathbf{Q}(z)$ *with continuous entries and* $\mathbf{Q}(0) = [0]$*,*
- *a non-constant,* $\mathcal{C}^\infty$ *smooth map* $\vec{g} : \mathbb{C} \to \mathbb{C}^2$*, such that:*

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and all derivatives of $\vec{g}$ *vanish at* $z = 0$*.* ∎

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This phenomenon can be used to construct a continuous J on $\mathbb{C}^4$ (with $J - J_{std}$ vanishing to infinite order) and $\mathcal{C}^\infty$ smooth J-holomorphic curves $\vec{f}$ without the SUCP property.

Counterexample for regularity: $\alpha \to 0^+$

Proposition

[C—Pan-Zhang$_{2017}$] *There exists a (real) differentiable function $V : \mathbb{C} \to \mathbb{C}$ such that $\frac{\partial V}{\partial \bar{z}}$ is continuous and $\frac{\partial V}{\partial z}$ is discontinuous.* $\blacksquare$

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Thank you!

Yuan Zhang's research supported by National Science Foundation grant DMS-1265330.

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