An example for Green's Theorem with discontinuous partial derivatives

Adam Coffman

Joint work with Yuan Zhang

Indiana University - Purdue University Fort Wayne

MAA MathFest in Washington, DC

August 2015

Statement of Green's Theorem

Theorem (G. Green, $\approx$ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

Then, for any closed rectangle R contained in Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Proof of Green's Theorem

Proof.

For $R = [a, b] \times [c, d]$,

$$\begin{aligned} & \int_{x=a}^{x=b} P(x, c)dx + \int_{y=c}^{y=d} Q(b, y)dy \\ & - \int_{x=a}^{x=b} P(x, d)dx - \int_{y=c}^{y=d} Q(a, y)dy \\ = \ & \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \frac{\partial Q}{\partial x} dx \right) dy - \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \frac{\partial P}{\partial y} dy \right) dx. \end{aligned}$$

∎

Proof of Green's Theorem

Proof.

For $R = [a, b] \times [c, d]$,

$$\int_{x=a}^{x=b} P(x,c)dx + \int_{y=c}^{y=d} Q(b,y)dy$$
$$- \int_{x=a}^{x=b} P(x,d)dx - \int_{y=c}^{y=d} Q(a,y)dy$$
$$= \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \frac{\partial Q}{\partial x} dx \right) dy - \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \frac{\partial P}{\partial y} dy \right) dx.$$

∎

Remark

Proof of Green's Theorem

Proof.

For $R = [a, b] \times [c, d]$,

$$\int_{x=a}^{x=b} P(x,c)dx + \int_{y=c}^{y=d} Q(b,y)dy$$
$$- \int_{x=a}^{x=b} P(x,d)dx - \int_{y=c}^{y=d} Q(a,y)dy$$
$$= \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \frac{\partial Q}{\partial x} dx \right) dy - \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \frac{\partial P}{\partial y} dy \right) dx.$$

∎

Remark

Nope.

Proof of Green's Theorem

Proof.

For $R = [a, b] \times [c, d]$,

$$\begin{aligned}
&\int_{x=a}^{x=b} P(x, c)dx + \int_{y=c}^{y=d} Q(b, y)dy \\
&- \int_{x=a}^{x=b} P(x, d)dx - \int_{y=c}^{y=d} Q(a, y)dy \\
= \; &\int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \frac{\partial Q}{\partial x} dx \right) dy - \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \frac{\partial P}{\partial y} dy \right) dx.
\end{aligned}$$

∎

Remark

Nope. Something's wrong.

Proof of Green's Theorem

Proof.

For $R = [a, b] \times [c, d]$,

$$\int_{x=a}^{x=b} P(x, c)dx + \int_{y=c}^{y=d} Q(b, y)dy$$

$$- \int_{x=a}^{x=b} P(x, d)dx - \int_{y=c}^{y=d} Q(a, y)dy$$

$$= \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \frac{\partial Q}{\partial x} dx \right) dy - \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \frac{\partial P}{\partial y} dy \right) dx.$$

∎

Remark

Nope. Something's wrong. Let's go back to the statement of the Theorem.

Statement of Green's Theorem

Theorem (G. Green, $\approx$ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

Then, for any closed rectangle R contained in Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark

False!

Statement of Green's Theorem

Theorem (G. Green, $\approx$ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x, y) = (P(x, y), Q(x, y))$.

Then, for any closed rectangle R contained in Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark

False! What's missing?

Statement of Green's Theorem

Theorem (G. Green, $\approx$ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
If $\vec{F} \in \mathcal{C}^1(\Omega)$ (meaning that the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ are continuous on Ω),
then, for any closed rectangle R contained in Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark

Statement of Green's Theorem

Theorem (G. Green, ≈ 1828)

On an open subset $\Omega \subseteq \mathbb{R}^2$, *let* $\vec{F} : \Omega \to \mathbb{R}^2$ *be a vector field with components* $\vec{F}(x, y) = (P(x, y), Q(x, y))$.
If $\vec{F} \in \mathcal{C}^1(\Omega)$ *(meaning that the partial derivatives* $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ *are continuous on* Ω*),*
then, for any closed rectangle R *contained in* Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Remark

This is the usual calculus textbook statement of Green's Theorem.

Statement of Green's Theorem

Theorem (G. Green, ≈ 1828; P. Cohen, U. Chicago Thesis, ≈ 1958)

On an open subset $\Omega \subseteq \mathbb{R}^2$, let $\vec{F} : \Omega \to \mathbb{R}^2$ be a vector field with components $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
If P and Q are continuous on Ω, and the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ all **exist** *at every point in Ω except for countably many, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \in L^1_{loc}(\Omega)$,*
then, for any closed rectangle R contained in Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Statement of Green's Theorem

Theorem (G. Green, ≈ 1828; P. Cohen, U. Chicago Thesis, ≈ 1958)

On an open subset $\Omega \subseteq \mathbb{R}^2$, *let* $\vec{F} : \Omega \to \mathbb{R}^2$ *be a vector field with components* $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
If P *and* Q *are continuous on* Ω, *and the partial derivatives* $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ *all* **exist** *at every point in* Ω *except for countably many, and* $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \in L^1_{loc}(\Omega)$,
then, for any closed rectangle R *contained in* Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Proof.

Difficult — comparable to the Looman-Menchoff Theorem in complex analysis. ∎

Statement of Green's Theorem

Theorem (Green's Theorem, in Bruna and Cufí, *Complex Analysis*)

On an open subset $\Omega \subseteq \mathbb{R}^2$, *let* $\vec{F} : \Omega \to \mathbb{R}^2$ *be a vector field with components* $\vec{F}(x, y) = (P(x, y), Q(x, y))$.
If P *and* Q *are differentiable on* Ω, *and* $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ *is continuous on* Ω, *then, for any closed rectangle* R *contained in* Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Statement of Green's Theorem

Theorem (Green's Theorem, in Bruna and Cufí, *Complex Analysis*)

On an open subset $\Omega \subseteq \mathbb{R}^2$, *let* $\vec{F} : \Omega \to \mathbb{R}^2$ *be a vector field with components* $\vec{F}(x, y) = (P(x, y), Q(x, y))$.
If P *and* Q *are differentiable on* Ω, *and* $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ *is continuous on* Ω, *then, for any closed rectangle* R *contained in* Ω,

$$\int_{\partial R} \vec{F} = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

Proof.

Not as difficult — comparable to the Cauchy-Goursat Theorem in complex analysis.
<u>Differentiable</u> means: can be locally approximated by a linear function.
$\implies$ all partial derivatives exist, but not conversely. ∎

How about an example?

Is there some $\vec{F} = (P, Q)$ so that P and Q are differentiable, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous, but at least one of the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ is discontinuous?

How about an example?

Is there some $\vec{F} = (P, Q)$ so that P and Q are differentiable, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous, but at least one of the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ is discontinuous?

So the Bruna-Cufí version of Green's Theorem applies, but the usual calculus textbook version does not.

How about an example?

Is there some $\vec{F} = (P, Q)$ so that P and Q are differentiable, and $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous, but at least one of the partial derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ is discontinuous?

So the Bruna-Cufí version of Green's Theorem applies, but the usual calculus textbook version does not.

1-dimensional analogue: $f(x) = x^2 \sin(1/x^2)$ is differentiable with $f(0) = f'(0) = 0$, but $\lim_{x \to 0} f'(x)$ does not exist: f' is not locally bounded.

Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
$\vec{F}(0,0) = (0,0)$, so the following example is continuous:

Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
$\vec{F}(0,0) = (0,0)$, so the following example is continuous:

$$\begin{aligned}
P &= y\sqrt{-\ln(x^2+y^2)} \\
Q &= x\sqrt{-\ln(x^2+y^2)} \\
\frac{\partial P}{\partial y} &= \sqrt{-\ln(x^2+y^2)} + \frac{y^2}{x^2+y^2}\cdot\frac{-1}{\sqrt{-\ln(x^2+y^2)}} \\
\frac{\partial Q}{\partial x} &= \sqrt{-\ln(x^2+y^2)} + \frac{x^2}{x^2+y^2}\cdot\frac{-1}{\sqrt{-\ln(x^2+y^2)}}
\end{aligned}$$

Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
$\vec{F}(0,0) = (0,0)$, so the following example is continuous:

$$\begin{aligned}
P &= y\sqrt{-\ln(x^2+y^2)} \\
Q &= x\sqrt{-\ln(x^2+y^2)} \\
\frac{\partial P}{\partial y} &= \sqrt{-\ln(x^2+y^2)} + \frac{y^2}{x^2+y^2}\cdot\frac{-1}{\sqrt{-\ln(x^2+y^2)}} \\
\frac{\partial Q}{\partial x} &= \sqrt{-\ln(x^2+y^2)} + \frac{x^2}{x^2+y^2}\cdot\frac{-1}{\sqrt{-\ln(x^2+y^2)}}
\end{aligned}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y^2-x^2}{x^2+y^2}\cdot\frac{1}{\sqrt{-\ln(x^2+y^2)}}$$

Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
$\vec{F}(0,0) = (0,0)$, so the following example is continuous:

$$\begin{aligned}
P &= y\sqrt{-\ln(x^2+y^2)} \\
Q &= x\sqrt{-\ln(x^2+y^2)} \\
\frac{\partial P}{\partial y} &= \sqrt{-\ln(x^2+y^2)} + \frac{y^2}{x^2+y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2+y^2)}} \\
\frac{\partial Q}{\partial x} &= \sqrt{-\ln(x^2+y^2)} + \frac{x^2}{x^2+y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2+y^2)}}
\end{aligned}$$

Remark

Close!

Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
$\vec{F}(0,0) = (0,0)$, so the following example is continuous:

$$\begin{aligned} P &= y\sqrt{-\ln(x^2+y^2)} \\ Q &= x\sqrt{-\ln(x^2+y^2)} \\ \frac{\partial P}{\partial y} &= \sqrt{-\ln(x^2+y^2)} + \frac{y^2}{x^2+y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2+y^2)}} \\ \frac{\partial Q}{\partial x} &= \sqrt{-\ln(x^2+y^2)} + \frac{x^2}{x^2+y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2+y^2)}} \end{aligned}$$

Remark

Close! Cohen's version of Green's Theorem applies. But $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ DNE at $(0,0)$.

Example 1

Let $\Omega =$ unit disk in $\mathbb{R}^2$. $\vec{F}(x,y) = (P(x,y), Q(x,y))$.
$\vec{F}(0,0) = (0,0)$, so the following example is continuous:

$$\begin{aligned}
P &= y\sqrt{-\ln(x^2+y^2)} \\
Q &= x\sqrt{-\ln(x^2+y^2)} \\
\frac{\partial P}{\partial y} &= \sqrt{-\ln(x^2+y^2)} + \frac{y^2}{x^2+y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2+y^2)}} \\
\frac{\partial Q}{\partial x} &= \sqrt{-\ln(x^2+y^2)} + \frac{x^2}{x^2+y^2} \cdot \frac{-1}{\sqrt{-\ln(x^2+y^2)}}
\end{aligned}$$

Remark

Close! Cohen's version of Green's Theorem applies. But $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ DNE at $(0,0)$. $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ has a removable discontinuity.

Plotting $\vec{F}$ as a vector field:

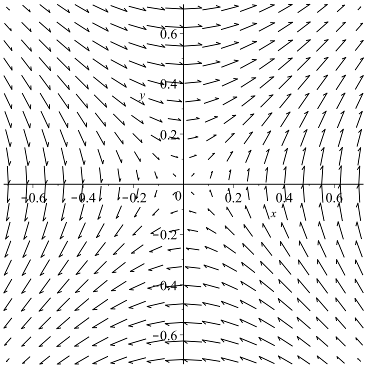

Figure: `fieldplot([P, Q], x = -.7 ..  .7, y = -.7 ..  .7);`

Plotting the magnitude $\|\vec{F}\|$ as a scalar:

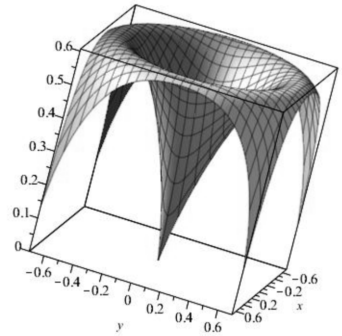

Figure: `plot3d(sqrt(P^2+Q^2), x = -.7 .. .7, y = -.7 .. .7);`

Plotting the partial derivatives as scalars:

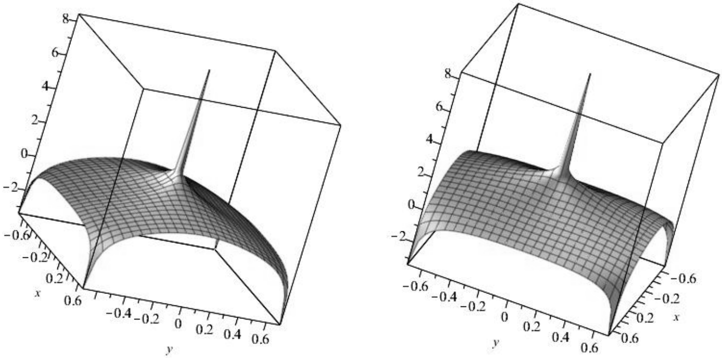

Figure: `plot3d(diff(P,y), x = -.7 ..  .7, y = -.7 ..  .7);`
`plot3d(diff(Q,x), x = -.7 ..  .7, y = -.7 ..  .7);`

Graphics by Maple 18

Plotting $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ as a scalar:

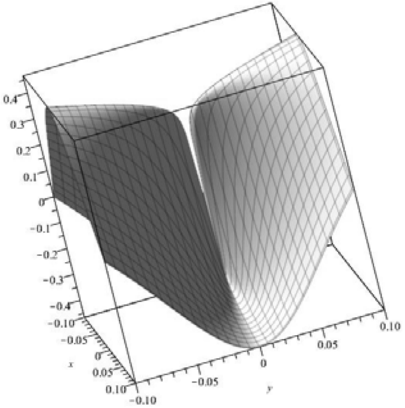

Figure: `plot3d(diff(Q,x)-(diff(P,y)),x=-.1 .. .1,y=-.1 .. .1);`

Example 2

We want to modify Example 1 to get differentiable P and Q, so $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ exist everywhere, but are discontinuous, while $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous.

Example 2

We want to modify Example 1 to get differentiable P and Q, so $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ exist everywhere, but are discontinuous, while $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous.

STEP 1: Introduce a parameter $0 < t < \frac{1}{2}$:
$\vec{F}_t(x, y) = (P_t(x, y), Q_t(x, y))$.
$\vec{F}_t(0, 0) = (0, 0)$,

Example 2

We want to modify Example 1 to get differentiable P and Q, so $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ exist everywhere, but are discontinuous, while $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is continuous.

STEP 1: Introduce a parameter $0 < t < \frac{1}{2}$:
$\vec{F}_t(x, y) = (P_t(x, y), Q_t(x, y))$.
$\vec{F}_t(0, 0) = (0, 0)$,

$$\begin{aligned} P_t &= y\sqrt{-\ln(x^2+y^2)} \cdot (x^2+y^2)^t \\ Q_t &= x\sqrt{-\ln(x^2+y^2)} \cdot (x^2+y^2)^t \end{aligned}$$

so Example 1 is the $t \to 0^+$ limit.

Example 2

P_t and Q_t are $\mathcal{C}^1$ ($\implies$ differentiable) with $\frac{\partial P_t}{\partial y}\Big]_{(0,0)} = \frac{\partial Q_t}{\partial x}\Big]_{(0,0)} = 0$, and:

$$\begin{aligned}
P_t &= y\sqrt{-\ln(x^2+y^2)}\cdot(x^2+y^2)^t \\
Q_t &= x\sqrt{-\ln(x^2+y^2)}\cdot(x^2+y^2)^t \\
\frac{\partial P_t}{\partial y} &= \sqrt{-\ln(x^2+y^2)}(x^2+y^2)^t - \frac{y^2(1+2t\ln(x^2+y^2))}{(x^2+y^2)^{1-t}\cdot\sqrt{-\ln(x^2+y^2)}} \\
\frac{\partial Q_t}{\partial x} &= \sqrt{-\ln(x^2+y^2)}(x^2+y^2)^t - \frac{x^2(1+2t\ln(x^2+y^2))}{(x^2+y^2)^{1-t}\sqrt{-\ln(x^2+y^2)}}
\end{aligned}$$

Example 2

P_t and Q_t are $\mathcal{C}^1$ ($\implies$ differentiable) with $\frac{\partial P_t}{\partial y}\Big]_{(0,0)} = \frac{\partial Q_t}{\partial x}\Big]_{(0,0)} = 0$, and:

$$\begin{aligned}
P_t &= y\sqrt{-\ln(x^2+y^2)}\cdot(x^2+y^2)^t \\
Q_t &= x\sqrt{-\ln(x^2+y^2)}\cdot(x^2+y^2)^t \\
\frac{\partial P_t}{\partial y} &= \sqrt{-\ln(x^2+y^2)}(x^2+y^2)^t - \frac{y^2(1+2t\ln(x^2+y^2))}{(x^2+y^2)^{1-t}\cdot\sqrt{-\ln(x^2+y^2)}} \\
\frac{\partial Q_t}{\partial x} &= \sqrt{-\ln(x^2+y^2)}(x^2+y^2)^t - \frac{x^2(1+2t\ln(x^2+y^2))}{(x^2+y^2)^{1-t}\sqrt{-\ln(x^2+y^2)}}
\end{aligned}$$

Let $r^2 = x^2 + y^2$.

Example 2

P_t and Q_t are $\mathcal{C}^1$ ($\implies$ differentiable) with $\frac{\partial P_t}{\partial y}\Big]_{(0,0)} = \frac{\partial Q_t}{\partial x}\Big]_{(0,0)} = 0$, and:

$$\begin{aligned}
P_t &= y\sqrt{-\ln(x^2+y^2)}\cdot(x^2+y^2)^t \\
Q_t &= x\sqrt{-\ln(x^2+y^2)}\cdot(x^2+y^2)^t \\
\frac{\partial P_t}{\partial y} &= \sqrt{-\ln(x^2+y^2)}(x^2+y^2)^t - \frac{y^2(1+2t\ln(x^2+y^2))}{(x^2+y^2)^{1-t}\cdot\sqrt{-\ln(x^2+y^2)}} \\
\frac{\partial Q_t}{\partial x} &= \sqrt{-\ln(x^2+y^2)}(x^2+y^2)^t - \frac{x^2(1+2t\ln(x^2+y^2))}{(x^2+y^2)^{1-t}\sqrt{-\ln(x^2+y^2)}}
\end{aligned}$$

Let $r^2 = x^2+y^2$. Max. Value of $f(r) = \sqrt{-\ln(r^2)}r^{2t}$ is $f(e^{\frac{-1}{4t}}) = \frac{1}{\sqrt{2et}}$.

Example 2

STEP 2: Multiply by a smooth "cutoff." Define $\kappa : (0, \infty) \to [0, 1]$ to be a weakly decreasing, $\mathcal{C}^\infty$ function:

$$\kappa(r) = \begin{cases} 1 & \text{for } 0 < r < 0.5 \\ 0 & \text{for } r > 0.6 \end{cases}$$

Example 2

STEP 2: Multiply by a smooth "cutoff." Define $\kappa : (0, \infty) \to [0, 1]$ to be a weakly decreasing, $\mathcal{C}^\infty$ function:

$$\kappa(r) = \begin{cases} 1 & \text{for } 0 < r < 0.5 \\ 0 & \text{for } r > 0.6 \end{cases}$$

$\vec{F}_t(x, y) = (P_t(x, y), Q_t(x, y))$, now with domain $\mathbb{R}^2$
$\vec{F}_t(0, 0) = (0, 0)$,

$$\begin{aligned} P_t &= y\sqrt{-\ln(x^2+y^2)} \cdot (x^2+y^2)^t \cdot \kappa(\sqrt{x^2+y^2}) \\ Q_t &= x\sqrt{-\ln(x^2+y^2)} \cdot (x^2+y^2)^t \cdot \kappa(\sqrt{x^2+y^2}) \end{aligned}$$

Still $\mathcal{C}^1$, with large $\frac{\partial P_t}{\partial y}$, $\frac{\partial Q_t}{\partial x}$ just off-center for small t.

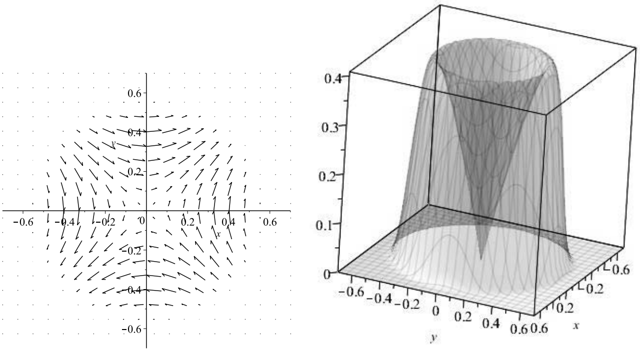

Figure: Plotting the vector field and its magnitude

STEP 3: Pick any sequence of disjoint disks in Quadrant I, with center (R_k, R_k) and radius $0 < r_k < \frac{R_k}{\sqrt{2}}$, so that $R_k \to 0^+$ as $k \to \infty$

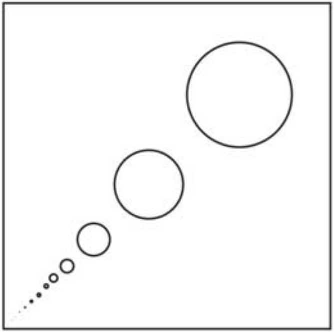

Figure: non-overlapping disks approaching the origin in $\mathbb{R}^2$

Graphics by Maple 18

STEP 4: Re-scale x, y, z directions in the graph of $\vec{F}_t$ by the same factor, r_k, by the formula: $r_k \vec{F}_t(\frac{x}{r_k}, \frac{y}{r_k})$

Graphics by Maple 18

STEP 4: Re-scale x, y, z directions in the graph of $\vec{F}_t$ by the same factor, r_k, by the formula: $r_k \vec{F}_t(\frac{x}{r_k}, \frac{y}{r_k})$

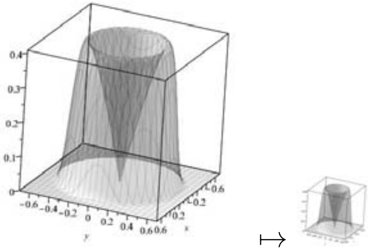

Figure: shrinking the domain and the height

The same large $\frac{\partial P_t}{\partial y}$, $\frac{\partial Q_t}{\partial x}$ just off-center for small t.

Example 2

STEP 5: Last step! For each $k = 1, 2, 3, \ldots$, re-center a shrunken $\vec{F}_t$ onto disk $\#k$ with center R_k, with $t = 2^{-4k} \to 0^+$.

Example 2

STEP 5: Last step! For each $k = 1, 2, 3, \ldots$, re-center a shrunken $\vec{F}_t$ onto disk $\#k$ with center R_k, with $t = 2^{-4k} \to 0^+$.
Also shrink the height again by a factor of 2^{-k}, so that $\frac{\partial \mathbf{P}}{\partial y}$, $\frac{\partial \mathbf{Q}}{\partial x}$ have max. value $2^{-k} \frac{1}{\sqrt{2et}} = \frac{2^k}{\sqrt{2e}} \to \infty$:

$$\vec{\mathbf{F}}(x, y) = \sum_{k=1}^{\infty} 2^{-k} r_k \vec{F}_{2^{-4k}} \left(\frac{x - R_k}{r_k}, \frac{y - R_k}{r_k} \right)$$

Example 2

STEP 5: Last step! For each $k = 1, 2, 3, \ldots$, re-center a shrunken $\vec{F}_t$ onto disk $\#k$ with center R_k, with $t = 2^{-4k} \to 0^+$.
Also shrink the height again by a factor of 2^{-k}, so that $\frac{\partial \mathbf{P}}{\partial y}$, $\frac{\partial \mathbf{Q}}{\partial x}$ have max. value $2^{-k}\frac{1}{\sqrt{2et}} = \frac{2^k}{\sqrt{2e}} \to \infty$:

$$\vec{\mathbf{F}}(x,y) = \sum_{k=1}^{\infty} 2^{-k} r_k \vec{F}_{2^{-4k}}\left(\frac{x - R_k}{r_k}, \frac{y - R_k}{r_k}\right)$$

Exercise

Still need to check:

- $\vec{\mathbf{F}} = (\mathbf{P}, \mathbf{Q})$ *is differentiable everywhere, including the origin.*
- $\frac{\partial \mathbf{Q}}{\partial x} - \frac{\partial \mathbf{P}}{\partial y}$ *is continuous everywhere, including the origin.*

Graphics by Maple 18

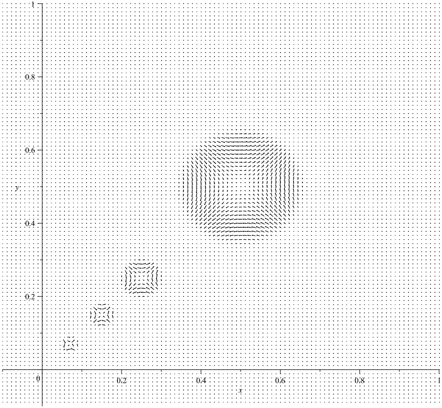

Figure: Easy to check partial derivatives exist at origin because $\vec{\mathbf{F}} \equiv 0$ along axes!

References

Dr. Y. Zhang's research supported by National Science Foundation grant DMS-1265330.

- J. Bruna and J. Cufí, *Complex Analysis*, Eur. Math. Soc., 2013. MR 3076702.
- A. Coffman, Y. Pan, and Y. Zhang, *Continuous solutions of nonlinear Cauchy-Riemann equations and pseudoholomorphic curves in normal coordinates*, preprint.
- P. Cohen, *Topics in the Theory of Uniqueness of Trigonometrical Series*, dissertation, University of Chicago, 1958. MR 2611474.
- P. Cohen, *On Green's theorem*, Proc. AMS **10** (1959), 109–112. MR 0104249.
- J. Cufí and J. Verdera, *A general form of Green's Formula and the Cauchy Integral Theorem*, Proc. AMS. (5) **143** (2015), 2091–2102. MR 3314118.
- J. Gray and S. Morris, *When is a function that satisfies the Cauchy-Riemann equations analytic?*, The American Mathematical Monthly (4) **85** (Apr., 1978), 246–256. MR 0470179.