Notes from a first course on complex analysis

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These notes supplement the textbook [CB], which I've used when teaching MA 525, a first course on complex variables for upper-level undergraduates or graduate students in the M.S. program at Purdue Fort Wayne. Some of these miscellaneous topics appeared on class handouts and this compilation is not intended to be a self-contained reference.

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1 Complex differentiability

Let $z_0 = x_0 + iy_0$ be a point in $\mathbb{C}$, and suppose z_0 is in the domain of a complex-valued function $f(z) = u(x, y) + iv(x, y)$.

These first two Propositions are sufficient conditions for $\mathbb{C}$-differentiability of f at the point z_0.

Proposition 1.1 ([CB], §22, p. 66)**.** *If u_x, u_y, v_x, v_y are continuous at z_0, and satisfy the Cauchy-Riemann equations at that point: $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, then f is $\mathbb{C}$-differentiable at z_0.*

Remark. In particular, the definition of two-variable continuity requires that u_x, u_y, v_x, v_y must exist in some neighborhood of z_0, not just at z_0. The idea is that the continuity of the partial derivatives (called the "$\mathcal{C}^1$" property) implies the real differentiability property of f at z_0, so the next Proposition applies. ∎

Proposition 1.2. *If u and v are $\mathbb{R}$-differentiable at (x_0, y_0) and the partial derivatives u_x, u_y, v_x, v_y satisfy the Cauchy-Riemann equations at that point: $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$, then f is $\mathbb{C}$-differentiable at z_0.*

Remark. Recall a real-valued two-variable function $u(x, y)$ is $\mathbb{R}$-differentiable at (x_0, y_0) means there exist real constants a, c so that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{|u(x, y) - [u(x_0, y_0) + a(x - x_0) + c(y - y_0)]|}{|(x, y) - (x_0, y_0)|} = 0.$$

Again, the two-dimensional limit requires that (x_0, y_0) is an interior point of the domain of u. This is equivalent to the properties (2) – (4) from [CB] §22, p. 67. ∎

The next two Propositions are sufficient conditions for f to be analytic on an open set (meaning, f is $\mathbb{C}$-differentiable at every point in the set).

Proposition 1.3 (The Looman-Menchoff Theorem). *If $f(z)$ is continuous on an open set D, and the partial derivatives satisfy the Cauchy-Riemann equations at every point of D: $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$, then f is analytic on D.* ∎

Proposition 1.4 (Montel, Tolstoff). *If $f(z)$ is locally bounded on an open set D, and the partial derivatives satisfy the Cauchy-Riemann equations at every point of D: $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$, then f is analytic on D.*

Remark. A function is locally bounded on a set D means that for each $w \in D$, there is some neighborhood N_w, $w \in N_w \subseteq D$, and some bound M_w, so that $|f(z)| \leq M_w$ for all $z \in N_w$. Every continuous function is locally bounded (for each w, $|f(z)| < |f(w)| + \epsilon$ for all z within some δ of w), so this result improves the Looman-Menchoff Theorem by requiring less in the hypothesis. For more about these Propositions, see [GM]. ∎

Exercise 1.5. In any of the above Propositions, $\mathbb{C}$-differentiability of f at a point does not follow from checking only the Cauchy-Riemann equations at one point, without any further hypothesis. For example, let

$$f(z) = \begin{cases} z^5/|z|^4 & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

Show that $f = u + iv$ has the following properties:

- On the set $\{z \neq 0\}$, expand u and v as rational functions of x, y (so f is continuous for $z \neq 0$).
- Calculate a limit as $z \to 0$ to show that f is continuous at 0. (Hint: Use [CB] Exercise #18.9., p. 56.)
- Using the limit definition of real partial derivatives at $(0,0)$, show that u and v satisfy the Cauchy-Riemann equations at $(0,0)$.
- Show that u and v do not satisfy the Cauchy-Riemann equations at any point other than $(0,0)$.
- Using the limit definition of complex derivative, show that f is not $\mathbb{C}$-differentiable at $z_0 = 0$. (Hint: this is related to [CB] Exercise #20.9., p. 63.)

Exercise 1.6. In any of the above Propositions, $\mathbb{C}$-differentiability of f at a point does not follow only from checking the Cauchy-Riemann equations on an open set, even on all of $\mathbb{C}$, without any further hypothesis. For example, let

$$f(z) = \begin{cases} e^{-1/z^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}.$$

Show that $f = u + iv$ has the following properties:

- On the set $\{z \neq 0\}$, show that f is $\mathbb{C}$-differentiable by using the rules for derivatives to find $f'(z)$. (So, the C-R equations are satisfied at every point $z \neq 0$ by the Theorem from [CB] §20, p. 65.)

- Using the limit definition of real partial derivatives at $(0,0)$, show that u and v satisfy the Cauchy-Riemann equations at $(0,0)$. (Hint: one step in a limit calculation could involve this substitution:)
$$\lim_{x\to 0^+} \frac{e^{-1/x^4}}{x} = \lim_{X\to +\infty} \frac{e^{-1/(1/X)^4}}{(1/X)}.$$
- Show that f is not continuous at $z = 0$, by showing that $\lim f(z) = +\infty$ along some direction (and so f is not locally bounded, and f is not $\mathbb{C}$-differentiable at $z = 0$ by the remark in [CB] §15, p. 59).

2 Bonus exercises

The following exercise is similar to [CB] Exercise #26.1, p. 81.

Exercise 2.1. For the following functions $u(x,y)$ with domain $D \subseteq \mathbb{C}$, check that u is harmonic on D, and find a "harmonic conjugate" $v(x,y)$ on D, using the method from [CB] Example 26.5, page 81.

1. $u = e^x \sin(y)$
2. $u = 2x^3 - 3x^2y - 6xy^2 + y^3$
3. $u = \dfrac{3x^2 + 8xy - 3y^2}{(x^2+y^2)^2}$, domain $D = \{(x,y) \neq (0,0)\}$. (some computer algebra might help on this one)
4. $u = \tan^{-1}\left(\dfrac{y}{x}\right)$, domain $D = \{x > 0\}$ (this is related to, but not the same as, [CB] Exercise #26.6, p. 82).

Exercise 2.2. For the principal branch of the logarithm, $\text{Log}(z) = \ln|z| + i\theta$, for $z = |z|\exp(i\theta)$, $-\pi < \theta < \pi$, define the analytic function $f(z) = \dfrac{z}{\text{Log}(z)}$, on the domain $D = \{|z-1| < 1\}$. Find the derivatives $f'(z)$ and $f''(z)$ on D, in terms of $\text{Log}(z)$. Prove the following statements about limits as z approaches 0 but stays in the domain D:

$$\begin{aligned}
\lim_{z\in D,\ z\to 0} f(z) &= 0. \\
\lim_{z\in D,\ z\to 0} f'(z) &= 0, \\
\lim_{z\in D,\ z\to 0} |f''(z)| &= +\infty.
\end{aligned}$$

Exercise 2.3. Using the exponential formula (1) from [CB] §34, p. 105, find infinitely many different complex solutions of the equation $\sin(z) = 5$. (You do not have to find all the solutions.)

3 Review of parametric curve calculus

The following result on real curves in $\mathbb{R}^n$ states that for a sufficiently smooth curve $\vec{r}$ with finite arclength, there is a change of parameter so that the composite has the same image but constant speed 1.

Theorem 3.1. *Given $\vec{r}(t)$ which has finite arclength L on $[a, b]$, and such that $\frac{d}{dt}\vec{r}(t)$ is continuous and nonvanishing on (a, b), there exists a function $f(s)$ on the domain $[0, L]$ so that $(\vec{r} \circ f)(0) = \vec{r}(a)$, $(\vec{r} \circ f)(L) = \vec{r}(b)$, and $\vec{r} \circ f$ has constant speed 1 on $(0, L)$.*

Proof. Let $s = \ell(t)$ be the arclength from the start point $\vec{r}(a)$ to the point on the curve at time t, $\vec{r}(t)$. By the formula for arclength,

$$s = \ell(t) = \int_a^t \left| \frac{d}{dx}\vec{r}(x) \right| dx, \tag{3.1}$$

and assuming the total arclength on the interval $[a, b]$ exists, we can conclude that the integral on the subinterval $[a, t]$ exists for every $t \in [a, b]$.

The Fundamental Theorem of Calculus applies for t in (a, b):

$$\frac{d}{dt}\ell(t) = \frac{d}{dt}\int_a^t \left| \frac{d}{dx}\vec{r}(x) \right| dx = \left| \frac{d}{dx}\vec{r}(x) \right| \Bigg]_{x=t} = |\vec{r}\,'(t)|\,.$$

From the assumption that $\vec{r}\,' \neq \vec{0}$ on (a, b), we can conclude that $|\vec{r}\,'(t)| > 0$ on (a, b), so $\ell(t)$ is the integral from a to t of a positive, continuous function, and therefore $s = \ell(t)$ is an increasing function on $[a, b]$. It follows that ℓ is invertible: there exists an inverse function $t = \ell^{-1}(s)$, so that if s is the arclength, then t is the unique time at which the plot $\vec{r}$ gets to length s.

From $(\ell \circ \ell^{-1})(s) = s$, we can $\frac{d}{ds}$ both sides to show that the derivative of the composite is constant: $\frac{d}{ds}(\ell \circ \ell^{-1})(s) = 1$. Applying the Chain rule,

$$\begin{aligned} 1 &= \frac{d}{ds}((\ell \circ \ell^{-1})(s)) = \ell'(\ell^{-1}(s)) \cdot \left(\frac{d}{ds}(\ell^{-1}(s)) \right) \\ \implies \frac{d}{ds}(\ell^{-1}(s)) &= \frac{1}{\ell'(\ell^{-1}(s))} = \frac{1}{\ell'(t)}. \end{aligned}$$

Combining the above two equations gives:

$$\frac{d}{ds}(\ell^{-1}(s)) = \frac{1}{\ell'(t)} = \frac{1}{|\vec{r}\,'(t)|} > 0.$$

Let s be the input parameter, $0 \leq s \leq L$, and consider the composition $(\vec{r} \circ (\ell^{-1}))(s) = \vec{r}(\ell^{-1}(s))$. This composite takes input s, gives $\ell^{-1}(s)$, which is the time at which $\vec{r}$ plots an arc of length s, and plugs this time into the function $\vec{r}$. So, $\vec{r}(\ell^{-1}(s))$ is the position on the curve at which the arclength is s. This change of parameter is called a parametrization by arclength, and the claim is that the function f from the statement of the Theorem can be chosen to be the function ℓ^{-1} that we've constructed.

Returning to the calculations, we want to show that $\vec{r}\circ(\ell^{-1})$ has constant speed 1 with respect to the parameter s, $0 < s < L$. The velocity of the composite is given by the Chain Rule:

$$\frac{d}{ds}\big((\vec{r}\circ(\ell^{-1}))(s)\big) = \vec{r}\,'(\ell^{-1}(s))\cdot\frac{d}{ds}(\ell^{-1}(s)) = \vec{r}\,'(t)\cdot\frac{d}{ds}(\ell^{-1}(s)),$$

and the speed is the magnitude:

$$\left|\frac{d}{ds}\big((\vec{r}\circ(\ell^{-1}))(s)\big)\right| = \left|\vec{r}\,'(t)\cdot\frac{d}{ds}(\ell^{-1}(s))\right| = |\vec{r}\,'(t)|\cdot\left|\frac{d}{ds}(\ell^{-1}(s))\right|,$$

and from the above equation $\frac{d}{ds}(\ell^{-1}(s)) = \frac{1}{|\vec{r}\,'(t)|}$, this product cancels to exactly 1. ∎

Lemma 3.2. *Given $\vec{r}(t):[a,b]\to\mathbb{R}^n$ which satisfies $\lim_{t\to a^+}\vec{r}(t)=\vec{r}(a)$, and $\lim_{t\to a^+}\left(\frac{d}{dt}\vec{r}(t)\right)=\vec{V}$, the following limit also exists: $\lim_{t\to a^+}\frac{\vec{r}(t)-\vec{r}(a)}{t-a}=\vec{V}$.*

Proof. The existence of the one-sided derivative follows from the Mean Value Theorem (applied to the components $\vec{r}(t)=(r_1(t),\ldots,r_n(t))$). ∎

So, if $\vec{r}$ is continuous on $[a,b]$ and the derivative extends continuously to the value $\vec{V}$ at endpoint a, then $\vec{V}$ is also equal to the one-sided derivative at $\vec{a}$.

Theorem 3.3. *Given $\vec{r}(t)$ which is continuous on $[a,b]$, and such that $\frac{d}{dt}\vec{r}(t)$ extends to a continuous and nonvanishing function on the closed interval $[a,b]$, there exists a function $f(s)$ on the domain $[0,L]$ so that $(\vec{r}\circ f)(0)=\vec{r}(a)$, $(\vec{r}\circ f)(L)=\vec{r}(b)$, $\vec{r}\circ f$ is continuous on $[0,L]$, has constant speed 1 on $(0,L)$, and the one-sided derivatives are also unit vectors, so that the velocity $\frac{d}{ds}(r\circ f)$ also extends continuously to $[0,L]$:*

$$\lim_{s\to0^+}\frac{(\vec{r}\circ f)(s)-(\vec{r}\circ f)(0)}{s} = \lim_{s\to0^+}\left(\frac{d}{ds}(r\circ f)\right),$$

and similarly for the other endpoint.

Proof. The finiteness of the arclength, L, is a consequence of the continuity of $\frac{d}{dt}\vec{r}(t)$ on the closed interval $[a,b]$. Use the same $f=\ell^{-1}$ constructed in the Proof of Theorem 3.1; since ℓ is continuous on $[a,b]$, f is continuous on $[0,L]$. Let $\lim_{t\to a^+}\vec{r}\,'(t)=\vec{V}\neq\vec{0}$. The following calculation, using the Composite Limit Theorem, establishes the existence of the limit.

$$\begin{aligned}\lim_{s\to0^+}\left(\frac{d}{ds}(r\circ\ell^{-1})\right) &= \lim_{s\to0^+}\left(\vec{r}\,'(\ell^{-1}(s))\cdot\frac{d}{ds}(\ell^{-1}(s))\right)\\ &= \lim_{s\to0^+}\left(\vec{r}\,'(\ell^{-1}(s))\right)\cdot\lim_{s\to0^+}\left(\frac{d}{ds}(\ell^{-1}(s))\right)\\ &= \vec{V}\cdot\lim_{s\to0^+}\frac{1}{|\vec{r}\,'(\ell^{-1}(s))|} = \vec{V}\cdot\frac{1}{|\vec{V}|}.\end{aligned}$$

The equality of this unit vector with the one-sided derivative is Lemma 3.2, using the continuity of $\vec{r}$. The other endpoint is considered similarly. ∎

Lemma 3.4. *Given $\vec{r} : [a, b] \to \mathbb{R}^n$, if $\vec{r}(t)$ is continuous on $[a, b]$, and $\frac{d}{dt}\vec{r}(t)$ exists on (a, b) and extends to a continuous and nonvanishing function on $[a, b)$, with*

$$\lim_{t \to a^+} \left(\frac{d}{dt}\vec{r}(t) \right) = \vec{V},$$

then for any c, $0 < c < |\vec{V}|$, there exists some $\delta > 0$ so that for $a \le t < a + \delta$,

$$c \cdot (t - a) \le |\vec{r}(t) - \vec{r}(a)| \le (2|\vec{V}| - c) \cdot (t - a).$$

Proof. By Lemma 3.2,

$$\lim_{t \to a^+} \frac{\vec{r}(t) - \vec{r}(a)}{t - a} = \vec{V}.$$

Corresponding to $\epsilon = |\vec{V}| - c > 0$, there is some $\delta > 0$ so that for $0 < t - a < \delta$,

$$\left| \frac{\vec{r}(t) - \vec{r}(a)}{t - a} - \vec{V} \right| < |\vec{V}| - c.$$

By the triangle inequality,

$$\begin{aligned} |\vec{r}(t) - \vec{r}(a)| &\ge |(t - a)\vec{V}| - \left| \vec{r}(t) - \vec{r}(a) - (t - a)\vec{V} \right| \\ &> (t - a)|\vec{V}| - (t - a)(|\vec{V}| - c) \\ &= c \cdot (t - a), \end{aligned}$$

and

$$\begin{aligned} |\vec{r}(t) - \vec{r}(a)| &= \left| \frac{\vec{r}(t) - \vec{r}(a)}{t - a} - \vec{V} + \vec{V} \right| \cdot |t - a| \\ &< (|\vec{V}| - c + |\vec{V}|) \cdot |t - a|. \end{aligned}$$

∎

Theorem 3.5. *Given $\vec{r}(t)$ which is continuous on $[a, b]$, and such that $\frac{d}{dt}\vec{r}(t)$ extends to a continuous and nonvanishing function on $[a, b)$ and which has arclength function $\ell(t)$ as in (3.1), then for any c, $0 < c < 1$, there exists some $\delta > 0$ so that for $a \le t < a + \delta$,*

$$c \cdot \ell(t) \le |\vec{r}(t) - \vec{r}(a)| \le \ell(t).$$

Proof. By Theorem 3.3 (possibly applied to some shorter interval $[a, b_0]$, $b_0 \le b$), there exists a function $f(s)$ on the domain $[0, L]$ so that $(\vec{r} \circ f)(0) = \vec{r}(a)$, $\vec{r} \circ f$ is continuous on $[0, L]$, has constant speed 1 on $(0, L)$, and the one-sided derivative at a is also a unit vector. Lemma 3.4 applies to any c, $0 < c < 1$, to give a lower bound, and there is a better upper bound:

$$c \cdot s \le |\vec{r}(f(s)) - \vec{r}(f(0))| \le s$$

for s in some interval $[0, \delta_1)$, depending on $0 < c < 1$. From the Proof of Theorem 3.1, $f = \ell^{-1}$, so if $0 < s = \ell(t) < \delta_1$, then

$$c \cdot \ell(t) \le |\vec{r}(t) - \vec{r}(a)| \le \ell(t)$$

for $0 < t < \ell^{-1}(\delta_1)$. ∎

The conclusion from the Theorem is that for some initial interval, the magnitude of the displacement is comparable to the arclength.

Given a continuous function $\vec{r}(t) : [a,b] \to \mathbb{R}^n$, the composite $|\vec{r}(t) - \vec{r}(a)|$ is a continuous function $[a,b] \to \mathbb{R}$. If there is some interval (a,c) on which $|\vec{r}(t) - \vec{r}(a)|$ is nonvanishing and $\frac{d}{dt}\vec{r}(t)$ exists (for example, $(a, a+\delta)$ from Lemma 3.4), then:

$$\begin{aligned}
\frac{d}{dt}(|\vec{r}(t) - \vec{r}(a)|) &= \frac{d}{dt}\sqrt{\sum_{k=1}^{n}(r_k(t) - r_k(a))^2} \\
&= \frac{1}{2} \cdot \frac{\sum_{k=1}^{n} 2(r_k(t) - r_k(a)) \cdot \frac{dr_k(t)}{dt}}{\sqrt{\sum_{k=1}^{n}(r_k(t) - r_k(a))^2}} \\
&= \frac{1}{|\vec{r}(t) - \vec{r}(a)|}(\vec{r}(t) - \vec{r}(a)) \cdot \frac{d\vec{r}}{dt} \\
&= \cos(\alpha(t))\left|\frac{d\vec{r}}{dt}\right|.
\end{aligned}$$

The cosine appears from the dot product formula, where $\alpha(t)$ is the angle between the direction vector $\vec{r}(t) - \vec{r}(a)$ and the velocity vector $\frac{d\vec{r}}{dt}$. If $\vec{r}$ happens to have unit speed for $a < t < c$, then $\frac{d}{dt}(|\vec{r}(t) - \vec{r}(a)|) = \cos(\alpha(t))$.

Exercise 3.6. Let F be an analytic function on a domain containing the closed unit disk $\{|z| \le 1\}$, with derivative $f(z) = F'(z)$. If $F(1)$ is real and $f(1) = 1$, then the squared modulus of the values of F on the unit circle, given by the real function

$$g(\theta) = |F(e^{i\theta})|^2,$$

has a critical point at $\theta = 0$.

4 Cauchy integrals

Notation 4.1. For $r > 0$ and $z_0 \in \mathbb{C}$, let $D(z_0, r)$ denote the Euclidean disk with center z_0 and radius r, and as the special case with $z_0 = 0$, abbreviate $D(0,r) = D_r$.

Notation 4.2. By a smooth arc, we mean a continuous parametric map $z : [0,1] \to \mathbb{C}$ with image Γ, which is one-to-one on $[0,1]$ with the possible exception of $z(0) = z(1)$, and differentiable on $(0,1)$ with $\frac{dz}{dt}$ extending to a continuous, non-vanishing function on $[0,1]$. By a piecewise smooth arc, we mean a continuous parametric map $z : [0,1] \to \mathbb{C}$ with image Γ, which is one-to-one on $[0,1]$, with the possible exception of $z(0) = z(1)$ (in this special case we say piecewise smooth contour), so that the domain has a partition $0 = t_0 < t_1 < \ldots < t_N = 1$, where z restricted to each $[t_i, t_{i+1}]$ is, after a suitable re-scaling of the domain, a smooth arc. In any case, the notation can be abused by referring only to Γ, with the parametrization (and the induced orientation) understood.

By Lemma 3.2, the one-sided derivatives are defined at the endpoints of a smooth arc.

Notation 4.3. For φ integrable on a piecewise smooth arc Γ (meaning, as in the previous Notation, $(\varphi\circ z)(t)\cdot z'(t)$ is integrable on $[0,1]$), define a function $\Phi:\mathbb{C}\setminus\Gamma\to\mathbb{C}$ by the formula:

$$\Phi(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(\zeta)}{\zeta-z}d\zeta.$$

More generally, for $n=1,2,3,4,\ldots$, define

$$\Phi_n(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(\zeta)}{(\zeta-z)^n}d\zeta,$$

so $\Phi_1=\Phi$.

Theorem 4.4. *For a piecewise smooth arc Γ and φ continuous on Γ, Φ_n is complex differentiable on the complement of Γ, with $\frac{d}{dz}\Phi_n=n\cdot\Phi_{n+1}$.*

Proof. Note that the derivative formula is just a special case of interchanging derivative and integral signs: formally, $\frac{d}{dz}\int_\Gamma F(z,w)dw=\int_\Gamma\frac{d}{dz}F(z,w)dw$. We will prove only the special case and not make any more general claim; the outline of the proof follows [A].

Step 1: a factoring trick.

$$\begin{aligned}
&(w-a)^n-(w-z)^n\\
=\;&(w-a)^n-(w-z)(w-a)^{n-1}+(w-z)(w-a)^{n-1}-(w-z)^n\\
=\;&(w-a)^{n-1}((w-a)-(w-z))+(w-z)(w-a)^{n-1}-(w-z)^n &\quad(4.1)\\
=\;&(w-a)^{n-1}(z-a)+(z-a)Q_{n-1}(z,w,a), &\quad(4.2)
\end{aligned}$$

where the $(w-z)(w-a)^{n-1}-(w-z)^n$ quantity in (4.1) is a polynomial in z of degree n, which has value 0 at $z=a$, so it factors as in (4.2). When $n=1$, Q_0 is identically zero.

Dividing by $(w-z)^n(w-a)^n$,

$$\begin{aligned}
&\frac{1}{(w-z)^n}-\frac{1}{(w-a)^n}\\
=\;&\frac{z-a}{(w-z)^n(w-a)}+\frac{1}{(w-z)^{n-1}(w-a)}-\frac{1}{(w-a)^n} &\quad(4.3)\\
=\;&(z-a)\cdot\frac{(w-a)^{n-1}+Q_{n-1}(z,w,a)}{(w-z)^n(w-a)^n}. &\quad(4.4)
\end{aligned}$$

Step 2: Φ_n is continuous on the complement of Γ. To show this, fix $a\notin\Gamma$, and we want $\lim_{z\to a}\Phi_n(z)=\Phi_n(a)$. Since the complement is open, we can find some radius $\rho>0$ so that $D(a,\rho)$ is contained in the complement, and for all $w\in\Gamma$, $|w-a|>\rho$ and $|w-z|>\rho/2$ for $z\in D(a,\rho/2)$. Using (4.4),

$$\begin{aligned}
\Phi_n(z)-\Phi_n(a)\;&=\;\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(w)}{(w-z)^n}dw-\frac{1}{2\pi i}\int_\Gamma\frac{\varphi(w)}{(w-a)^n}dw\\
&=\;\frac{1}{2\pi i}\int_\Gamma\varphi(w)\cdot(z-a)\cdot\frac{(w-a)^{n-1}+Q_{n-1}(z,w,a)}{(w-z)^n(w-a)^n}dw. &\quad(4.5)
\end{aligned}$$

Taking absolute value and using L for the arclength of Γ,

$$\begin{aligned}
&|\Phi_n(z)-\Phi_n(a)|\\
= \quad & \frac{|z-a|}{|2\pi i|}\cdot\left|\int_\Gamma \varphi(w)\cdot\frac{(w-a)^{n-1}+Q_{n-1}(z,w,a)}{(w-z)^n(w-a)^n}dw\right|\\
\leq \quad & \frac{|z-a|}{2\pi}\frac{1}{(\rho/2)^n}\frac{1}{\rho^n}\max_{w\in\Gamma}\{|\varphi(w)|\}\max\{\left|(w-a)^{n-1}+Q_{n-1}(z,w,a)\right|\}\cdot L,
\end{aligned}$$

where the second max is over the compact product space $\{(z,w): w\in\Gamma, |z-a|\leq\rho/2\}$. Every factor in the last line except $|z-a|$ depends only on Γ, φ, ρ, a, and n, but not on z, and this is enough to establish the claimed continuity.

Step 3: Φ_1 is complex differentiable on $\mathbb{C}\setminus\Gamma$. Again, fix $a\notin\Gamma$. Using the $n=1$ case of (4.5), where $Q_0\equiv 0$,

$$\frac{\Phi_1(z)-\Phi_1(a)}{z-a} = \frac{1}{2\pi i}\int_\Gamma\frac{\varphi(w)}{(w-z)(w-a)}dw \tag{4.6}$$

$$= \frac{1}{2\pi i}\int_\Gamma\frac{\left(\frac{\varphi(w)}{(w-a)}\right)}{(w-z)}dw. \tag{4.7}$$

In (4.7), the numerator $\frac{\varphi(w)}{w-a}$ is continuous on Γ, so the $n=1$ case of Step 2 applies with this fraction substituted for $\varphi(w)$, and quantity (4.7) is, as a function of z, continuous at a. Taking the $z\to a$ limit of (4.6)=(4.7) gives:

$$\Phi_1'(a)=\frac{1}{2\pi i}\int_\Gamma\frac{\left(\frac{\varphi(w)}{(w-a)}\right)}{(w-a)}dw=\Phi_2(a).$$

The equation $\Phi_1'=1\cdot\Phi_2$ is the $n=1$ case of the claimed derivative formula, and also the start of an induction on n.

Step 4: The inductive step. Assume, for any continuous φ on Γ, $\Phi_{n-1}'=(n-1)\cdot\Phi_n$ on $\mathbb{C}\setminus\Gamma$.

$$\frac{\Phi_n(z)-\Phi_n(a)}{z-a} \tag{4.8}$$

$$= \frac{\int_\Gamma\frac{\varphi(w)}{(w-z)^n}dw-\int_\Gamma\frac{\varphi(w)}{(w-a)^n}dw}{2\pi i(z-a)}$$

$$= \frac{\int_\Gamma\varphi(w)\left(\frac{z-a}{(w-z)^n(w-a)}+\frac{1}{(w-z)^{n-1}(w-a)}-\frac{1}{(w-a)^n}\right)dw}{2\pi i(z-a)} \tag{4.9}$$

$$= \frac{1}{2\pi i}\int_\Gamma\frac{\left(\frac{\varphi(w)}{w-a}\right)}{(w-z)^n}dw+\frac{\int_\Gamma\frac{\left(\frac{\varphi(w)}{w-a}\right)}{(w-z)^{n-1}}dw-\int_\Gamma\frac{\left(\frac{\varphi(w)}{w-a}\right)}{(w-a)^{n-1}}dw}{2\pi i(z-a)}$$

$$= \tilde\Phi_n(z)+\frac{\tilde\Phi_{n-1}(z)-\tilde\Phi_{n-1}(a)}{z-a}, \tag{4.10}$$

where (4.9) uses (4.3), and $\tilde\Phi_n$ in (4.10) denotes the Φ_n expression with the continuous function $\frac{\varphi(w)}{w-a}$ substituted for $\varphi(w)$, as in the previous step. Taking the $z\to a$ limit, quantity (4.8) has limit $\Phi_n'(a)$, and in (4.10), the first term $\tilde{\Phi}_n(z)$ is continuous at a by Step 2, and the limit of the second term is $\tilde{\Phi}_{n-1}'(a) = (n-1) \cdot \tilde{\Phi}_n(a)$, by the inductive hypothesis. The conclusion is

$$\begin{aligned}
\Phi_n'(a) &= \tilde{\Phi}_n(a) + (n-1)\tilde{\Phi}_n(a) \\
&= n\tilde{\Phi}_n(a) = \frac{n}{2\pi i}\int_\Gamma \frac{\left(\frac{\varphi(w)}{w-a}\right)}{(w-a)^n}dw \\
&= n \cdot \Phi_{n+1}(a). \quad \blacksquare
\end{aligned}$$

Corollary 4.5. *If $f(z)$ is complex differentiable on an open set D, then $f'(z)$ is also complex differentiable on D.*

Proof. Given any $a \in D$, there is some disk $D(a,r) \subseteq D$; let Γ be the circle $\{|z-a| = r/2\}$, and let φ be the (continuous) restriction of f to Γ. Theorem 4.4 applies to z in $D(a, r/2)$, where

$$\Phi_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^1}dw = f(z)$$

by the Cauchy Integral Formula, and the conclusion of the Theorem is that the derivative of $f = \Phi_1$ on $D(a, r/2)$ is:

$$f'(z) = \Phi_2(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^2}dw,$$

which is complex differentiable on $D(a, r/2)$. $\blacksquare$

Repeating the above construction shows that all higher derivatives $f^{(n)}$ exist, with the formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}dw.$$

5 Principal values

Definition 5.1. For Γ as in Notation 4.3, a point $\tau \in \Gamma$, and $\varepsilon > 0$, let Γ_ε denote the complement $\Gamma \setminus D(\tau, \varepsilon)$, which for small ε is a pair of piecewise smooth arcs (or possibly one arc), parametrized by restricting the parametrization of Γ. If, for φ as in Notation 4.3, the limit

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{\varphi(\zeta)}{\zeta - \tau}d\zeta \tag{5.1}$$

exists, then it is called the Principal Value of the Cauchy integral Φ at τ, $P.V.\Phi(\tau)$.

Definition 5.2. For any constant $0 < \alpha < 1$ and any set $B \subseteq \mathbb{C}$, a function $\varphi : B \to \mathbb{C}$ is Hölder continuous with exponent α on B means: there is a constant C so that for any $z_1, z_2 \in B$, $|\varphi(z_1) - \varphi(z_2)| < C|z_1 - z_2|^\alpha$. This property is abbreviated $\varphi \in \mathcal{C}^{0,\alpha}(B)$.

Theorem 5.3. *For $0 < \alpha < 1$, and a piecewise smooth contour Γ, if $\varphi \in \mathcal{C}^{0,\alpha}(\Gamma)$, then:*

- *The Principal Value of Φ exists at any point $\tau \in \Gamma$;*
- *For any smooth point $\tau \in \Gamma$, the Principal Value of Φ is equal to*

$$\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau}d\zeta + \frac{1}{2}\varphi(\tau).$$

Proof. The first step is to show that for a piecewise smooth arc Γ,

$$\int_\Gamma \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau}d\zeta$$

exists, as a complex valued improper integral, in the following sense. On Γ_ϵ, the quantity being integrated is continuous, so the only parts requiring attention are the one or two arcs with endpoint τ in the disk $D(\tau,\varepsilon)$. It is enough to show that there is some smooth arc reparametrizing a piece of Γ, $z(t) \in \Gamma \cap D(\tau,\varepsilon)$, $t \in [0,1]$, with $z(0)=\tau$, with the property: for any $\eta > 0$ there is some $\delta > 0$ so that for all $0 < a < \delta$:

$$\left|\int_{z(a)}^{z(\delta)} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau}d\zeta\right| < \eta.$$

By Lemma 3.4, we can choose a small enough sub-arc of $\Gamma \cap D(\tau,\varepsilon)$, parametrized by $z(t)$ on $[0,B]$, $z(0)=\tau$, $|z'(t)|=1$ on $(0,B)$, and $|z(t)-z(0)| \geq ct$ for some $c > 0$. (This is not yet related to η.)

Then, for any $0 < b < B$,

$$\begin{aligned}
& \int_b^B \frac{|\varphi(z(t))-\varphi(z(0))|}{|z(t)-z(0)|}dt \\
\leq\ & \int_b^B \frac{C|z(t)-z(0)|^\alpha}{|z(t)-z(0)|}dt \\
=\ & \int_b^B \frac{C}{|z(t)-z(0)|^{1-\alpha}}dt \\
\leq\ & C\int_b^B \frac{1}{(ct)^{1-\alpha}}dt \\
=\ & \frac{C}{c^{1-\alpha}}\frac{t^\alpha}{\alpha}\Big]_b^B < \frac{CB^\alpha}{\alpha c^{1-\alpha}}.
\end{aligned}$$

Since the quantity being integrated is nonnegative and has bounded integral for all b, as b decreases to 0, $\int_b^B$ weakly increases to some finite least upper bound $0 \leq U \leq \frac{CB^\alpha}{\alpha c^{1-\alpha}}$. In particular, for any η, there is some small $\delta > 0$ so that for all $0 < a \leq \delta$, $U - \eta < \int_a^B \leq U$. So, for all $0 < a < \delta$,

$$\begin{aligned}
\left|\int_{z(a)}^{z(\delta)} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} d\zeta\right| &= \left|\int_{t=a}^{t=\delta} \frac{\varphi(z(t))-\varphi(z(0))}{z(t)-z(0)} z'(t)dt\right| \\
&\leq \int_a^\delta \frac{|\varphi(z(t))-\varphi(z(0))|}{|z(t)-z(0)|} dt \\
&= \int_\delta^B - \int_a^B < U - (U-\eta) = \eta.
\end{aligned}$$

and the claim follows.

The previous part only required that Γ is a piecewise smooth arc; the next step uses the assumption that it is a contour, with global parametrization $z(t)$. We make the simplifying assumption that $\tau = z(t_0)$ for some $t_0 \neq 0, 1$; otherwise, the argument can be easily modified.

For a smooth point τ, and sufficiently small $\varepsilon > 0$, there are numbers $t_1 < t_2$ so that Γ_ε is a connected arc, that is parametrized by $z(t)$ in two sub-arcs, one for the restricted domain $[0, t_1]$, and the other for $[t_2, 1]$. As a function of ζ, the expression $\frac{1}{\zeta-\tau}$ has an antiderivative $\log(\zeta - \tau)$, away from some branch cut starting at τ and avoiding Γ_ε by staying in the connected exterior of Γ. Considering the quantity

$$\begin{aligned}
\int_{\Gamma_\varepsilon} \frac{d\zeta}{\zeta-\tau} &= \int_{z(0)}^{z(t_1)} \frac{d\zeta}{\zeta - z(t_0)} + \int_{z(t_2)}^{z(1)} \frac{d\zeta}{\zeta - z(t_0)} \\
&= (\log(z(t_1)-\tau) - \log(z(0)-\tau)) \\
&\quad + (\log(z(1)-\tau) - \log(z(t_2)-\tau)) \\
&= \log\left(\frac{z(t_1)-\tau}{z(t_2)-\tau}\right) \\
&= \ln\left|\frac{z(t_1)-\tau}{z(t_2)-\tau}\right| + i \arg\left(\frac{z(t_1)-\tau}{z(t_2)-\tau}\right),
\end{aligned}$$

as $\varepsilon \to 0^+$, the integral approaches $0 + i\pi$. If τ is a <u>corner</u> point (a shared endpoint of arcs in the contour) with angle ρ, then the above difference in arguments approaches ρ and the integral approaches $i\rho$. The claimed formula for the smooth point follows from the add-and-subtract trick:

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{\varphi(\zeta)}{\zeta-\tau} d\zeta &= \lim_{\varepsilon\to 0^+} \left(\frac{1}{2\pi i}\int_{\Gamma_\varepsilon} \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} d\zeta + \frac{\varphi(\tau)}{2\pi i}\int_{\Gamma_\varepsilon} \frac{d\zeta}{\zeta-\tau}\right) \\
&= \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} d\zeta + \frac{1}{2}\varphi(\tau). \quad \blacksquare
\end{aligned}$$

Notation 5.4. Given a $\mathcal{C}^{0,\alpha}$ function φ on a piecewise smooth arc Γ, and any point $\tau \in \Gamma$, define Ψ_τ by

$$\Psi_\tau(z) = \int_\Gamma \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta - z} d\zeta.$$

It was checked in the above proof that $\Psi_\tau(\tau)$ is well-defined as an improper integral, and $\Psi_\tau(z)$ is analytic on $\mathbb{C} \setminus \Gamma$ by Theorem 4.4.

Theorem 5.5. *For any limit with z approaching τ non-tangentially,*

$$\lim_{z\to\tau} \Psi_\tau(z) = \Psi_\tau(\tau).$$

Proof. The arc Γ and the point τ are given, and we also fix an arc on which z will approach τ, that has the following property: There is some $\delta_1 > 0$ and some $c > 0$ so that if z is on the approach arc and $0 < |z - \tau| < \delta_1$, then for all $\zeta \in \Gamma$, $|z - \zeta| > c|z - \tau|$. We take this as the definition of non-tangential, although if we choose some other approach arc, the constants δ_1 and c may be different.

Given $\eta > 0$, we want to show that there is some $\delta > 0$ so that if z is on the approach arc and $0 < |z - \tau| < \delta$, then $|\Psi_\tau(z) - \Psi_\tau(\tau)| < \eta$. By definition,

$$\begin{aligned}\Psi_\tau(z) - \Psi_\tau(\tau) &= \int_\Gamma \frac{\varphi(\zeta) - \varphi(\tau)}{\zeta - z} d\zeta - \int_\Gamma \frac{\varphi(\zeta) - \varphi(\tau)}{\zeta - \tau} d\zeta \\ &= \int_\Gamma \frac{(\varphi(\zeta) - \varphi(\tau))(z - \tau)}{(\zeta - z)(\zeta - \tau)} d\zeta.\end{aligned}$$

From the Proof of Theorem 5.3, there is some $\varepsilon > 0$ and some parametrization $\zeta(t)$ of Γ near τ so that the improper integral satisfies:

$$\left| \int_{\Gamma\setminus\Gamma_\varepsilon} \frac{\varphi(\zeta) - \varphi(\tau)}{\zeta - \tau} d\zeta \right| \le \int_{t_0-\delta_2}^{t_0+\delta_3} \frac{|\varphi(\zeta(t)) - \varphi(\tau)|}{|\zeta(t) - \tau|} dt < \frac{\eta c}{2}.$$

The ε depends on the constant c from the fixed approach arc, but not on the point z. For z on the approach arc such that $0 < |z - \tau| < \delta_1$, $\frac{|z-\tau|}{|\zeta - z|} < \frac{1}{c}$, so

$$\left| \int_{\Gamma\setminus\Gamma_\varepsilon} \frac{(\varphi(\zeta) - \varphi(\tau))(z - \tau)}{(\zeta - z)(\zeta - \tau)} d\zeta \right| < \frac{\eta}{2}.$$

Suppose $0 < |z - \tau| < \varepsilon/2$, so $|\zeta - z| > \varepsilon/2$ for all $\zeta \in \Gamma_\varepsilon$. Let L_ε be the arclength of Γ_ε. Then

$$\begin{aligned}\left| \int_{\Gamma_\varepsilon} \frac{\varphi(\zeta) - \varphi(\tau)}{(\zeta - z)(\zeta - \tau)} d\zeta \right| &\le \max \left| \frac{\varphi(\zeta) - \varphi(t)}{(\zeta - z)(\zeta - \tau)} \right| L_\varepsilon \\ &< \frac{2}{\varepsilon} \max \left| \frac{\varphi(\zeta) - \varphi(\tau)}{\zeta - \tau} \right| L_\varepsilon,\end{aligned}$$

where the max is over $\zeta \in \Gamma_\varepsilon$.

If $\max \left| \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} \right| L_\varepsilon = 0$, then the claim is established, with $\delta = \min\{\delta_1, \varepsilon/2\}$. Otherwise, if

$$0 < |z - \tau| < \frac{\eta\varepsilon}{4 \max \left| \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta-\tau} \right| L_\varepsilon},$$

then

$$\begin{aligned}\left| \int_{\Gamma_\varepsilon} \frac{(\varphi(\zeta) - \varphi(\tau))(z - \tau)}{(\zeta - z)(\zeta - \tau)} d\zeta \right| &= |z - \tau| \left| \int_{\Gamma_\varepsilon} \frac{\varphi(\zeta) - \varphi(\tau)}{(\zeta - z)(\zeta - \tau)} d\zeta \right| \\ &< \frac{\eta}{2}. \quad \blacksquare\end{aligned}$$

For a piecewise smooth contour Γ as in Theorem 5.3, let D^+ denote the interior region of Γ and let D^- denote the exterior. For φ and Φ as in Notation 4.3, define $\Phi^\pm : \Gamma \to \mathbb{C}$ as a non-tangential limit, if it exists:

$$\Phi^\pm(\tau) = \lim_{z\to\tau,\ z\in D^\pm} \Phi(z).$$

Theorem 5.6 (Plemelj Jump)**.** *Given a $\mathcal{C}^{0,\alpha}$ function φ on Γ, the functions $\Phi^\pm$ are well-defined at every smooth point τ, and satisfy*

$$\Phi^\pm(\tau) = P.V.\Phi(\tau) \pm \frac{1}{2}\varphi(\tau).$$

Proof. At any smooth point, there is a non-tangential approach arc on either side. Use the add-and-subtract trick again.

$$\begin{aligned} \Phi^\pm(\tau) &= \lim_{z\to\tau,\ z\in D^\pm} \frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi i}\lim_{z\to\tau,\ z\in D^\pm}\left(\int_\Gamma \frac{\varphi(\zeta)-\varphi(\tau)}{\zeta - z}d\zeta + \varphi(\tau)\int_\Gamma \frac{d\zeta}{\zeta - z}\right). \end{aligned}$$

So, using the Cauchy Integral Formula,

$$\Phi^+(\tau) = \frac{1}{2\pi i}\lim_{z\to\tau,\ z\in D^+} \Psi_\tau(z) + \varphi(\tau),$$

and

$$\Phi^-(\tau) = \frac{1}{2\pi i}\lim_{z\to\tau,\ z\in D^-} \Psi_\tau(z).$$

The claimed result follows from the limit in Theorem 5.5 and the formula for the Principal Value in Theorem 5.3. ∎

6 Power series

Definition 6.1. An infinite series of the form $\sum_{n=0}^{\infty} c_n(z-a)^n$ is called a <u>power series</u>. The c_n are called the <u>coefficients</u>, and a is called the <u>center</u> of the power series.

The coefficients c_n, the center a, and the variable z can all be complex numbers. The index n usually starts at 0 (the constant term is $c_0 z^0 = c_0$), or, if the first few coefficients are 0, n may start at any positive integer. (This definition of power series excludes negative or non-integer exponents n.)

Definition 6.2. The <u>domain of convergence</u> of a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ is the set of all (complex) numbers z so that the series is convergent.

Note that the domain of convergence cannot be the empty set, since any power series always converges at its center, $z = a$: $\sum_{n=0}^{\infty} c_n(a-a)^n = c_0 + c_1 0^1 + c_2 0^2 + \cdots = c_0$.

Proposition 6.3 ("Abel's Lemma"). *For a power series centered at a,* $\sum_{n=0}^{\infty} c_n(z-a)^n$, *exactly one of the following holds:*

- $\sum_{n=0}^{\infty} c_n(z-a)^n$ *converges to* c_0 *at* $z = a$, *and diverges for all other* z.
- $\sum_{n=0}^{\infty} c_n(z-a)^n$ *is absolutely convergent for all* $z \in \mathbb{C}$.
- *There is a real number* $R > 0$ *so that* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *is absolutely convergent for* $|z-a| < R$, *and the series is divergent for* $|z-a| > R$.

Proof. For a proof, see [C]. ∎

Definition 6.4. The number R is the radius of convergence of the power series, and it must be nonnegative. The first two cases are referred to as $R = 0$ and $R = \infty$.

Note that Proposition 6.3 is inconclusive when both $0 < R < \infty$ and $|z-a| = R$. (Geometrically, this is the case where the domain of convergence is a disk in $\mathbb{C}$ with positive radius, and z is on the circular boundary of the disk.) The power series could be divergent, absolutely convergent, or conditionally convergent for z on the boundary. In the case where the center a is on the real number line, then the real values of z for which the series is convergent form an interval centered at a (the intersection of the disk and the real axis), and the points on the boundary are the two endpoints, $a - R$ and $a + R$.

Proposition 6.5. *If* $\sum_{n=0}^{\infty} c_n(z-a)^n$ *has radius of convergence* R, *then* $\sum_{n=1}^{\infty} c_n n(z-a)^{n-1}$ *also has radius of convergence* R.

Proof. For a proof, see [C]. ∎

Theorem 6.6. *The function defined by* $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ *is analytic on the disk* $\{|z-a| < R\}$, *with* $f'(z) = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1}$.

Proof. The following steps give an elementary ϵ/δ argument, based on a proof from [BC]; unlike [CB] Chapter 5, we do not use integration and we do not use the general theory of uniform convergence, only what is specifically needed here.

First pick a specific point z in the disk $\{|z-a| < R\}$. Then choose some ρ so that $|z-a| < \rho < R$.

We want to check the definition of limit appearing in the definition of derivative at the point z, $f'(z)$, so we want to show:

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} = \sum_{n=1}^{\infty} c_n n(z-a)^{n-1}.$$

The absolute convergence of the RHS series is Proposition 6.5. So, given $\epsilon > 0$, we want to find $\delta > 0$ so that

$$0 < |w - z| < \delta \implies \left| \frac{f(w) - f(z)}{w - z} - \sum_{n=1}^{\infty} c_n n (z-a)^{n-1} \right| < \epsilon. \tag{6.1}$$

The δ may depend on ϵ, a, c_n, R, ρ, and z, but not on w.

For integer $N = 1, 2, 3, \ldots$, define:

$$\begin{aligned} S_N(z) &= \sum_{n=0}^{N} c_n (z-a)^n \\ R_N(z) &= \sum_{n=N+1}^{\infty} c_n (z-a)^n. \end{aligned}$$

So, for any N, $f(z) + S_N(z) + R_N(z)$, and $\lim_{N\to\infty} S_N(z) = f(z)$ and $\lim_{N\to\infty} R_N(z) = 0$. We will choose a specific N later, in a way that may depend on ϵ, a, c_n, R, ρ, and z, but not on w and not on the δ we have not yet found. The quantity from (6.1) satisfies the following inequality for any N:

$$\begin{aligned} & \left| \frac{f(w) - f(z)}{w - z} - \sum_{n=1}^{\infty} c_n n (z-a)^{n-1} \right| \\ = & \left| \frac{(S_N(w) + R_N(w)) - (S_N(z) + R_N(z))}{z - w} \right. \\ & \left. - \left(\left(\sum_{n=1}^{N} c_n n (z-a)^{n-1} \right) + \left(\sum_{n=N+1}^{\infty} c_n n (z-a)^{n-1} \right) \right) \right| \\ \leq & \left| \frac{S_N(w) - S_N(z)}{w - z} - \sum_{n=1}^{N} c_n n (z-a)^{n-1} \right| \qquad (6.2) \\ & + \left| \sum_{n=N+1}^{\infty} c_n n (z-a)^{n-1} \right| \qquad (6.3) \\ & + \left| \frac{R_N(w) - R_N(z)}{w - z} \right|. \qquad (6.4) \end{aligned}$$

The term (6.3) is the tail end of the convergent series from Proposition 6.5: there's some N_1 so that if $N > N_1$, then the term (6.3) is less than $\epsilon/3$. This cutoff N_1 depends on ϵ, a, c_n, and z, but not on w.

The term (6.4) can be re-arranged:

$$\begin{aligned}
\frac{R_N(w)-R_N(z)}{w-z} &= \frac{\left(\displaystyle\sum_{n=N+1}^{\infty} c_n(w-a)^n\right) - \left(\displaystyle\sum_{n=N+1}^{\infty} c_n(z-a)^n\right)}{w-z} \\
&= \sum_{n=N+1}^{\infty} c_n \frac{(w-a)^n-(z-a)^n}{(w-a)-(z-a)} \\
&= \sum_{n=N+1}^{\infty} c_n \sum_{k=0}^{n-1} (z-a)^k (w-a)^{n-1-k} \qquad (6.5)
\end{aligned}$$

The step (6.5) follows from the polynomial identity

$$(w-a)^n-(z-a)^n = ((w-a)-(z-a))\sum_{k=0}^{n-1}(z-a)^k(w-a)^{n-1-k},$$

a telescoping sum similar to the geometric series formula, which holds for all z, w, and a. Assuming $|w-a|<\rho$, the following estimate holds:

$$\begin{aligned}
\left| c_n \sum_{k=0}^{n-1}(z-a)^k(w-a)^{n-1-k}\right| &\le |c_n|\sum_{k=0}^{n-1}|z-a|^k|w-a|^k \\
&< |c_n|\sum_{k=0}^{n-1}\rho^k\rho^{n-1-k} = |c_n|n\rho^{n-1}.
\end{aligned}$$

The series $\displaystyle\sum_{n=0}^{\infty} c_n \sum_{k=0}^{n-1}(z-a)^k(w-a)^{n-1-k}$ is absolutely convergent, by the Comparison Test ([C]) applied to

$$\sum_{n=0}^{\infty}\left| c_n\sum_{k=0}^{n-1}(z-a)^k(w-a)^{n-1-k}\right| \le \sum_{n=0}^{\infty}|c_n|n\rho^{n-1},$$

which is absolutely convergent by Proposition 6.5 (with $z=\rho+a$). So (6.5) is the tail end of a convergent series, and there is some there is some N_2 so that if $N>N_2$ then

$$\left|\sum_{n=N+1}^{\infty} c_n\sum_{k=0}^{n-1}(z-a)^k(w-a)^{n-1-k}\right| \le \sum_{n=N+1}^{\infty}|c_n|n\rho^{n-1} < \epsilon/3.$$

This cutoff N_2 depends on ϵ, c_n, and ρ, but not on the specific values of z or w, only that they both are in the disk $D(a,\rho)$, and in particular w does not have to be close to z. It follows that if $N>N_2$, $w\ne z$, $|z-a|<\rho$, and $|w-a|<\rho$, then $\left|\frac{R_N(w)-R_N(z)}{w-z}\right|<\epsilon/3$.

Now, fix a number $N>\max\{N_1,N_2\}$, and consider the term (6.2). $S_N(z)=\displaystyle\sum_{n=0}^{N}c_n(z-a)^n$ is a polynomial of degree N, with derivative

$$S_N'(z)=\lim_{w\to z}\frac{S_N(w)-S_N(z)}{w-z}=\sum_{n=1}^{N}c_n n(z-a)^{n-1},$$

so corresponding to $\epsilon/3 > 0$, there is some $\delta_1 > 0$ so that if $0 < |w - z| < \delta_1$, then the (6.2) quantity is less than $\epsilon/3$.

To bound all three terms at the same time, let $\delta = \min\{\delta_1, \rho - |z - a|\} > 0$. Then $|w - a| = |w - a + z - z| \leq |w - z| + |z - a|$, and if $0 < |w - z| < \delta$, then $|w - a| \leq \rho$, which, together with $N > N_2$, is all we need for the $\epsilon/3$ bound on the (6.4) term. The (6.3) term is bounded by $\epsilon/3$ because $N > N_1$, and the (6.2) term is bounded by $\epsilon/3$ because $0 < |w - z| < \delta \leq \delta_1$. ∎

The following property of analytic functions may be useful in answering [CB] Exercises #59.12.*a.*, p. 195, and #66.4 and #66.5, p. 219.

Lemma 6.7. *Given an open set $D \subseteq \mathbb{C}$, a point $z_0 \in D$, and a number $n = 1, 2, 3, \ldots$, if $h(z)$ is analytic on D and there is some disk $\{z : |z - z_0| < r\}$ where h has a series expansion:*

$$h(z) = \sum_{k=n}^{\infty} c_k (z - z_0)^k,$$

then this function f is also analytic on D:

$$f(z) = \left\{ \begin{array}{cc} \dfrac{h(z)}{(z - z_0)^n} & \text{for } z \neq z_0 \\ c_n & \text{for } z = z_0 \end{array} \right\}.$$

Proof. The series for h has a positive radius of convergence $r > 0$ by Taylor's Theorem from [CB] §57, p. 189 (possibly $r = \infty$); the hypothesis of the Lemma is that the coefficients $c_0, \ldots, c_{n-1}$ are all $= 0$, so the series starts with coefficient c_n (which may or may not be $= 0$).

The function $f(z)$ is complex differentiable at every point in $D \setminus \{z_0\}$, by the quotient rule for derivatives. We only need to check that f is complex differentiable at z_0 to get the claimed conclusion. Consider the series:

$$g(z) = \sum_{k=n}^{\infty} c_k (z - z_0)^{k-n}.$$

This is a power series with all non-negative exponents, and defines some function g so that $g(z_0) = c_n = f(z_0)$. For any point z with $0 < |z - z_0| < r$, the series is convergent, because it is equal to a scalar multiple of the convergent series for h:

$$\begin{aligned} g(z) &= \sum_{k=n}^{\infty} \left(c_k (z - z_0)^k \cdot \frac{1}{(z - z_0)^n} \right) \\ &= \frac{1}{(z - z_0)^n} \cdot \sum_{k=n}^{\infty} c_k (z - z_0)^k = \frac{1}{(z - z_0)^n} h(z). \end{aligned}$$

The scalar multiple rule applies because $\frac{1}{(z - z_0)^n}$ does not depend on the summation index k. We can conclude that the series for $g(z)$ converges for all z with $|z - z_0| < r$, so $g(z)$ is complex differentiable at z_0 by Theorem 6.6 (or the Corollary from [CB] §65, p. 215). The above construction shows $g(z) = f(z)$ for all z with $|z - z_0| < r$, so f is also complex differentiable at z_0 and has a series expansion centered at z_0 given by the $g(z)$ series. ∎

7 Rational functions

Definition 7.1. A <u>rational function</u> is defined by $f(z) = P(z)/Q(z)$, where P and Q are polynomials and $Q \not\equiv 0$.

By the Fundamental Theorem of Algebra, there are finitely many points r where $Q(r) = 0$, so the domain of a rational function is $\{z \in \mathbb{C} : Q(z) \neq 0\}$, the open, connected, non-empty complement of a finite set.

Lemma 7.2. *Given any complex number w and polynomials $P(z)$ and $Q(z)$ so that the rational function $f(z) = P(z)/Q(z)$ is non-constant on the domain $D = \{z \in \mathbb{C} : Q(z) \neq 0\}$, the following are equivalent:*

1. *There exists a solution $z = s \in D$ of the equation $f(z) = w$.*
2. *There exists $s \in D$ and a polynomial $F(z) \not\equiv 0$ so that $P(z) - wQ(z) = (z - s)F(z)$.*

Proof. First, $P(z) - wQ(z)$ is not the constant zero polynomial; otherwise, if $P(z) - wQ(z) \equiv 0$, then for any $t \in D$, $P(t) = wQ(t) \iff P(t)/Q(t) = w = f(t)$, which contradicts the initial assumption that f is not a constant function in D.

For (2) $\implies$ (1), plugging in $z = s \in D$ gives $P(s) - wQ(s) = (s - s)F(s) = 0$ and $Q(s) \neq 0$, so $P(s) = wQ(s)$ and $f(s) = P(s)/Q(s) = w$.

Conversely, for (1) $\implies$ (2), the hypothesis is that $f(s) = P(s)/Q(s) = w$, with $Q(s) \neq 0$; this is equivalent to $P(s) = wQ(s) \iff P(s) - wQ(s) = 0$, so s is a root of the polynomial $P(z) - wQ(z)$. $P(z) - wQ(z)$ cannot be any constant polynomial: it was already shown that it can't be $\equiv 0$, and it also cannot be any non-zero constant, because it has a root s. So $P(z) - wQ(z)$ has degree ≥ 1 and root s, and it factors as $(z - s)F(z)$ for some (possibly constant but not $\equiv 0$) polynomial $F(z)$. ∎

Theorem 7.3. *For any non-constant rational function $f(z) = P(z)/Q(z)$, the set of numbers w such that $f(z) = w$ has no solution is a finite set.*

Proof. Let P have degree M and let Q have degree N. Let $f(z) = P(z)/Q(z)$ have domain $D \subseteq \mathbb{C}$ as in Lemma 7.2.

Case 1. $Q(z)$ is a constant function. $Q \not\equiv 0$ by definition, so $f(z) = P(z)/Q(z)$ is a non-constant polynomial and for any w, there is at least one root of the polynomial $f(z) - w$, by the Fundamental Theorem of Algebra. So the set of w with no solutions of $f(z) = w$ is the empty set.

Case 2. $Q(z)$ is a non-constant polynomial of degree $N > 0$ and, by the Fundamental Theorem of Algebra, factors as $Q(z) = q_N(z - r_1)(z - r_2)\cdots(z - r_N)$, for leading coefficient $q_N \neq 0$ and some possibly repeating list of roots $(r_1, \ldots, r_N)$. For any ordered list of non-negative integers $\vec{v} = (v_1, v_2, \ldots, v_N)$, define $R_{\vec{v}}(z) = (z - r_1)^{v_1}(z - r_2)^{v_2}\cdots(z - r_N)^{v_N}$, so $R_{\vec{v}}$ has leading coefficient 1 and degree $v_1 + \cdots + v_N$, and every root of $R_{\vec{v}}$ is one of the roots of Q.

If w is a number such that $f(z) = w$ has no solution in D, then neither of the equivalent conditions in Lemma 7.2 holds. In particular, $P(z) - wQ(z)$ cannot be equal to any polynomial of the form $(z - s)F(z)$ for $s \in D$. We want to show that given P and Q, there are only finitely many such w.

Case 2.a. Consider the set of w such that $P(z) - wQ(z)$ is a constant function. For any w in this set, applying the N^{th} derivative to both sides of $P(z) - wQ(z) \equiv C$ gives

$$P^{(N)}(z) - wq_N N! \equiv 0,$$

so $P^{(N)}(z)$ is a constant function and $w = P^{(N)}(z)/(q_N N!)$ is uniquely determined by P and Q and does not depend on C (or z). The set of such w is either empty or has only this one element.

Case 2.b. Consider the set of w such that $P(z) - wQ(z)$ is non-constant and not of the form $(z - s)F(z)$ for any $s \in D$. So any root of $P(z) - wQ(z)$ must be a root of $Q(z)$, and we can conclude $P(z) - wQ(z)$ must be equal to $C_{\vec{v}} R_{\vec{v}}(z)$ for some non-zero leading coefficient $C_{\vec{v}}$ and some vector $\vec{v}$ with non-negative integer entries satsifying $0 < v_1 + \cdots + v_N \leq \max\{M, N\}$. There are only finitely many such vectors $\vec{v}$; we will show that for each $\vec{v}$, there is at most one pair $(w, C_{\vec{v}})$ such that $P(z) - wQ(z) = C_{\vec{v}} R_{\vec{v}}(z)$.

Pick any t in the non-empty set D, so $Q(t) \neq 0$, $R_{\vec{v}}(t) \neq 0$, and w and $C_{\vec{v}}$ satisfy $P(t) - wQ(t) = C_{\vec{v}} R_{\vec{v}}(t) \neq 0$. Using this constant t, define a new polynomial

$$d(z) = \det \begin{bmatrix} Q(t) & R_{\vec{v}}(t) \\ Q(z) & R_{\vec{v}}(z) \end{bmatrix} = Q(t) R_{\vec{v}}(z) - R_{\vec{v}}(t) Q(z).$$

If $d(z) \equiv 0$ then $R_{\vec{v}}(z) = \left(\frac{R_{\vec{v}}(t)}{Q(t)}\right) Q(z)$, and $P(z) - wQ(z) = C_{\vec{v}} \left(\frac{R_{\vec{v}}(t)}{Q(t)}\right) Q(z)$, so $P(z) - \left(w - \frac{C_{\vec{v}} R_{\vec{v}}(t)}{Q(t)}\right) Q(z) \equiv 0$. However, as in the Proof of Lemma 7.2, this implies $f(z) = P(z)/Q(z)$ is constant on D, contradicting the assumption. So $d(z) \not\equiv 0$, and there is some $u \in \mathbb{C}$ with $d(u) \neq 0$. This means the pair $(w, C_{\vec{v}})$ satisfies $P(u) - wQ(u) = C_{\vec{v}} R_{\vec{v}}(u)$ and also the linear system

$$\begin{bmatrix} Q(t) & R_{\vec{v}}(t) \\ Q(u) & R_{\vec{v}}(u) \end{bmatrix} \begin{bmatrix} w \\ C_{\vec{v}} \end{bmatrix} = \begin{bmatrix} P(t) \\ P(u) \end{bmatrix},$$

where the constant coefficient matrix has non-zero determinant $d(u)$, and there is exactly one solution $(w, C_{\vec{v}})$ satisfying this system. There may be other constraints on $(w, C_{\vec{v}})$ but checking $P(z) - wQ(z) = C_{\vec{v}} R_{\vec{v}}(z)$ at these two points $z = t$ and $z = u$ already rules out the existence of more than one solution for $(w, C_{\vec{v}})$. ∎

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