ANALYTIC STABILITY OF THE CR CROSS-CAP

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For $m < n$, any real analytic m-submanifold of complex n-space with a nondegenerate CR singularity is shown to be locally equivalent, under a holomorphic coordinate change, to a fixed real algebraic variety defined by linear and quadratic polynomials. The situation is analogous to Whitney's stability theorem for cross-cap singularities of smooth maps. The complex analyticity of the normalizing transformation is proved using a rapid convergence argument.

1. Introduction

For $m \leq n$, if a real m-manifold M is embedded in $\mathbb{C}^n$, then for each point $\mathbf{x}$ on M there are two possibilities: the tangent m-plane at $\mathbf{x}$ may contain a complex line, so M is said to be "CR singular" at $\mathbf{x}$, or it may not, so M is said to be "totally real" at $\mathbf{x}$. This article will consider the local extrinsic geometry of a real analytically embedded M near a CR singular point, in the case when the CR singularity satisfies some natural nondegeneracy properties and $\frac{2}{3}(n+1) \leq m < n$ (so $(m,n) = (4,5)$ is the case of lowest dimension). The main result is an algebraizability property: there exists a holomorphic coordinate change in a neighborhood of $\mathbf{x}$ so that M is real algebraic in the new coordinate system. In fact, M will be biholomorphically equivalent to a fixed normal form variety, so that, unlike the well-known $m = n$ case, nondegenerate CR singularities have no continuous invariants under biholomorphisms.

The analysis of normal forms near CR singular points is part of the program of studying the local equivalence problem for real m-submanifolds of $\mathbb{C}^n$, as described in the survey [**BER**]. Normal forms for CR singular real n-manifolds in $\mathbb{C}^n$, $m = n \geq 2$, have been the subject of much study, e.g., by [**Bishop**], [**M**], [**MW**], [**Webster**]. Real surfaces in $\mathbb{C}^n$ ($m = 2$, $n \geq 3$) have been considered by [$\mathbf{H}_1$], [$\mathbf{H}_2$], [$\mathbf{C}_4$], and real threefolds in $\mathbb{C}^4$ by [$\mathbf{C}_5$]. A formal normal form for a CR singular real 4-manifold in $\mathbb{C}^5$ was found by [**Beloshapka**] and [$\mathbf{C}_1$] — it was shown that there exists a transformation (not unique) defined by formal power series, taking M to the normal form.

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The new result here is the existence of a normalizing transformation defined by series that are convergent in a neighborhood of the singularity.

2. Topological considerations

We briefly recall some topological properties of CR singularities. We could consider real submanifolds of any complex manifold, but since the main result on the normalization is about the local geometry, we can begin by assuming M is a smoothly immersed real m-manifold in $\mathbb{C}^n$.

The most basic invariant of a CR singularity at a point $\mathbf{x} \in M$ is the number $\mathbf{j}(\mathbf{x}) = \dim_{\mathbb{C}} T_{\mathbf{x}} \cap J_{\mathbf{x}} T_{\mathbf{x}}$, where $T_{\mathbf{x}}$ is the real tangent space of M at $\mathbf{x}$ and $J_{\mathbf{x}}$ is the complex structure operator corresponding to scalar multiplication by i on the tangent space of the ambient complex manifold. The number $\mathbf{j}(\mathbf{x})$ is the dimension of the largest complex subspace tangent to M at $\mathbf{x}$, so $0 \leq \mathbf{j}(\mathbf{x}) \leq m/2$.

One way to keep track of $\mathbf{j}(\mathbf{x})$ is the following construction. For $m \leq n$, let G be the grassmannian variety of real m-subspaces in $\mathbb{C}^n \cong \mathbb{R}^{2n}$ (see [**G**], [$\mathbf{C}_1$]), and then the real m-subspaces T such that $\dim_{\mathbb{C}} T \cap iT \geq j$ form a subvariety $\mathcal{D}_j$ of real codimension $2j(n-m+j)$ in G. The occurrence of complex tangents of an immersion corresponds to the intersection of $\mathcal{D}_j$ with the image of the Gauss map $M \to G : \mathbf{x} \mapsto T_{\mathbf{x}}$, and the immersion could be called "generic" if the Gauss map meets each stratum $\mathcal{D}_j \setminus \mathcal{D}_{j+1}$ transversely. So, generic immersions of M in $\mathbb{C}^n$ are totally real outside a codimension $2(n-m+1)$ subset of M, and if $m < \frac{2}{3}(n+1)$, then a generic immersion of M is totally real everywhere. This resembles the bounds in Whitney's embedding and immersion theorems ([$\mathbf{W}_2$], [$\mathbf{W}_3$]). In the range $\frac{2}{3}(n+1) \leq m \leq n$, CR singularities are topologically stable — small smooth perturbations of a generic immersion with a CR singular point will still have a CR singular point. For compact real submanifolds of complex manifolds, there are topological obstructions to the property of being totally real at every point, and the CR singularities can be enumerated by characteristic class formulas (see [$\mathbf{D}_1$], [$\mathbf{D}_2$], [$\mathbf{C}_1$], and their references on this topic).

The case addressed by this paper is $\frac{2}{3}(n+1) \leq m < n$, and $\mathbf{j}(\mathbf{x}) = 1$ — only points $\mathbf{x}$ where exactly one complex line is tangent at $\mathbf{x}$ will be considered, and only in dimension cases where the CR singularity is stable under smooth perturbations of the immersion. As mentioned in the Introduction, the $m = n$ case has a qualitatively different local geometry than the $m < n$ case and is not considered here. The cases $(m,n) = (2,3)$ or $(3,4)$ considered in [$\mathbf{C}_4$], [$\mathbf{C}_5$], fall outside the topological stability range. The case $(m,n) = (4,5)$, considered by [**Beloshapka**], [$\mathbf{D}_1$], [$\mathbf{C}_1$], [$\mathbf{C}_2$], has the lowest dimensions in the range, and the generic singularity is isolated (codimension 4 in M).

3. The quadratic normal form

Let the ambient complex space be $\mathbb{C}^n$, with coordinates $(z_1, \ldots, z_n)$. The real and imaginary parts of the coordinate functions are labeled $z_j = x_j + iy_j$ for $j = 1, \ldots, n$. Let M be a real analytic m-dimensional submanifold embedded in $\mathbb{C}^n$, with $m < n$, and let $\mathbf{x}$ be a point on M at which M is tangent to a complex line but not to any complex 2-plane — in terms of the previous Section, $\mathbf{j}(\mathbf{x}) = 1$, which we regard as a nondegeneracy assumption, since for M in general position, the points where $\mathbf{j}(\mathbf{x}) > 1$ form a subset of higher codimension.

By a translation that moves $\mathbf{x}$ to the origin $\vec{0}$, and then a complex linear transformation of $\mathbb{C}^n$, the tangent space $T = T_{\vec{0}}$ of M can be assumed to be the $(x_1, y_1, x_2, \ldots, x_{m-1})$-space, which contains the z_1-axis. Then there is some neighborhood Δ of the origin in $\mathbb{C}^n$ so that the defining equations of M in Δ are in the form of a graph over a neighborhood of the origin in T:

$$
\begin{aligned}
y_s &= H_s(z_1, \bar{z}_1, x_2, \ldots, x_{m-1}) \\
z_u &= h_u(z_1, \bar{z}_1, x_2, \ldots, x_{m-1}),
\end{aligned} \tag{1}
$$

where for $s = 2, \ldots, m-1$, H_s is a real valued real analytic function, and for $u = m, \ldots, n$, h_u is a complex valued real analytic function, with H_s and h_u defined in a neighborhood of the origin in T, and vanishing to second order at $(x_1, y_1, x_2, \ldots, x_{m-1}) = (0, \ldots, 0)$. The expression "$x_2, \ldots, x_{m-1}$" is abbreviated as just x. So, the defining functions are of the following form:

$$
\begin{aligned}
H_s(z_1, \bar{z}_1, x) &= \alpha_s z_1^2 + \beta_s z_1 \bar{z}_1 + \gamma_s \bar{z}_1^2 + \sum \delta_s^{s_1} z_1 x_{s_1} \\
&\quad + \sum \epsilon_s^{s_1} \bar{z}_1 x_{s_1} + \sum \theta_s^{s_1 s_2} x_{s_1} x_{s_2} + E_s(z_1, \bar{z}_1, x) \\
h_u(z_1, \bar{z}_1, x) &= \alpha_u z_1^2 + \beta_u z_1 \bar{z}_1 + \gamma_u \bar{z}_1^2 + \sum \delta_u^{s_1} z_1 x_{s_1} \\
&\quad + \sum \epsilon_u^{s_1} \bar{z}_1 x_{s_1} + \sum \theta_u^{s_1 s_2} x_{s_1} x_{s_2} + e_u(z_1, \bar{z}_1, x),
\end{aligned}
$$

with E_s, e_u having terms of degree three or higher. Each of these functions can be expressed as the restriction to $\{(z_1, \zeta, x) \in \mathbb{C}^m : \zeta = \bar{z}_1, x = \bar{x}\}$ of an m-variable series with complex coefficients:

$$
\begin{aligned}
H_s(z_1, \zeta, x) &= \alpha_s z_1^2 + \beta_s z_1 \zeta + \gamma_s \zeta^2 + \sum \delta_s^{s_1} z_1 x_{s_1} \\
&\quad + \sum \epsilon_s^{s_1} \zeta x_{s_1} + \sum \theta_s^{s_1 s_2} x_{s_1} x_{s_2} + \sum_{a+b+I \geq 3} E_s^{abI} z_1^a \zeta^b x^I \\
h_u(z_1, \zeta, x) &= \alpha_u z_1^2 + \beta_u z_1 \zeta + \gamma_u \zeta^2 + \sum \delta_u^{s_1} z_1 x_{s_1} \\
&\quad + \sum \epsilon_u^{s_1} \zeta x_{s_1} + \sum \theta_u^{s_1 s_2} x_{s_1} x_{s_2} + \sum_{a+b+I \geq 3} e_u^{abI} z_1^a \zeta^b x^I,
\end{aligned}
$$

where x^I abbreviates $x_2^{i_2}x_3^{i_3}\cdots x_{m-1}^{i_{m-1}}$ and $a+b+I$ abbreviates $a+b+i_2+\ldots+i_{m-1}$. Each of the series in (z_1,ζ,x) converges on some set $\{(z_1,\zeta,x):|z_1|<r,|\zeta|<r,|x_s|<r\}$, $r>0$, to a complex analytic function, with $\gamma_s=\overline{\alpha_s}$, $\epsilon_s^{s_1}=\overline{\delta_2^{s_1}}$, etc., so that $H_s(z_1,\bar z_1,x)$ and $E_s(z_1,\bar z_1,x)$ are real valued.

Definition 3.1. A (formal, with complex coefficient C) monomial of the form $Cz_1^a\zeta^bx^I$ has "degree" $a+b+I$. A power series (convergent or formal) in m variables $e(z_1,\zeta,x)=\sum e^{abI}z_1^a\zeta^bx^I$ is said to have "degree" d if $e^{abI}=0$ for all (a,b,I) such that $a+b+I<d$. Sometimes a series of degree d will be abbreviated $O(d)$. An ordered k-tuple of series $(e_1,\ldots,e_k)$ has degree d if all its components have degree d.

Definition 3.2. Similarly for n variables, a monomial $Cz_1^{a_1}\cdots z_n^{a_n}$ has degree $a_1+\cdots+a_n$, but we will also work with the "weight" $a_1+\cdots+a_{m-1}+2a_m+\cdots+2a_n$. A series $p(\vec z)=\sum p^{a_1\ldots a_n}z_1^{a_1}\ldots z_n^{a_n}$ has "weight" W if $p^{a_1\ldots a_n}=0$ when $a_1+\cdots+a_{m-1}+2a_m+\cdots+2a_n<W$.

We consider two coordinate systems for a neighborhood of the origin in $\mathbb{C}^n$: the previously mentioned $\vec z=(z_1,\ldots,z_n)$, and a new system $\tilde z=(\tilde z_1,\ldots,\tilde z_n)$, with $\tilde z_j=\tilde x_j+\tilde y_j$. The two systems are related by the following "change of coordinates" transformation:

$$\tilde z=\vec z+\vec p(\vec z), \tag{2}$$

where $\vec p(\vec z)=(p_1(\vec z),\ldots,p_n(\vec z))$ and each component p_j is a holomorphic function of $z_1,\ldots,z_n$, whose series expansion has weight 2, and for $j\ge m$, also degree 2. Such a transformation of $\mathbb{C}^n$ has invertible linear part, so it is invertible on some neighborhood of $\vec 0$. In the calculations of this Section, we will neglect considering the size of that neighborhood, and consider only points close enough to the origin, but the size of the domain of $\vec p$ will be important information in later Sections.

The goal of this Section is to establish some nondegeneracy conditions on the defining equations (1), by using complex linear transformations and nonlinear transformations of the form (2) to put the quadratic terms of (1) into a normal form. Similar calculations have already been done in the $m=n$ case, and the $(m,n)=(2,3)$, $(3,4)$, and $(4,5)$ cases, by [**Bishop**], [$\mathbf{C}_4$], [$\mathbf{C}_5$], and [**Beloshapka**], respectively, so we will skip some of the computational details.

As the first special case of a transformation of the form (2) to be used, let $p_1=0$ and let $p_2,\ldots,p_n$ be homogeneous quadratic polynomials in $z_1,\ldots,z_{m-1}$. Using such a transformation, the quadratic terms in h_u that are products of z_1 and x only, without a $\bar z_1$ factor, can be eliminated in the new coordinate system, or their coefficients (α_u, $\delta_u^{s_1}$, $\theta_u^{s_1s_2}$) can be altered to attain any complex values, by a suitable choice of $\vec p$. This transformation may also change some higher-degree terms but does not alter the coefficients β_u, γ_u, $\epsilon_u^{s_1}$. Similarly, the quadratic terms without $\bar z_1$ in each H_s can also be eliminated by a transformation $\tilde{z} = \vec{z} + \vec{p}$, which simultaneously eliminates their conjugates (using $\gamma_s = \overline{\alpha_s}$), leaving only the mixed term $\beta_s z_1 \bar{z}_1$.

The result of this preliminary normalization is that for any CR singular submanifold M of the form (1), there exists a quadratic coordinate transformation of the form (2) with $p_1 = 0$, so that M has the following general normal form. In a local coordinate system $\vec{z}$ in some neighborhood of the CR singularity, the defining equations of M are of the form (1), with

$$\begin{aligned} y_s = H_s(z_1, \bar{z}_1, x) &= \beta_s z_1 \bar{z}_1 + O(3) \\ z_u = h_u(z_1, \bar{z}_1, x) &= \beta_u z_1 \bar{z}_1 + \gamma_u \bar{z}_1^2 + \sum \epsilon_u^{s_1} \bar{z}_1 x_{s_1} + O(3). \end{aligned}$$

At this point we consider which invertible complex linear transformations of $\mathbb{C}^n$ fix the tangent plane T with coordinates (z_1, x). The matrix representation of such a transformation must be of the form $\tilde{z} = \mathbf{A}z$, where:

$$\mathbf{A} = \begin{pmatrix} a_1 & a_2 \dots a_{m-1} & a_m \dots a_n \\ 0 & \mathbf{R} & * \\ 0 & 0 & \mathbf{C} \end{pmatrix}.$$

The entries $a_1, \dots, a_n$ are complex, with $a_1 \neq 0$, the $(m-2) \times (m-2)$ block $\mathbf{R}$ has real entries and a nonzero determinant, and the $(n-m+1) \times (n-m+1)$ block $\mathbf{C}$ has complex entries and a nonzero determinant.

The <u>first nondegeneracy condition</u> is that the $(n-m+1) \times 2$ block of coefficients β_u, γ_u in the functions h_u satisfies:

$$\text{rank} \begin{pmatrix} \beta_m & \gamma_m \\ \vdots & \vdots \\ \beta_n & \gamma_n \end{pmatrix} = 2. \tag{3}$$

In particular, this requires $m < n$. In this nondegenerate case, there is a linear transformation of $\mathbb{C}^n$ which uses the block $\mathbf{C}$ in the above complex matrix to put these coefficients into a row echelon form:

$$\begin{aligned} z_t = h_t(z_1, \bar{z}_1, x) &= \sum \epsilon_t^{s_1} \bar{z}_1 x_{s_1} + O(3) \\ z_{n-1} = h_{n-1}(z_1, \bar{z}_1, x) &= \bar{z}_1^2 + \sum \epsilon_{n-1}^{s_1} \bar{z}_1 x_{s_1} + O(3) \\ z_n = h_n(z_1, \bar{z}_1, x) &= z_1 \bar{z}_1 + \sum \epsilon_n^{s_1} \bar{z}_1 x_{s_1} + O(3), \end{aligned}$$

for $t = m \dots, n-2$, or there are no h_t expressions if $m = n-1$.

As a consequence of the first nondegeneracy condition on the functions h_u, the functions H_s can also be simplified. There is a linear transformation with components $\tilde{z}_s = z_s - i\beta_s z_n$ (where for $s = 2, \dots, m-1$, the complex coefficients $-i\beta_s$ are entries from the $*$ block of the matrix $\mathbf{A}$) that eliminates the $\beta_s z_1 \bar{z}_1$ terms from each function H_s. This may introduce more terms of the form $z_1 x_{s_1}$ or $\bar{z}_1 x_{s_1}$ in the H_s functions, which can be eliminated by $\tilde{z}_s = z_s + p_s$ quadratic transformations as done previously, without re-introducing any $z_1\bar{z}_1$ terms.

A linear transformation of the form $\tilde{z}_1 = z_1 + \sum a_s z_s$, using the block $a_2, \ldots, a_{m-1}$ from the matrix $\mathbf{A}$, can eliminate the terms of the form $\bar{z}_1 x_{s_1}$ from either the h_{n-1} quantity or the h_n quantity, but generally not both at once — we make the choice to eliminate the terms $\sum \epsilon_{n-1}^{s_1} \bar{z}_1 x_{s_1}$ from h_{n-1}. This may introduce more terms of the form $z_1 x_{s_1}$ or $x_{s_1} x_{s_2}$ in the other h_u functions, which can be eliminated by $\tilde{z}_u = z_u + p_u$ quadratic transformations as done previously.

The real and imaginary parts of the coefficients $\epsilon_u^{s_1}$, for $u = m, \ldots, n-2$ and $u = n$, on the terms $\bar{z}_1 x_{s_1}$, $s_1 = 2, \ldots, m-1$, form a real $2(n-m) \times (m-2)$ coefficient matrix, in this expression where the LHS is a column $(n-m)$-vector:

$$\left(\sum \epsilon_u^{s_1} \bar{z}_1 x_{s_1}\right)_{u=m,\ldots,n-2,n} = \tag{4}$$
$$\begin{pmatrix} 1 & i \ldots 0 & 0 \\ \vdots & & \vdots \\ 0 & 0 \ldots 1 & i \end{pmatrix} \begin{pmatrix} \mathrm{Re}(\epsilon_m^2) & \mathrm{Re}(\epsilon_m^3) \ldots & \mathrm{Re}(\epsilon_m^{m-1}) \\ \mathrm{Im}(\epsilon_m^2) & \mathrm{Im}(\epsilon_m^3) \ldots & \mathrm{Im}(\epsilon_m^{m-1}) \\ \vdots & & \vdots \\ \mathrm{Re}(\epsilon_{n-2}^2) & \mathrm{Re}(\epsilon_{n-2}^3) \ldots & \mathrm{Re}(\epsilon_{n-2}^{m-1}) \\ \mathrm{Im}(\epsilon_{n-2}^2) & \mathrm{Im}(\epsilon_{n-2}^3) \ldots & \mathrm{Im}(\epsilon_{n-2}^{m-1}) \\ \mathrm{Re}(\epsilon_n^2) & \mathrm{Re}(\epsilon_n^3) \ldots & \mathrm{Re}(\epsilon_n^{m-1}) \\ \mathrm{Im}(\epsilon_n^2) & \mathrm{Im}(\epsilon_n^3) \ldots & \mathrm{Im}(\epsilon_n^{m-1}) \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_{m-1} \end{pmatrix} \bar{z}_1$$

The <u>second nondegeneracy condition</u> is that this real matrix has rank $2(n-m)$. It follows that the number of x_s directions, $m-2$, must be greater than or equal to the number $2(n-m)$, and this is equivalent to $m \geq \frac{2}{3}(n+1)$, exactly the lower bound of the dimensions of topological stability, as discussed in Section 2.

When the second nondegeneracy condition holds, the real $\mathbf{R}$ block of the matrix $\mathbf{A}$ can transform the x_s variables to put the above real matrix into echelon form, transforming the real and imaginary parts of the $\epsilon_u^{s_1}$ coefficients, without altering the $\bar{z}_1^2$ and $z_1\bar{z}_1$ terms. We get the following quadratic normal form for a nondegenerate CR singularity:

$$\begin{aligned} y_s = H_s(z_1, \bar{z}_1, x) &= E_s(z_1, \bar{z}_1, x) = O(3) \\ z_t = h_t(z_1, \bar{z}_1, x) &= \bar{z}_1 x_{2(t-m+2)} + i\bar{z}_1 x_{2(t-m+2)+1} + e_t(z_1, \bar{z}_1, x) \\ z_{n-1} = h_{n-1}(z_1, \bar{z}_1, x) &= \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x) \\ z_n = h_n(z_1, \bar{z}_1, x) &= z_1\bar{z}_1 + \bar{z}_1 x_2 + i\bar{z}_1 x_3 + e_n(z_1, \bar{z}_1, x), \end{aligned} \tag{5}$$

with $s = 2, \ldots, m-1$, $t = m, \ldots, n-2$, or, again, there are no h_t expressions if $m = n-1$. If $m > \frac{2}{3}(n+1)$, then the $x_2, \ldots, x_{2n-2m+1}$ variables appear in the quadratic part of the normal form but the variables $x_{2n-2m+2}, \ldots, x_{m-1}$ do not. In fact, near the origin, the locus of CR singularities (with $\mathbf{j}(\mathbf{x}) = 1$)

is a codimension $2(n-m+1)$ submanifold of M whose tangent space at the origin is the real subspace with coordinates $x_{2n-2m+2}, \ldots, x_{m-1}$.

Having stated these two nondegeneracy conditions, we are now ready to state the main result:

Proposition 3.3. *Given $\frac{2}{3}(n+1) \leq m < n$, let M be a real analytic m-submanifold of $\mathbb{C}^n$ with a CR singularity at $\mathbf{x}$, with $\mathbf{j}(\mathbf{x}) = 1$. If its local defining equations (of the form* (1)*) satisfy both nondegeneracy conditions (the full rank of coefficient matrices* (3, 4)*) so that they can be put into the form* (5)*, then there exists a holomorphic coordinate change $\tilde{z} = \vec{z} + \vec{p}$ as in* (2)*, in a neighborhood of $\vec{0} \in \mathbb{C}^n$, transforming the equations* (5) *into the following real algebraic normal form:*

$$
\begin{aligned}
(6) \qquad \tilde{y}_s &= 0, \quad s = 2\ldots, m-1 \\
\tilde{z}_t &= \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}), \quad t = m\ldots, n-2 \\
\tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 \\
\tilde{z}_n &= \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3).
\end{aligned}
$$

∎

The real algebraic variety defined by (6) is denoted $\widetilde{M}^{m,n}$, or more briefly $\widetilde{M}$. The example $\widetilde{M}^{4,5}$ is exactly the normal form of [**Beloshapka**]. The Proposition states that any real analytic M satisfying only $\mathbf{j}(\mathbf{x}) = 1$ at a point and both quadratic nondegeneracy conditions is locally biholomorphically equivalent to the real algebraic model. This is the "analytic stability" mentioned in the title, and it is apparently analogous to stability theorems in the singularity theory of smooth maps, where any sufficiently nondegenerate singularity is equivalent under a change of coordinates to a unique polynomial model. The equations for $\widetilde{M}$ resemble the normal forms for smooth maps with cross-cap (or "S_1") singularities, as in [$\mathbf{W}_4$], [**Haefliger**], [**GG**] §VII.4, and $\widetilde{M}$ and the images of the singular maps also have similar structures as a cartesian product when the singularity is not isolated. The main difference between Whitney's normal forms and (6) is that the quantities in (6) are not monomials, and cannot be simultaneously transformed into monomials by holomorphic coordinate transformations in the nondegenerate case. More will be said about the analogies with singularity theory in Section 8.

In the case $m = \frac{2}{3}(n+1)$ when the singularity is isolated, some of the topological invariants mentioned in Section 2 depend on an orientation of M, so it may be useful to consider normalizing transformations that fix a given orientation of the tangent plane T. This corresponds to the real block $\mathbf{R}$ of matrix $\mathbf{A}$ having a positive determinant, and the last equation of the normal form (5) falls into two cases: $z_n = \bar{z}_1(z_1 + x_2 \pm ix_3)$. The two normal forms are equivalent under the biholomorphic transformation $\tilde{z}_3 = -z_3$, but this reverses the orientation of T. In the remaining Sections we will not be concerned with the orientation.

4. A functional equation

To show the existence of a normalizing transformation, we will set up a system of nonlinear functional equations, so that any solution $\vec{p}$ of the system will define a normalizing transformation $\tilde{z} = \vec{z} + \vec{p}$ as in (2). In addition to finding a formal power series solution, we will also have to show that the solution is convergent in some neighborhood of the origin. The method of proof is the rapid convergence technique, as used in [**M**] and [$\mathbf{C}_4$]. Rather than trying to solve the system of equations directly, we first find an approximate solution by solving a related system of linear equations. Iteration of this process gives a sequence of approximations that approach an exact solution. The issue of the domain of convergence of the exact solution was not addressed by [**Beloshapka**], and was left open by [$\mathbf{C}_1$]. In [$\mathbf{C}_1$], each approximate solution in the sequence was constructed only on a domain a fraction of the size of the previous one in the sequence — when the domains shrink to a point, the limit is an exact formal series solution, but no conclusion can be drawn about its analyticity. The new step here, which is crucial for the method of [**M**] to be applicable, is the construction of a sequence of approximate solutions whose domains shrink slowly enough so their diameters are bounded below by a positive constant.

Starting with the quadratic part of the defining equations in normal form (5), we consider the effect of a coordinate change (2). As previously mentioned, the $\tilde{z} = \vec{z} + \vec{p}$ transformation is (at least formally) invertible near $\vec{0}$, and it may be useful to think of $\tilde{z} = \vec{z} + \vec{p}$ as having identity linear part, although there could be linear terms with weight 2, for example, $\tilde{z}_1 = z_1 + a_n z_n$.

In terms of $\tilde{z}$ and $\vec{z}$, consider the system of equations

$$
\begin{aligned}
(7)\quad 0 &= \operatorname{Im}(\tilde{z}_s) = \operatorname{Im}(z_s + p_s(\vec{z})) \\
0 &= \tilde{z}_t - (\bar{\tilde{z}}_1 \tilde{x}_{2(t-m+2)} + i\bar{\tilde{z}}_1 \tilde{x}_{2(t-m+2)+1}) \\
&= z_t + p_t(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}\operatorname{Re}(z_{2(t-m+2)} + p_{2(t-m+2)}(\vec{z})) \\
&\quad - i\overline{(z_1 + p_1(\vec{z}))}\operatorname{Re}(z_{2(t-m+2)+1} + p_{2(t-m+2)+1}(\vec{z})) \\
0 &= \tilde{z}_{n-1} - \bar{\tilde{z}}_1^2 \\
&= z_{n-1} + p_{n-1}(\vec{z}) - \overline{(z_1 + p_1(\vec{z}))}^2 \\
0 &= \tilde{z}_n - \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) \\
&= z_n + p_n(\vec{z}) \\
&\quad - \overline{(z_1 + p_1(\vec{z}))}(z_1 + p_1(\vec{z}) + \operatorname{Re}(z_2 + p_2(\vec{z})) + i\operatorname{Re}(z_3 + p_3(\vec{z}))).
\end{aligned}
$$

In order to get (6) to be the defining equations for M in the $\tilde{z}$ coordinates, the above equalities must hold for points $\vec{z}$ on M and near $\vec{0}$. So, we can replace the $\vec{z}=(z_1,\dots,z_n)$ expressions in (7) by the defining functions (5):

$$\vec{z}=(z_1,x_2+iH_2(z_1,\bar{z}_1,x),\dots,h_n(z_1,\bar{z}_1,x)), \tag{8}$$

to get a system of equations where the RHS functions depend only on z_1, $\bar{z}_1$, x:

$$\begin{aligned}
0 &= \mathrm{Im}(x_s+iH_s+p_s(\vec{z}))=E_s(z_1,\bar{z}_1,x)+\mathrm{Im}(p_s(\vec{z})) \\
0 &= e_t(z_1,\bar{z}_1,x)+p_t(\vec{z})-\overline{p_1(\vec{z})}(x_{2(t-m+2)}+ix_{2(t-m+2)+1}) \\
&\quad -\bar{z}_1(\mathrm{Re}(p_{2(t-m+2)}(\vec{z}))+i\mathrm{Re}(p_{2(t-m+2)+1}(\vec{z}))) \\
&\quad -\overline{p_1(\vec{z})}(\mathrm{Re}(p_{2(t-m+2)}(\vec{z}))+i\mathrm{Re}(p_{2(t-m+2)+1}(\vec{z}))) \\
0 &= e_{n-1}(z_1,\bar{z}_1,x)+p_{n-1}(\vec{z})-2\bar{z}_1\overline{p_1(\vec{z})}-\overline{p_1(\vec{z})}^2 \\
0 &= e_n(z_1,\bar{z}_1,x)+p_n(\vec{z})-\bar{z}_1(p_1(\vec{z})+\mathrm{Re}(p_2(\vec{z}))+i\mathrm{Re}(p_3(\vec{z}))) \\
&\quad -\overline{(p_1(\vec{z}))}(z_1+x_2+ix_3) \\
&\quad -\overline{(p_1(\vec{z}))}(p_1(\vec{z})+\mathrm{Re}(p_2(\vec{z}))+i\mathrm{Re}(p_3(\vec{z}))).
\end{aligned} \tag{9}$$

The components of $\vec{e}=(E_2,\dots,E_{m-1},e_m,\dots,e_n)$ appear in two ways — as terms in each equation of (9), and also in the $\vec{z}$ input (8) for each $p_j(\vec{z})$ in (9), $j=1,\dots,n$. So, given $\vec{e}$, if we happen to have an exact solution $\vec{p}$ of the above system of functional equations, the conclusion of Proposition 3.3 holds and we are done. However, (9) is a nonlinear system in the unknown quantity $\vec{p}$, where in addition to the composition with the given defining functions (8), there are products of the components p_j and their complex conjugates.

As a first step in solving for $\vec{p}$ in terms of $\vec{e}$, consider the following system of simpler equations:

$$\begin{aligned}
0 &= E_s(z_1,\bar{z}_1,x)+\mathrm{Im}(p_s(\vec{z})) \\
0 &= e_t(z_1,\bar{z}_1,x)+p_t(\vec{z})-\overline{p_1(\vec{z})}(x_{2(t-m+2)}+ix_{2(t-m+2)+1}) \\
&\quad -\bar{z}_1(\mathrm{Re}(p_{2(t-m+2)}(\vec{z}))+i\mathrm{Re}(p_{2(t-m+2)+1}(\vec{z}))) \\
0 &= e_{n-1}(z_1,\bar{z}_1,x)+p_{n-1}(\vec{z})-2\bar{z}_1\overline{p_1(\vec{z})} \\
0 &= e_n(z_1,\bar{z}_1,x)+p_n(\vec{z}) \\
&\quad -\bar{z}_1(p_1(\vec{z})+\mathrm{Re}(p_2(\vec{z}))+i\mathrm{Re}(p_3(\vec{z})))-\overline{(p_1(\vec{z}))}(z_1+x_2+ix_3),
\end{aligned} \tag{10}$$

where the $\vec{z}$ input for each p_j is:

$$\vec{z}=(z_1,x_2,\dots,x_{m-1},\bar{z}_1(x_4+ix_5),\dots,\bar{z}_1^2,\bar{z}_1(z_1+x_2+ix_3)). \tag{11}$$

This simplifies $p_j(\vec{z})$ by considering only the linear and quadratic parts of the input (8). Also, the products of p_j are dropped, so that these are (real) linear equations.

To see how the new equations are related to the original system, suppose $\vec{e}$ has degree $d\ge 3$, and that $\vec{p}$ is a solution of (10–11) so that $p_1,\dots,p_{2n-2m+1}$ have weight $\ge d-1$, and $p_{2n-2m+2},\dots,p_n$ have weight $\ge d$. Evaluating the

RHS of (9) with this solution for $\vec{p}$ evidently results in expressions of degree $\geq 2d-2$. Converting these expressions in z_1, $\bar{z}_1$, x to $\tilde{z}_1$, $\bar{\tilde{z}}_1$, $\tilde{x}$ and equating them to the $\tilde{z}$ expressions in (7) gives the higher-order terms of the new defining equations for M in the $\tilde{z}$ coordinate system. (It will be shown later (Theorem 6.5) that in fact for $\vec{z} \in M$ close enough to $\vec{0}$, z_1, $\bar{z}_1$, x are real analytic functions of $\tilde{z}_1$, $\bar{\tilde{z}}_1$, $\tilde{x}$.) So, while a solution $\vec{p}$ of the linearized equations is just an approximation to the solution of the original system, using such a $\vec{p}$ to define a coordinate transformation does have the effect of nearly doubling the order of vanishing of the $\vec{e}$ quantity.

5. A solution of the linear equation

The goal of this Section is to construct a solution $\vec{p}$ of the system of linear equations (10–11), given the higher-order terms of the defining equations, $\vec{e}$. Considering $\vec{p}$ and $\vec{e}$ as formal power series, such a solution exists but is not unique — this fact, together with the approximate doubling of the degree mentioned in the previous Section and iteration of the linearization procedure, is enough to show the (already known, as mentioned previously) formal equivalence of M and $\widetilde{M}$. The solution $\vec{p}$ constructed here will be an n-tuple of series in $\vec{z} = (z_1, \ldots, z_n)$ with the following properties: the size of the domain of convergence of $\vec{p}$ is comparable in a certain sense to the size of the domain of $\vec{e}$, and also a suitable norm of $\vec{p}$ is bounded in terms of a suitable norm of $\vec{e}$.

Notation 5.1. For $\mathbf{r} = (r_1, \ldots, r_N) \in \mathbb{R}^N$, with all $r_j > 0$, define a polydisc in $\mathbb{C}^N$ by

$$\mathbb{D}_{\mathbf{r}} = \{(z_1, \ldots, z_N) : |z_j| < r_j\}.$$

As special cases, let

$$D_r = \mathbb{D}_{(r,r,\ldots,r)} \subseteq \mathbb{C}^m$$

and

$$\Delta_r = \mathbb{D}_{(r,\ldots,r,2r^2,\ldots,2r^2,r^2,3r^2)} \subseteq \mathbb{C}^n,$$

where there are $m-1$ radius lengths r and $n-m-1$ radius lengths $2r^2$, in the $z_m, \ldots, z_{n-2}$ coordinate directions.

The initial assumption on the defining equations is that $\vec{e}(z_1, \bar{z}_1, x) = (E_2, \ldots, E_{m-1}, e_m, \ldots, e_n)$ is real analytic, so there is some $r > 0$ so that each component of $\vec{e}$ is the restriction to $\{\zeta = \bar{z}_1, x = \bar{x}\}$ of a multivariable power series in (z_1, ζ, x) with center $(0, 0, \ldots, 0)$ and complex coefficients which converges on a complex polydisc $D_r \subseteq \mathbb{C}^m$ (or, equivalently, a complex analytic function on D_r).

Notation 5.2. For a complex valued function $e(z_1, \zeta, x)$ of m complex variables, which is defined on some set containing the polydisc D_r, define the norm

$$|e|_r = \sup_{(z_1,\zeta,x)\in D_r} |e(z_1,\zeta,x)|.$$

For an $(n-1)$-tuple $\vec{e} = (E_2, \dots, e_n)$, define

$$|\vec{e}|_r = |E_2|_r + \cdots + |e_n|_r.$$

For a complex valued function $p(z_1, \dots, z_n)$ of n complex variables, which is defined on some set containing the polydisc Δ_r, define the norm

$$\|p\|_r = \sup_{\vec{z}\in\Delta_r} |p(\vec{z})|.$$

With this notation, we can further assume $r > 0$ is small enough so that $|\vec{e}(z_1,\zeta,x)|_r$ is finite. Given $\vec{e}$ with degree ≥ 3, the eventual goal is to find some $\tilde{r}$, $0 < \tilde{r} \leq r$, and a holomorphic map $\vec{p} : \Delta_{\tilde{r}} \to \mathbb{C}^n$, so that the transformation $\tilde{z} = \vec{z} + \vec{p}(\vec{z})$ is a biholomorphism with domain $\Delta_{\tilde{r}}$ taking M to $\widetilde{M}$. That is, if $\vec{z} \in M \cap \Delta_{\tilde{r}}$, then $\tilde{z}$ satisfies (6). However, in this Section we are only looking for $\vec{p}$ that is a solution of (10–11).

Some steps of the Proof of Theorem 5.6 will decompose series into subseries and their complex conjugates, where these preliminary Lemmas on the $|e|_r$ norm will be useful.

Lemma 5.3. *Given $0 < R < r$ and complex coefficients a_{jkI}, b_{jkI}, if*

$$\left|\sum a_{jkI} z_1^j \zeta^k x^I\right|_r \leq K$$

and for complex x with $|x_s| < r$, $j,k = 0,1,2,3,\dots$,

$$\left|\sum_I b_{jkI} x^I\right| \leq \left|\sum_I a_{jkI} x^I\right|,$$

then

$$\left|\sum b_{jkI} z_1^j \zeta^k x^I\right|_R \leq \frac{Kr^2}{(r-R)^2}.$$

Proof. For $(z_1,\zeta,x) \in D_r$, these series are absolutely convergent and equal:

$$\sum a_{jkI} z_1^j \zeta^k x^I = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\left(\sum_I a_{jkI} x^I\right)\zeta^k\right) z_1^j.$$

Using Cauchy's Estimate ([**A**]) twice,

$$\left|\sum_{k=0}^{\infty}\left(\sum_I a_{jkI} x^I\right)\zeta^k\right| \leq \frac{K}{r^j}, \quad \text{and} \quad \left|\sum_I a_{jkI} x^I\right| \leq \frac{K}{r^k r^j}.$$

For $(z_1,\zeta,x)\in D_r$, the series $\sum b_{jkI}z_1^j\zeta^kx^I$ is absolutely convergent, and for $(z_1,\zeta,x)\in D_R$:

$$\begin{aligned}
\left|\sum b_{jkI}z_1^j\zeta^kx^I\right| &= \left|\sum_{j=0}^\infty\left(\sum_{k=0}^\infty\left(\sum_I b_{jkI}x^I\right)\zeta^k\right)z_1^j\right|\\
&\le \sum_{j=0}^\infty\left(\sum_{k=0}^\infty\left|\sum_I b_{jkI}x^I\right||\zeta|^k\right)|z_1|^j\\
&\le \sum_{j=0}^\infty\left(\sum_{k=0}^\infty\left|\sum_I a_{jkI}x^I\right||\zeta|^k\right)|z_1|^j\\
&\le \sum_{j=0}^\infty\left(\sum_{k=0}^\infty\frac{K}{r^kr^j}|\zeta|^k\right)|z_1|^j=\sum_{j,k}K\left(\frac{|\zeta|}{r}\right)^k\left(\frac{|z_1|}{r}\right)^j\\
&= K\frac{1}{1-\frac{|\zeta|}{r}}\frac{1}{1-\frac{|z_1|}{r}}=\frac{Kr^2}{(r-|\zeta|)(r-|z_1|)}<\frac{Kr^2}{(r-R)^2}.
\end{aligned}$$

∎

In the applications of the Lemma, for each pair (j,k), the coefficients b_{jkI} will either be zero for all I or equal to a_{jkI} for all I, so the estimate in the hypothesis is satisfied.

Notation 5.4. On the complex vector space of formal power series, define the following real structure operator:

$$e=\sum e^{abI}z_1^a\zeta^bx^I\mapsto e'=\sum\overline{e^{abI}}\zeta^az_1^bx^I.$$

Lemma 5.5. *Given $r>0$, the restriction of the above map to the subspace $\{e:|e|_r<\infty\}$ is an isometry.*

Proof. The equality of norms uses a change of variables which does not change the radius length r.

$$\begin{aligned}
|e'|_r &= \sup_{(z_1,\zeta,x)\in D_r}\left|\sum\overline{e^{abI}}\zeta^az_1^bx^I\right|\\
&= \sup_{(\zeta',z_1',x')=(\bar z_1,\bar\zeta,\bar x)\in D_r}\left|\sum\overline{e^{abI}}\,\overline{(z_1')}^a\,\overline{(\zeta')}^b\,\overline{(x')}^I\right|\\
&= \sup_{(\zeta',z_1',x')\in D_r}\left|\overline{\sum e^{abI}(z_1')^a(\zeta')^b(x')^I}\right|\\
&= \sup_{(z_1',\zeta',x')\in D_r}\left|\sum e^{abI}(z_1')^a(\zeta')^b(x')^I\right|=|e|_r.
\end{aligned}$$

∎

Of course, this map is a representation of complex conjugation: given a series $e(z_1, \bar{z}_1, x)$ for real x, which "complexifies" to $e = e(z_1, \zeta, x)$ for $(z_1, \zeta, x) \in D_r$ for the purposes of finding its norm as in Notation 5.2, expanding $\overline{e(z_1, \bar{z}_1, x)}$ as a series in $(z_1, \bar{z}_1, x)$ and then complexifying gives $e' = e'(z_1, \zeta, x)$.

In an attempt to simplify the notation by avoiding an excess of indices in an already intricate calculation, the following Theorem will focus on one particular dimension pair (m, n). In order to represent the most general behavior, we want $m < n - 1$, so there is a z_t equation in (5), and also $m > \frac{2}{3}(n + 1)$, so there is a variable x_{m-1} that does not appear in the quadratic part of the defining equations. The smallest pair where both conditions occur is $m = 7$, $n = 9$, so we will be considering a real 7-manifold in $\mathbb{C}^9$, where the coordinates of the tangent plane are $z_1, x_2, \ldots, x_6$, and the CR singular locus in M near $\vec{0}$ is a real curve tangent to the x_6 axis at the origin.

Theorem 5.6. *Given $r > 0$ and $\vec{e}(z_1, \zeta, x)$ convergent on D_r with $|\vec{e}|_r < \infty$ and degree $d \geq 3$, there exists $\vec{p}$ which is convergent on Δ_r, and which solves this case of the system of equations* (10–11)*:*

$$
\begin{aligned}
(12)\quad 0 &= E_s(z_1, \bar{z}_1, x) + \mathrm{Im}(p_s(\vec{z})),\ s = 2, \ldots, 6 \\
(13)\quad 0 &= e_7(z_1, \bar{z}_1, x) + p_7(\vec{z}) \\
&\quad -\overline{p_1(\vec{z})}(x_4 + ix_5) - \bar{z}_1(\mathrm{Re}(p_4(\vec{z})) + i\mathrm{Re}(p_5(\vec{z}))) \\
(14)\quad 0 &= e_8(z_1, \bar{z}_1, x) + p_8(\vec{z}) - 2\bar{z}_1\overline{p_1(\vec{z})} \\
(15)\quad 0 &= e_9(z_1, \bar{z}_1, x) + p_9(\vec{z}) \\
&\quad -\bar{z}_1(p_1(\vec{z}) + \mathrm{Re}(p_2(\vec{z})) + i\mathrm{Re}(p_3(\vec{z}))) - \overline{(p_1(\vec{z}))}(z_1 + x_2 + ix_3),
\end{aligned}
$$

where

$$
\vec{z} = (z_1, x_2, x_3, x_4, x_5, x_6, \bar{z}_1(x_4 + ix_5), \bar{z}_1^2, \bar{z}_1(z_1 + x_2 + ix_3)), \tag{16}
$$

and which satisfies:

$$
\|p_1\|_r \leq \frac{3}{2r}|e_8|_r,\ \ \|p_8\|_r \leq 4|e_8|_r,
$$

and for any $0 < R < r$,

$$\begin{aligned}
\|p_2\|_R &\leq \frac{3|e_9|_r + 18|e_8|_r}{R} + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_2|_r + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_3|_r \\
\|p_3\|_R &\leq \frac{3|e_9|_r + 18|e_8|_r}{R} + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_2|_r + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_3|_r \\
\|p_4\|_R &\leq \frac{3|e_7|_r + 9|e_8|_r}{R} + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_4|_r + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_5|_r \\
\|p_5\|_R &\leq \frac{3|e_7|_r + 9|e_8|_r}{R} + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_4|_r + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_5|_r \\
\|p_6\|_R &\leq \frac{20r^2}{(r-R)^2}|E_6|_r \\
\|p_7\|_R &\leq 4|e_7|_r + 12|e_8|_r + 8R\left(\frac{r^2}{(r-R)^2} + 1\right)(|E_4|_r + |E_5|_r) \\
\|p_9\|_R &\leq 4|e_9|_r + 24|e_8|_r + 8R\left(\frac{r^2}{(r-R)^2} + 1\right)(|E_2|_r + |E_3|_r).
\end{aligned}$$

Proof. First, notice that if $\vec{p}(z_1, \ldots, z_9)$ is a formal series solution of (12–16), it does not follow that $\vec{p}$ is convergent at any point (other than the origin). For example, with any component p_j, the series expressions $p_j(\vec{z})$ and

$$p_j(\vec{z}) + ((z_1 + z_2 + iz_3)^2 z_8 - z_9^2) \cdot \mathrm{Q}(\vec{z}) \tag{17}$$

are formally the same when restricted to $\vec{z}$ as in (16), for any (possibly divergent) series Q. So, if one formal solution $\vec{p}$ exists, then there exist infinitely many divergent solutions. There may also exist formal series solutions that are convergent only on some neighborhood of the origin much smaller than that claimed in the Theorem.

Continuing with the abbreviation $x = x_2, x_3, x_4, x_5, x_6$, and also using $z = z_2, z_3, z_4, z_5, z_6$, the following choice of normalization will simplify the construction of the solution $\vec{p}$ satisfying the claimed convergence and bounds:

$$\begin{aligned}
p_1(\vec{z}) &= p_1(z_1, z, z_8) \\
p_j(\vec{z}) &= p_j^E(z_1, z, z_8) + z_9 p_j^O(z_1, z, z_8),
\end{aligned}$$

for $j = 2, \ldots, 9$. Note that $\vec{p}$ does not depend on z_7, and the first component p_1 does not depend on z_9. We may make the further assumption that p_1 is an even function of z_1: $p_1(z_1, z, z_8) = p_1(-z_1, z, z_8)$. The remaining components, p_j, have some terms not depending on z_9, labeled p_j^E, and other terms which have exactly one linear factor of z_9. The p_j^E and p_j^O terminology corresponds to even and odd powers of $\bar{z}_1$ which appear after the substitution of (16) into $\vec{p}$. The choice that $\vec{p}$ has at most linear terms in $z_9 = \bar{z}_1(z_1 + x_2 + ix_3)$ is made to avoid high powers of the non-monomial quantity $\bar{z}_1(z_1 + x_2 + ix_3)$, since as in [$\mathbf{C}_1$], any multinomial coefficients in the series expansion of $\vec{p}(\vec{z})$ could be large enough to affect the size of the domain of convergence.

We begin with the e_8 equation of the system, (14). If the series expansion of e_8 had only even powers of $\bar{z}_1$, then it would be a very simple matter to compare the coefficients of $e_8(z_1,\bar{z}_1,x)$ and $p_8^E(z_1,x,\bar{z}_1^2)$, and get a solution of the equation with $p_8^O = p_1 = 0$. The odd powers of $\bar{z}_1$ in e_8 make the p_8^O and p_1 quantities necessary to solve the equation. The consideration of the terms of the components of the given quantity $\vec{e}$ which are even or odd in $\bar{z}_1$ was part of the analysis of [**Beloshapka**] and [$\mathbf{C}_1$] of the formal normal form problem, and even/odd decompositions also appeared in analogous calculations in [$\mathbf{W}_1$]. However, to deal with the non-monomial property of the quadratic normal form, there will be some re-arrangements of the terms in the series which were not required in [$\mathbf{W}_1$]. First, decompose e_8 into even and odd parts e_{8A}, e_{8B}, e_{8C}, and then apply an add-and-subtract trick to e_{8C}, as follows:

$$\begin{aligned}
e_8 &= \sum e_8^{abI} z_1^a \bar{z}_1^b x^I = e_{8A}+e_{8B}+e_{8C}\\
e_{8A} &= \sum_{b \text{ even}} e_8^{abI} z_1^a \bar{z}_1^b x^I\\
e_{8B} &= \sum_{a \text{ even},\, b \text{ odd}} e_8^{abI} z_1^a \bar{z}_1^b x^I\\
e_{8C} &= \sum_{a,\, b \text{ odd}} e_8^{abI} z_1^a \bar{z}_1^b x^I = e_{8D}+e_{8E}\\
e_{8D} &= \bar{z}_1(z_1+x_2+ix_3)\sum_{a,\, b \text{ odd}} e_8^{abI} z_1^{a-1} \bar{z}_1^{b-1} x^I\\
e_{8E} &= -(x_2+ix_3)\sum_{a,\, b \text{ odd}} e_8^{abI} z_1^{a-1} \bar{z}_1^{b} x^I.
\end{aligned}$$

Let $f_8 = e_{8B}+e_{8E}$, so $f_8(z_1,\bar{z}_1,x)$ is even in z_1 and odd in $\bar{z}_1$. Then, combining $e_8 = e_{8A}+e_{8D}+f_8$ with the normalization for p_1 and p_8 in the e_8 equation (14), a straightforward (by construction) comparison of coefficients yields the following equalities:

$$\begin{aligned}
0 &= e_{8A}(z_1,\bar{z}_1,x)+p_8^E(z_1,x,\bar{z}_1^2)\\
0 &= e_{8D}(z_1,\bar{z}_1,x)+\bar{z}_1(z_1+x_2+ix_3)p_8^O(z_1,x,\bar{z}_1^2)\\
0 &= f_8(z_1,\bar{z}_1,x)-2\bar{z}_1\overline{p_1(z_1,x,\bar{z}_1^2)}.
\end{aligned}$$

If $p_8^E(z_1,z,z_8)=\sum p_8^{acI}z_1^az_8^cz^I$, then the coefficient p_8^{acI} must be equal to $-e_8^{a,2c,I}$, and we get an estimate for the norm of p_8^E on the polydisc $\Delta_r\subseteq\mathbb{C}^9$:

$$\begin{aligned}\|p_8^E\|_r &= \sup_{\vec{z}\in\Delta_r}|p_8^E(\vec{z})| = \sup_{|z_1|<r,|x_s|<r,|\zeta^2|<r^2}|p_8^E(z_1,x,\zeta^2)|\\ &= \sup_{(z_1,\zeta,x)\in D_r}|-e_{8A}(z_1,\zeta,x)| = |e_{8A}|_r\\ &= \left|\frac{1}{2}(e_8(z_1,\zeta,x)+e_8(z_1,-\zeta,x))\right|_r \leq |e_8|_r.\end{aligned}$$

Note that by using the averaging formula to extract the even part of e_8, we can just apply the triangle inequality to get the estimate for the subseries instead of Lemma 5.3. There is a similar estimate for the other component p_8^O, but this time the Schwarz Lemma ([**A**]) is used in two steps:

$$\begin{aligned}\|z_9p_8^O\|_r &\leq \|z_9\|_r\|p_8^O\|_r = 3r^2\sup_{|z_1|<r,|x_s|<r,|\zeta^2|<r^2}|p_8^O(z_1,x,\zeta^2)|\\ &= 3r^2\sup_{(z_1,\zeta,x)\in D_r}\left|\frac{-e_{8D}(z_1,\zeta,x)}{\zeta(z_1+x_2+ix_3)}\right|\\ &= 3r^2\sup_{(z_1,\zeta,x)\in D_r^*}\left|\frac{-e_{8C}(z_1,\zeta,x)}{z_1\zeta}\right| \leq 3r^2\sup_{(z_1,\zeta,x)\in D_r^*}\frac{\frac{|z_1|}{r}\sup_{|z_1|<r}|e_{8C}|}{|z_1||\zeta|}\\ &\leq 3r\sup_{(z_1,\zeta,x)\in D_r^*}\frac{\sup_{|z_1|<r}\left|\frac{|\zeta|}{r}\sup_{|\zeta|<r}|e_{8C}|\right|}{|\zeta|} = 3|e_{8C}|_r\\ &= 3\left|\frac{e_8(z_1,\zeta,x)-e_8(z_1,-\zeta,x)-e_8(-z_1,\zeta,x)+e_8(-z_1,-\zeta,x)}{4}\right|_r\\ &\leq 3|e_8|_r.\end{aligned}$$

In some of the above steps, we restricted to the open subset $D_r^*=D_r\setminus(\{z_1=0\}\cup\{\zeta=0\})$, which avoids division by 0 but by the maximum principle, does not affect the sup.

From $f_8=e_{8B}+e_{8E}$ and the Schwarz Lemma,

$$\begin{aligned}|f_8|_r &\leq |e_{8B}|_r+|e_{8E}|_r\\ &= \left|\frac{e_8(z_1,\zeta,x)-e_8(z_1,-\zeta,x)+e_8(-z_1,\zeta,x)-e_8(-z_1,-\zeta,x)}{4}\right|_r\\ &\quad+\left|-\frac{x_2+ix_3}{z_1}e_{8C}\right|_r\\ &\leq |e_8|_r+|x_2+ix_3|_r\cdot\frac{1}{r}|e_{8C}|_r \leq 3|e_8|_r.\end{aligned}$$

Solving for p_1 involves complex conjugation, so we take care to work out a few steps. By a comparison of the coefficients of f_8 and p_1, if

$$p_1(z_1, z, z_8) = \sum_{\alpha \text{ even}} p_1^{\alpha\beta I} z_1^{\alpha} z_8^{\beta} z^I,$$

then $p_1^{\alpha\beta I} = \frac{1}{2}\overline{f_8^{2\beta,\alpha+1,I}}$. Using the Schwarz Lemma and Lemma 5.5:

$$\begin{aligned}
\|p_1\|_r &= \sup_{(z_1,\zeta,x)\in D_r} \left| \sum_{\alpha \text{ even}} p_1^{\alpha\beta I} z_1^{\alpha} \zeta^{2\beta} x^I \right| = \sup_{(z_1,\zeta,x)\in D_r} \left| \sum_{\alpha \text{ even}} \frac{1}{2}\overline{f_8^{2\beta,\alpha+1,I}} z_1^{\alpha} \zeta^{2\beta} x^I \right| \\
&= \sup_{(z_1,\zeta,x)\in D_r} \left| \sum_{a \text{ even},\ b \text{ odd}} \frac{\overline{f_8^{abI}} \zeta^a z_1^b x^I}{2z_1} \right| \\
&= \left| \frac{f_8'(z_1,\zeta,x)}{2z_1} \right|_r \le \frac{1}{2r}|f_8'|_r = \frac{1}{2r}|f_8|_r \le \frac{3}{2r}|e_8|_r.
\end{aligned}$$

By construction, p_1 has weight $d-1$ and p_8 has weight d.

Moving next to the E_6 equation of (12), split the real valued series E_6 into subseries, some real and some in complex conjugate pairs:

$$\begin{aligned}
E_6 &= e_{6A} + \overline{e_{6A}} + E_{6B} + e_{6C} + \overline{e_{6C}} + e_{6D} + \overline{e_{6D}} + E_{6E} \\
e_{6A} &= \sum_{a>b,\ b \text{ even}} E_6^{abI} z_1^a \bar{z}_1^b x^I \\
E_{6B} &= \sum_{a \text{ even}} E_6^{aaI} z_1^a \bar{z}_1^a x^I \\
e_{6C} &= \sum_{a>b,\ a \text{ even},\ b \text{ odd}} E_6^{abI} z_1^a \bar{z}_1^b x^I \\
e_{6D} &= \sum_{a>b,\ a,\ b \text{ odd}} E_6^{abI} z_1^a \bar{z}_1^b x^I \\
E_{6E} &= \sum_{a \text{ odd}} E_6^{aaI} z_1^a \bar{z}_1^a x^I.
\end{aligned}$$

By Lemma 5.3, $|e_{6A}|_R \le \frac{r^2}{(r-R)^2}|E_6|_r$, and all the other subseries have the same bound. We rearrange two of these subseries in order to be able to compare coefficients with p_6:

$$\begin{aligned}
e_{6D} &= e_{6F} + e_{6G} \\
e_{6F} &= (z_1 + x_2 + ix_3)\bar{z}_1 \sum_{a>b,\ a,\ b \text{ odd}} E_6^{abI} z_1^{a-1} \bar{z}_1^{b-1} x^I \\
e_{6G} &= -(x_2 + ix_3) \sum_{a>b,\ a,\ b \text{ odd}} E_6^{abI} z_1^{a-1} \bar{z}_1^{b} x^I \\
E_{6E} &= e_{6H} + \overline{e_{6H}} + e_{6I} + \overline{e_{6I}} \\
e_{6H} &= \frac{1}{2}(z_1 + x_2 + ix_3)\bar{z}_1 \sum_{a \text{ odd}} E_6^{aaI} z_1^{a-1} \bar{z}_1^{a-1} x^I \\
e_{6I} &= -\frac{1}{2}(x_2 + ix_3) \sum_{a \text{ odd}} E_6^{aaI} z_1^{a-1} \bar{z}_1^{a} x^I,
\end{aligned}$$

and collect some of these subseries back together:

$$\begin{aligned}
f_{6A}(z_1, \bar{z}_1, x) &= e_{6A} + \overline{e_{6I}} = \sum_{a>b,\ b \text{ even}} f_{6A}^{abI} z_1^a \bar{z}_1^b x^I \\
f_{6C}(z_1, \bar{z}_1, x) &= \overline{e_{6C}} + \overline{e_{6G}} = \sum_{a<b,\ a \text{ odd},\ b \text{ even}} f_{6C}^{abI} z_1^a \bar{z}_1^b x^I,
\end{aligned}$$

so

$$E_6 = f_{6A} + \overline{f_{6A}} + E_{6B} + f_{6C} + \overline{f_{6C}} + e_{6F} + \overline{e_{6F}} + e_{6H} + \overline{e_{6H}}.$$

The unknown p_6 can also be expressed as a sum of subseries:

$$\begin{aligned}
p_6 &= p_6^E(z_1, z, z_8) + z_9 p_6^O(z_1, z, z_8) \\
p_6^E &= p_{6A} + p_{6B} + p_{6C} \\
p_{6A} &= \sum_{\alpha > 2\gamma} p_{6A}^{\alpha\gamma I} z_1^\alpha z^I z_8^\gamma \\
p_{6B} &= \sum p_{6B}^{\gamma I} z_1^{2\gamma} z^I z_8^\gamma \\
p_{6C} &= \sum_{\alpha < 2\gamma,\ \alpha \text{ odd}} p_{6C}^{\alpha\gamma I} z_1^\alpha z^I z_8^\gamma \\
p_6^O &= p_{6D} + p_{6E} \\
p_{6D} &= \sum_{\alpha > 2\gamma,\ \alpha \text{ even}} p_{6D}^{\alpha\gamma I} z_1^\alpha z^I z_8^\gamma \\
p_{6E} &= \sum p_{6E}^{\gamma I} z_1^{2\gamma} z^I z_8^\gamma.
\end{aligned}$$

Comparing coefficients, the equation $0 = E_6 + \frac{1}{2i}(p_6 - \overline{p_6})$ from (12) turns into these five equations and their complex conjugates:

$$\begin{aligned}
0 &= f_{6A} + \frac{1}{2i} p_{6A} \\
0 &= \frac{1}{2} E_{6B} + \frac{1}{2i} p_{6B} \\
0 &= f_{6C} + \frac{1}{2i} p_{6C} \\
0 &= e_{6F} + \frac{1}{2i}(z_1 + x_2 + ix_3)\bar{z}_1 p_{6D} \\
0 &= e_{6H} + \frac{1}{2i}(z_1 + x_2 + ix_3)\bar{z}_1 p_{6E}.
\end{aligned}$$

Solving for each component of p_6 gives a weight d quantity, and using Lemma 5.5, the Schwarz Lemma, and the previously mentioned estimates for the subseries of E_6, we get these estimates:

$$\begin{aligned}
\|p_{6A}\|_R &= |-2if_{6A}|_R = 2|e_{6A} + e'_{6I}|_R \leq 2(|e_{6A}|_R + |e_{6I}|_R) \\
&\leq 2\left(|e_{6A}|_R + \left|\frac{-(x_2 + ix_3)E_{6E}}{2z_1}\right|_R\right) \\
&\leq 2\left(\frac{r^2|E_6|_r}{(r-R)^2} + \frac{2R}{2}\cdot\frac{1}{R}\cdot\frac{r^2|E_6|_r}{(r-R)^2}\right) = \frac{4r^2}{(r-R)^2}|E_6|_r, \\
\|p_{6B}\|_R &= |-2i\cdot\frac{1}{2}E_{6B}|_R \leq \frac{r^2}{(r-R)^2}|E_6|_r, \\
\|p_{6C}\|_R &= |-2if_{6C}|_R = 2|e'_{6C} + e'_{6G}|_R \\
&\leq 2\left(|e_{6C}|_R + \left|\frac{-(x_2 + ix_3)e_{6D}}{z_1}\right|_R\right) \\
&\leq 2\left(\frac{r^2|E_6|_r}{(r-R)^2} + 2R\cdot\frac{1}{R}\cdot\frac{r^2|E_6|_r}{(r-R)^2}\right) = \frac{6r^2}{(r-R)^2}|E_6|_r, \\
\|z_9 p_{6D}\|_R &\leq \|z_9\|_R\|p_{6D}\|_R = 3R^2\left|-2i\frac{e_{6F}}{(z_1 + x_2 + ix_3)\zeta}\right|_R = 6R^2\left|\frac{e_{6D}}{z_1\zeta}\right|_R \\
&\leq 6|e_{6D}|_R \leq \frac{6r^2}{(r-R)^2}|E_6|_r, \\
\|z_9 p_{6E}\|_R &\leq 3R^2\left|-2i\frac{e_{6H}}{(z_1 + x_2 + ix_3)\zeta}\right|_R = 6R^2\left|\frac{e_{6E}}{2z_1\zeta}\right|_R \leq \frac{3r^2}{(r-R)^2}|E_6|_r.
\end{aligned}$$

Finding p_2, p_3, p_4, p_5, is a little trickier since each appears in more than one equation of (12–15). We will simultaneously solve for p_4, p_5, p_7, using a more involved comparison of coefficients, and similarly but independently, also p_2, p_3, p_9.

To find p_4, p_5, and p_7, we consider the E_4, E_5, e_7 equations of (12, 13), and use the previously found solution for p_1 to get the system with the unknowns on the LHS and the known $O(d)$ quantities on the RHS:

$$\mathrm{Im}(p_4) = -E_4 \tag{18}$$
$$\mathrm{Im}(p_5) = -E_5 \tag{19}$$
$$p_7 - \bar{z}_1\mathrm{Re}(p_4) - i\bar{z}_1\mathrm{Re}(p_5) = -e_7 + (x_4 + ix_5)\overline{p_1} \tag{20}$$

Starting with the RHS of (18), the following decomposition of E_4 is different from that of E_6:

$$\begin{aligned}
E_4 &= E_{4A} + e_{4B} + \overline{e_{4B}} + e_{4C} + \overline{e_{4C}} + E_{4D} \\
E_{4A} &= \sum_{a,\ b \text{ even}} E_4^{abI} z_1^a \bar{z}_1^b x^I \\
e_{4B} &= \sum_{a>b,\ a \text{ odd},\ b \text{ even}} E_4^{abI} z_1^a \bar{z}_1^b x^I \\
e_{4C} &= \sum_{a>b,\ a \text{ even},\ b \text{ odd}} E_4^{abI} z_1^a \bar{z}_1^b x^I \\
E_{4D} &= \sum_{a,\ b \text{ odd}} E_4^{abI} z_1^a \bar{z}_1^b x^I \\
&= e_{4E} + e_{4F} \\
e_{4E} &= (z_1 + x_2 + ix_3)\bar{z}_1 \sum_{a,\ b \text{ odd}} E_4^{abI} z_1^{a-1} \bar{z}_1^{b-1} x^I \\
e_{4F} &= -(x_2 + ix_3) \sum_{a,\ b \text{ odd}} E_4^{abI} z_1^{a-1} \bar{z}_1^b x^I.
\end{aligned}$$

The E_{4A} piece is simply an even part, so $|E_{4A}|_r \leq |E_4|_r$, and similarly for the odd part, $|E_{4D}|_r \leq |E_4|_r$. The other two subseries satisfy the estimate from Lemma 5.3: $|e_{4B}|_R \leq \frac{r^2}{(r-R)^2}|E_4|_r$ and $|e_{4C}|_R \leq \frac{r^2}{(r-R)^2}|E_4|_r$. We re-group some of these subseries:

$$\begin{aligned}
m_4(z_1, \bar{z}_1, x) &= \overline{e_{4B}} + e_{4C} + e_{4F} = \sum_{a \text{ even},\ b \text{ odd}} m_4^{abI} z_1^a \bar{z}_1^b x^I \\
f_4(z_1, \bar{z}_1, x) &= E_{4A} + e_{4B} + \overline{e_{4C}} + \overline{m_4} = \sum_{b \text{ even}} f_4^{abI} z_1^a \bar{z}_1^b x^I \\
\implies E_4 &= f_4 + e_{4E} + m_4 - \overline{m_4}.
\end{aligned}$$

Similarly, $E_5 = f_5 + e_{5E} + m_5 - \overline{m_5}$, where f_5 is even in $\bar{z}_1$ and m_5 is even in z_1 and odd in $\bar{z}_1$.

The estimates follow from Lemma 5.5 and the Schwarz Lemma:

$$\begin{aligned}
|m_4|_R &= |e'_{4B} + e_{4C} + e_{4F}|_R \le |e_{4B}|_R + |e_{4C}|_R + |e_{4F}|_R \\
&\le \frac{r^2}{(r-R)^2}|E_4|_r + \frac{r^2}{(r-R)^2}|E_4|_r + |x_2 + ix_3|_R \left|\frac{E_{4D}}{z_1}\right|_R \\
&\le \left(\frac{2r^2}{(r-R)^2} + 2\right)|E_4|_r \\
|f_4|_R &= |E_{4A} + 2e_{4B} + 2e'_{4C} + e'_{4F}|_R \le \left(\frac{4r^2}{(r-R)^2} + 3\right)|E_4|_r,
\end{aligned}$$

and similarly for m_5 and f_5.

From the RHS of (20), let $f_7(z_1, \bar{z}_1, x) = -e_7 + (x_4 + ix_5)\overline{p_1(z_1, x, \bar{z}_1^2)}$, so

$$|f_7|_r \le |-e_7|_r + \left|(x_4 + ix_5)\frac{f_8}{2\zeta}\right|_r \le |e_7|_r + 3|e_8|_r. \tag{21}$$

It splits into even and odd parts, $f_7 = f_{7A} + f_{7B} + f_{7C}$, with:

$$\begin{aligned}
f_{7A} &= \sum_{b \text{ even}} f_{7A}^{abI} z_1^a \bar{z}_1^b x^I \\
f_{7B} &= \sum_{a,\, b \text{ odd}} f_{7B}^{abI} z_1^a \bar{z}_1^b x^I \\
f_{7C} &= \sum_{a \text{ even},\, b \text{ odd}} f_{7C}^{abI} z_1^a \bar{z}_1^b x^I,
\end{aligned}$$

with $|f_{7A}|_r \le |f_7|_r$ and the same bound for f_{7B}, f_{7C}. Let:

$$\begin{aligned}
g_{7A} &= f_{7A} + i\bar{z}_1 m_4 - \bar{z}_1 m_5 = \sum_{b \text{ even}} g_{7A}^{abI} z_1^a \bar{z}_1^b x^I \\
g_{7B} &= f_{7B} - i\bar{z}_1\overline{m_4} + \bar{z}_1\overline{m_5} = \sum_{a,\, b \text{ odd}} g_{7B}^{abI} z_1^a \bar{z}_1^b x^I = g_{7C} + g_{7D} \\
g_{7C} &= (z_1 + x_2 + ix_3)\bar{z}_1 \sum_{a,\, b \text{ odd}} g_{7B}^{abI} z_1^{a-1} \bar{z}_1^{b-1} x^I \\
g_{7D} &= -(x_2 + ix_3) \sum_{a,\, b \text{ odd}} g_{7B}^{abI} z_1^{a-1} \bar{z}_1^b x^I.
\end{aligned}$$

Then

$$f_7 = g_{7A} + g_{7C} + g_{7D} + f_{7C} - \bar{z}_1(im_4 - i\overline{m_4} - m_5 + \overline{m_5})$$

is in a form that compares to the LHS of (20) to give three equations:

$$p_7^E = g_{7A} \tag{22}$$

$$(z_1 + x_2 + ix_3)\bar{z}_1 p_7^O = g_{7C} \tag{23}$$

$$-\bar{z}_1(\mathrm{Re}(p_4) + i\mathrm{Re}(p_5)) = g_{7D} + f_{7C} - \bar{z}_1(im_4 - i\overline{m_4} - m_5 + \overline{m_5}). \tag{24}$$

Equations (22), (23) determine p_7, with the estimates

$$\begin{aligned}
\|p_7^E\|_R &= |g_{7A}|_R = |f_{7A} + i\zeta m_4 - \zeta m_5|_R \\
&\leq |f_7|_r + R\cdot\left(\frac{2r^2}{(r-R)^2}+2\right)|E_4|_r + R\cdot\left(\frac{2r^2}{(r-R)^2}+2\right)|E_5|_r \\
\|z_9 p_7^O\|_R &\leq 3R^2\left|\frac{g_{7C}}{(z_1+x_2+ix_3)\zeta}\right|_R = 3R^2\left|\frac{g_{7B}}{z_1\zeta}\right|_R \\
&\leq 3|f_{7B} - i\zeta m_4' + \zeta m_5'|_R \\
&\leq 3|f_7|_r + 3R\left(\frac{2r^2}{(r-R)^2}+2\right)|E_4|_r + 3R\left(\frac{2r^2}{(r-R)^2}+2\right)|E_5|_r.
\end{aligned}$$

Dividing (24) by $-\bar{z}_1$, then considering the real and imaginary parts and recalling (18), (19), we get the system:

$$\begin{aligned}
\mathrm{Re}(p_4) &= \mathrm{Re}\left(\frac{g_{7D}+f_{7C}}{-\bar{z}_1}\right) + im_4 - i\overline{m_4} \\
\mathrm{Re}(p_5) &= \mathrm{Im}\left(\frac{g_{7D}+f_{7C}}{-\bar{z}_1}\right) + im_5 - i\overline{m_5} \\
\mathrm{Im}(p_4) &= -E_4 = -f_4 - e_{4E} - m_4 + \overline{m_4} \\
\mathrm{Im}(p_5) &= -E_5 = -f_5 - e_{5E} - m_5 + \overline{m_5}.
\end{aligned}$$

Note that it is at this point where the second nondegeneracy condition (the full rank of coefficient matrix (4)) is used — if the quadratic term $\epsilon_7^5\bar{z}_1x_5$ in h_7 had coefficient 0 instead of i, then $\mathrm{Re}(p_5)$ would not appear in (13), and (20) could not be solved this way.

Recombining the real and imaginary parts of p_4, p_5, there is (by construction) a convenient cancellation:

$$\begin{aligned}
p_4 &= \mathrm{Re}(p_4) + i\mathrm{Im}(p_4) = \mathrm{Re}\left(\frac{g_{7D}+f_{7C}}{-\bar{z}_1}\right) - if_4 - ie_{4E} \\
p_5 &= \mathrm{Re}(p_5) + i\mathrm{Im}(p_5) = \mathrm{Im}\left(\frac{g_{7D}+f_{7C}}{-\bar{z}_1}\right) - if_5 - ie_{5E}.
\end{aligned}$$

These equations were set up so that e_{4E} and e_{5E} are the only RHS terms with odd powers of $\bar{z}_1$, so $p_4 = p_4^E + z_9 p_4^O$ and $p_5 = p_5^E + z_9 p_5^O$ are each determined by a comparison of coefficients, and by construction, p_4 and p_5

have weight $d-1$ and satisfy the estimates:

$$\begin{aligned}
\|p_4^E\|_R &= \left|\frac{g_{7D}}{-2\zeta}+\frac{f_{7C}}{-2\zeta}+\frac{g'_{7D}}{-2z_1}+\frac{f'_{7C}}{-2z_1}-if_4\right|_R\\
&\leq \frac{1}{R}|g_{7D}|_R+\frac{1}{R}|f_{7C}|_R+|f_4|_R\\
&\leq \frac{1}{R}\left|-(x_2+ix_3)\frac{g_{7B}}{z_1}\right|_R+\frac{1}{R}|f_7|_R+|f_4|_R\\
&\leq \frac{3|e_7|_r+9|e_8|_r}{R}+\left(\frac{8r^2}{(r-R)^2}+7\right)|E_4|_r+\left(\frac{4r^2}{(r-R)^2}+4\right)|E_5|_r\\
\|p_5^E\|_R &= \left|\frac{g_{7D}}{-2i\zeta}+\frac{f_{7C}}{-2i\zeta}-\frac{g'_{7D}}{-2iz_1}-\frac{f'_{7C}}{-2iz_1}-if_5\right|_R\\
&\leq \frac{3|e_7|_r+9|e_8|_r}{R}+\left(\frac{4r^2}{(r-R)^2}+4\right)|E_4|_r+\left(\frac{8r^2}{(r-R)^2}+7\right)|E_5|_r
\end{aligned}$$

$$\begin{aligned}
\|z_9p_4^O\|_R &\leq \|z_9p_4^O\|_r\leq 3r^2\left|\frac{-ie_{4E}}{(z_1+x_2+ix_3)\zeta}\right|_r=3r^2\left|\frac{E_{4D}}{z_1\zeta}\right|_r\leq 3|E_4|_r\\
\|z_9p_5^O\|_R &\leq \|z_9p_5^O\|_r\leq 3r^2\left|\frac{-ie_{5E}}{(z_1+x_2+ix_3)\zeta}\right|_r=3r^2\left|\frac{E_{5D}}{z_1\zeta}\right|_r\leq 3|E_5|_r.
\end{aligned}$$

The method of finding p_2, p_3, p_9 can be copied from the solution of p_4, p_5, p_7. In the place of (18), (19), (20), the system to be solved is

$$\begin{aligned}
\mathrm{Im}(p_2) &= -E_2\\
\mathrm{Im}(p_3) &= -E_3\\
p_9-\bar z_1\mathrm{Re}(p_2)-i\bar z_1\mathrm{Re}(p_3) &= -e_9+(z_1+x_2+ix_3)\overline{p_1}+\bar z_1p_1,
\end{aligned}\tag{25}$$

and the RHS of the third equation can be abbreviated f_9, in analogy with f_7. The estimate (21) changes to:

$$|f_9|_r\leq|-e_9|_r+\left|(z_1+x_2+ix_3)\frac{f_8}{2\zeta}\right|_r+\left|\zeta\frac{f'_8}{2z_1}\right|_r\leq|e_9|_r+6|e_8|_r.$$

Both the construction of the solution and the estimates proceed by only changing the subscripts from $4,5,7$ to $2,3,9$, and adjusting the estimate for f_9 to get the claimed results — the second nondegeneracy condition on the quadratic part of h_9 is used here also in the same way. ∎

Corollary 5.7. *Given* $\frac{2}{3}(n+1)\leq m<n$, $r>0$, *and* $\vec e(z_1,\zeta,x)$ *convergent on* D_r *with* $|\vec e|_r<\infty$ *and degree* $d\geq 3$, *there exists* $\vec p$ *which is convergent on* Δ_r, *and which solves the system of equations* (10–11), *and which satisfies:*

$$\|p_1\|_r\leq\frac{3}{2r}|e_{n-1}|_r,\ \|p_{n-1}\|_r\leq 4|e_{n-1}|_r,$$

and for any $0 < R < r$,

$$
\begin{aligned}
\|p_2\|_R &\leq \frac{3|e_n|_r + 18|e_{n-1}|_r}{R} \\
&\quad + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_2|_r + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_3|_r \\
\|p_3\|_R &\leq \frac{3|e_n|_r + 18|e_{n-1}|_r}{R} \\
&\quad + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_2|_r + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_3|_r \\
\|p_n\|_R &\leq 4|e_n|_r + 24|e_{n-1}|_r + 8R\left(\frac{r^2}{(r-R)^2} + 1\right)(|E_2|_r + |E_3|_r).
\end{aligned}
$$

Further, if $m > \frac{2}{3}(n+1)$*, then for* $s = 2n - 2m + 2, \dots, m-1$,

$$
\|p_s\|_R \leq \frac{20r^2}{(r-R)^2}|E_s|_r,
$$

and if $m < n - 1$*, then for* $t = m, \dots, n-2$,

$$
\begin{aligned}
\|p_{2(t-m+2)}\|_R &\leq \frac{3|e_t|_r + 9|e_{n-1}|_r}{R} + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_{2(t-m+2)}|_r \\
&\quad + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_{2(t-m+2)+1}|_r \\
\|p_{2(t-m+2)+1}\|_R &\leq \frac{3|e_t|_r + 9|e_{n-1}|_r}{R} + \left(\frac{4r^2}{(r-R)^2} + 4\right)|E_{2(t-m+2)}|_r \\
&\quad + \left(\frac{8r^2}{(r-R)^2} + 10\right)|E_{2(t-m+2)+1}|_r \\
\|p_t\|_R &\leq 4|e_t|_r + 12|e_{n-1}|_r \\
&\quad + 8R\left(\frac{r^2}{(r-R)^2} + 1\right)(|E_{2(t-m+2)}|_r + |E_{2(t-m+2)+1}|_r).
\end{aligned}
$$

Proof. The method of solution from the Proof of Theorem 5.6 groups the system of equations into smaller sub-systems that can be solved sequentially, so the generalization from $(7,9)$ to (m,n) can be accomplished by a straightforward re-labeling of subscripts (described below), resulting in similar estimates as claimed. The nondegeneracy conditions remain essential for any (m,n).

The solution claimed by the Corollary can be chosen to have the following form, where now z abbreviates $z_2, \dots, z_{m-1}$ and again p_1 is even in z_1:

$$
\begin{aligned}
p_1(\vec{z}) &= p_1(z_1, z, z_{n-1}) \\
p_j(\vec{z}) &= p_j^E(z_1, z, z_{n-1}) + z_n p_j^O(z_1, z, z_{n-1}),
\end{aligned}
$$

for $j = 2, \dots, n$. For $m < n - 1$, this $\vec{p}$ does not depend on $z_m, \dots, z_{n-2}$.

The e_{n-1} equation from (10) determines p_{n-1} and p_1, exactly as in the solution of (14), replacing the subscript 8 with $n-1$ in the first part of the above Proof. The subscript 1 does not change.

If $m > \frac{2}{3}(n+1)$, then each of the $3m-2n-2$ individual E_s equations, $s = 2n-2m+2, \ldots, m-1$, independently determines p_s, in analogy with the solution for p_6 in terms of E_6 in the second part of the above Proof. If $m = \frac{2}{3}(n+1)$ (the case of an isolated singularity), there are no equations analogous to the above Proof's E_6 equation.

The sub-system of three equations determining p_2, p_3, p_n in terms of E_2, E_3, e_n, and p_1, can be solved in analogy with the above E_2, E_3, e_9 group of equations (25), only the subscript 9 needs to change to n. If $m = n-1$, then those three equations are the only remaining ones in the system.

If $m < n-1$, then there are $n-m-1$ more sub-systems of three equations, to be solved for $p_{2(t-m+2)}$, $p_{2(t-m+2)+1}$, p_t, $t = m, \ldots, n-2$, in terms of $E_{2(t-m+2)}$, $E_{2(t-m+2)+1}$, e_t, and p_1, in analogy with equations (18–20). Solving each of these sub-systems depends only on having solved for p_1, and not any other equations in the system (10). ∎

It is not yet claimed that using the solution $\vec{p}$ of Theorem 5.6 or Corollary 5.7 in (2) defines a local biholomorphism; this will be shown later (Theorem 6.4), under certain conditions on $\vec{e}$ and r. The most important property so far of the solution $\vec{p}$ is that the norms of its components can be estimated on Δ_R for R less than, but arbitrarily close to, r.

Corollary 5.8. *Given $\frac{2}{3}(n+1) \leq m < n$, there is a constant $c_1 > 0$ (depending only on m, n), such that for any $\vec{p}$ and $\vec{e}$ as in Corollary 5.7, and any radius lengths ρ, r with $\frac{1}{2} < \rho < r \leq 1$, the following hold:*

$$\max_{j=1,\ldots,n} \{\|p_j\|_\rho\} \leq \frac{c_1|\vec{e}|_r}{(r-\rho)^2}$$

$$\max_{j=1,\ldots,n} \left\{ \sum_{k=1}^{n} \left\| \frac{dp_k}{dz_j} \right\|_\rho \right\} \leq \frac{c_1|\vec{e}|_r}{(r-\rho)^3}.$$

Proof. Let $R = \frac{1}{2}(\rho + r)$. The bound on each p_j follows from $\|p_j\|_\rho \leq \|p_j\|_R$ and the bounds from the previous Corollary, using $\frac{1}{2} < R < r \leq 1$, and $16 < \frac{1}{(r-R)^2} = \frac{4}{(r-\rho)^2}$. The bounds for the derivatives of p_k follow from this consequence ([**A**]) of Cauchy's estimate:

If $0 < R_2 < R_1$ and $f(w)$ is holomorphic and bounded by K for $|w| < R_1$, then $\frac{df}{dw}$ is bounded by $\frac{K}{R_1 - R_2}$ for $|w| < R_2$.

This fact can be applied with $K = \|p_k\|_R$ and $R_1 - R_2 = R - \rho = \frac{1}{2}(r-\rho)$ for the $z_1, \ldots, z_{m-1}$ derivatives, $R_1 - R_2 = R^2 - \rho^2 > R - \rho = \frac{1}{2}(r-\rho)$ for the z_{n-1} derivatives, and $R_1 - R_2 = 3R^2 - 3\rho^2 > \frac{3}{2}(r-\rho)$ for the z_n derivatives. The $z_m, \ldots, z_{n-2}$ derivatives are zero by construction. ∎

The lower bound $r > \frac{1}{2}$ was important for the previous Corollary, but it is not a significant *a priori* restriction on the manifold M. By a real rescaling $\vec{z} \mapsto (a_1 z_1, \ldots, a_1 z_{m-1}, a_1^2 z_m, \ldots, a_1^2 z_n)$, $a_1 > 0$, the equations (5) can be assumed to define M for $|z_1| < 1$, $|x_s| < 1$, and for any $\eta > 0$, there is a rescaling making $|\vec{e}|_1 < \eta$.

6. The new defining equations and some estimates

To get a solution of the nonlinear equation (9) by iterating the solution of the linear equation, the rapid convergence technique will apply, closely following the methods used by [**M**] on a different CR singularity problem. Each step along the way to a proof of Proposition 3.3 is stated as a Theorem.

Substituting the linear equation's normalized solution $\vec{p}$ from Corollary 5.7 into $E_2, \ldots, e_n$ in the RHS of the nonlinear equation (9) gives a quantity $\vec{q}$ depending on z_1, $\bar{z}_1$, x. Let

$$\vec{z} = (z_1, x, \ldots, \bar{z}_1(x_{2(t-m+2)} + ix_{2(t-m+2)+1}), \ldots, \bar{z}_1^2, \bar{z}_1(z_1 + x_2 + ix_3)), \tag{26}$$

as in (11), let

$$\begin{aligned} \vec{z} + \vec{e} &= (z_1, x_2 + iE_2, \ldots, x_{m-1} + iE_{m-1}, \\ &\quad \ldots, \bar{z}_1(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) + e_t, \ldots, \\ &\quad \bar{z}_1^2 + e_{n-1}, \bar{z}_1(z_1 + x_2 + ix_3) + e_n), \end{aligned} \tag{27}$$

as in (8), and then define $\vec{q}(z_1, \bar{z}_1, x) = (Q_2, \ldots, Q_{m-1}, q_m, \ldots, q_n)$ by:

$$Q_s = \mathrm{Im}(p_s(\vec{z} + \vec{e}) - p_s(\vec{z})) \tag{28}$$

$$\begin{aligned} q_t &= p_t(\vec{z} + \vec{e}) - p_t(\vec{z}) \\ &\quad - (x_{2(t-m+2)} + ix_{2(t-m+2)+1})\overline{(p_1(\vec{z} + \vec{e}) - p_1(\vec{z}))} \\ &\quad - \bar{z}_1 \cdot \mathrm{Re}(p_{2(t-m+2)}(\vec{z} + \vec{e}) - p_{2(t-m+2)}(\vec{z})) \\ &\quad - i\bar{z}_1 \cdot \mathrm{Re}(p_{2(t-m+2)+1}(\vec{z} + \vec{e}) - p_{2(t-m+2)+1}(\vec{z})) \\ &\quad - \overline{p_1(\vec{z} + \vec{e})}(\mathrm{Re}(p_{2(t-m+2)}(\vec{z} + \vec{e})) + i\mathrm{Re}(p_{2(t-m+2)+1}(\vec{z} + \vec{e}))) \end{aligned} \tag{29}$$

$$\begin{aligned} q_{n-1} &= p_{n-1}(\vec{z} + \vec{e}) - p_{n-1}(\vec{z}) \\ &\quad - 2\bar{z}_1\overline{(p_1(\vec{z} + \vec{e}) - p_1(\vec{z}))} - \overline{(p_1(\vec{z} + \vec{e}))}^2 \end{aligned} \tag{30}$$

$$\begin{aligned} q_n &= p_n(\vec{z} + \vec{e}) - p_n(\vec{z}) - \bar{z}_1(p_1(\vec{z} + \vec{e}) - p_1(\vec{z})) \\ &\quad - \bar{z}_1(\mathrm{Re}(p_2(\vec{z} + \vec{e}) - p_2(\vec{z})) + i\mathrm{Re}(p_3(\vec{z} + \vec{e}) - p_3(\vec{z}))) \\ &\quad - (z_1 + x_2 + ix_3)\overline{(p_1(\vec{z} + \vec{e}) - p_1(\vec{z}))} \\ &\quad - \overline{(p_1(\vec{z} + \vec{e}))}(p_1(\vec{z} + \vec{e}) + \mathrm{Re}(p_2(\vec{z} + \vec{e})) + i\mathrm{Re}(p_3(\vec{z} + \vec{e}))). \end{aligned} \tag{31}$$

To outline the role of $\vec{q}$ in the argument, the next step (Theorem 6.2) will suppose $\vec{p}(z_1, \ldots, z_n)$ is complex analytic on Δ_ρ, and $|\vec{e}|_\sigma$ is small enough so that $\vec{z} \in \Delta_\sigma \implies \vec{z} + \vec{e} \in \Delta_\rho$, and so $\vec{q}$ is a real analytic function for $(z_1, \bar{z}_1, x) \in D_\sigma$. If $\vec{q}(z_1, \bar{z}_1, x)$ happens to be identically zero, then the manifold M has been brought to normal form by the functions $\vec{p}$. Otherwise, the degree of $\vec{q}$ is at least $2d-2$ by the construction of the solution $\vec{p}$, and defining $\vec{q}(z_1,\zeta,x)$ by (28–31), with ζ formally substituted for $\bar{z}_1$ and allowing complex x, the norm $|\vec{q}|_\sigma$ can be bounded in terms of the norm of $\vec{e}$. Then later, in the Proof of Theorem 6.6, converting $\vec{q}(z_1,\bar{z}_1,x)$ into an expression in $\tilde{z}_1$, $\bar{\tilde{z}}_1$, $\tilde{x}$ and equating it to the $\tilde{z}$ polynomial expression in (7) gives the defining equations of M in the $\tilde{z}$ coordinate system.

The $\tilde{N}=N$ case of the following Lemma is proved in [$\mathbf{C}_4$] (Lemma 4.1.).

Lemma 6.1. *Let $f=(f_1,\ldots,f_{\tilde{N}}):\mathbb{D}_{\mathbf{r}}\to\mathbb{C}^{\tilde{N}}$ be a holomorphic map with*

$$\max_{j=1,\ldots,N}\left\{\sum_{k=1}^{\tilde{N}}\sup_{\vec{z}\in\mathbb{D}_{\mathbf{r}}}\left|\frac{df_k}{dz_j}(\vec{z})\right|\right\}\le K.$$

Then, for $\vec{z}$, $\vec{z}\,'\in\mathbb{D}_{\mathbf{r}}$,

$$\sum_{k=1}^{\tilde{N}}|f_k(\vec{z}\,')-f_k(\vec{z})|\le K\sum_{j=1}^{N}|z_j'-z_j|.$$

∎

Theorem 6.2. *There are some constants $c_2>0$ and $\delta_1>0$ (depending on m, n) such that if $\frac{1}{2}<\sigma<r\le 1$, and $\vec{e}$ is as in Corollary 5.7, with $|\vec{e}|_r\le\delta_1(r-\sigma)$, then*

$$|\vec{q}|_\sigma\le\frac{c_2|\vec{e}|_r^2}{(r-\sigma)^3}.$$

Proof. Let $\rho=\frac{1}{2}(r+\sigma)$. Note that if $\delta_1\le\frac{1}{2}$, the formal series for $\vec{q}$ is convergent on D_σ, since then, for $(z_1,\zeta,x)\in D_r$, $|x_s+iE_s|<\sigma+\delta_1(r-\sigma)\le\sigma+(\rho-\sigma)=\rho$, $|\zeta(x_{2(t-m+2)}+ix_{2(t-m+2)+1})+e_t|<2\sigma^2+(\rho-\sigma)<2\sigma^2+(\rho-\sigma)(2(\rho+\sigma))=2\rho^2$, and similarly $|\zeta^2+e_{n-1}|<\rho^2$ and $|(z_1+x_2+ix_3)\zeta+e_n|<3\rho^2$, so $\vec{z}+\vec{e}\in\Delta_\rho$, which is contained in the domain of $\vec{p}$ by Corollary 5.7. The $N=n$, $\tilde{N}=1$, $\mathbb{D}_{\mathbf{r}}=\Delta_\rho$ case of Lemma 6.1 applies to $p_k:\Delta_\rho\to\mathbb{C}$, with

$$\max_{j=1,\ldots,n}\left\{\left\|\frac{dp_k}{dz_j}\right\|_\rho\right\}\le K=\frac{c_1|\vec{e}|_r}{(r-\rho)^3},$$

by Corollary 5.8, and $\vec{z}\,'=\vec{z}+\vec{e}\in\Delta_\rho$, so the conclusion is:

$$|p_k(\vec{z}+\vec{e})-p_k(\vec{z})|\le K(|E_2|_r+\cdots+|e_n|_r)=\frac{c_1|\vec{e}|_r}{(r-\rho)^3}|\vec{e}|_r=\frac{8c_1|\vec{e}|_r^2}{(r-\sigma)^3}.$$

This provides bounds for the differences that appear in (28–31), and the remaining terms are the products, where we can use $\frac{1}{2}<\sigma<\rho<r\le 1$, the bound of Theorem 5.6 on the p_1 factor, and the bounds of Corollary 5.8 on

the other factors. For example, in a case of (29) where $t=m<n-1$, part of the expression is the product:

$$\begin{aligned}&\sup_{D_\sigma}\left|(p_1(\vec{z}+\vec{e}))'\frac{p_4(\vec{z}+\vec{e})+(p_4(\vec{z}+\vec{e}))'+ip_5(\vec{z}+\vec{e})+i(p_5(\vec{z}+\vec{e}))'}{2}\right|\\ \le\ & \|p_1\|_\rho(\|p_4\|_\rho+\|p_5\|_\rho)\le\frac{3}{2r}|e_{n-1}|_r\left(\frac{2c_1|\vec{e}|_r}{(r-\rho)^2}\right)<\frac{6c_1|\vec{e}|_r^2}{(r-\rho)^2}<\frac{12c_1|\vec{e}|_r^2}{(r-\sigma)^3}.\end{aligned}$$

∎

The following Lemma on inverse functions will be used twice, in the construction of the new coordinate system and the new defining equations; a proof by a standard iteration procedure is sketched in ([**C**$_4$]).

Lemma 6.3. *Suppose* $0<R_{2,k}<R_{1,k}$ *for* $k=1,\ldots,N$, *so that*

$$\mathbb{D}^2=\mathbb{D}_{(R_{2,1},\ldots,R_{2,N})}\subseteq\mathbb{D}^1=\mathbb{D}_{(R_{1,1},\ldots,R_{1,N})}.$$

Let $f(\vec{z})=(f_1(z_1,\ldots,z_N),\ldots,f_N(z_1,\ldots,z_N))$ *be holomorphic on* $\mathbb{D}^1$, *with*

$$\max_{j=1,\ldots,N}\left\{\sum_{k=1}^N\sup_{\vec{z}\in\mathbb{D}^1}\left|\frac{df_k}{dz_j}(\vec{z})\right|\right\}\le K<1,$$

and

$$\sum_{k=1}^N\sup_{\vec{z}\in\mathbb{D}^2}|f_k(\vec{z})|\le(1-K)\min_{k=1,\ldots,N}\{R_{1,k}-R_{2,k}\}.$$

Then, given $\vec{w}\in\mathbb{D}^2$, *there exists a unique solution* $\vec{z}\in\mathbb{D}^1$ *of the equation*

$$\vec{w}=\vec{z}+f(\vec{z}),$$

and this solution satisfies

$$\sum_{k=1}^N|z_k-w_k|\le\frac{1}{1-K}\sum_{k=1}^N|f_k(\vec{w})|.$$

∎

Theorem 6.4. *There is some constant* $\delta_2>0$ *(depending on* m, n*) so that for any radius lengths* $\frac{1}{2}<\sigma<r\le1$, *and* $\vec{e}$, $\vec{p}$ *as in Corollary* 5.7, *with* $|\vec{e}|_r\le\delta_2(r-\sigma)^3$ *and* $\rho=\frac{1}{2}(r+\sigma)$, *the transformation*

$$\Psi:\vec{z}=(z_1,\ldots,z_n)\mapsto\tilde{z}=(z_1+p_1(\vec{z}),\ldots,z_n+p_n(\vec{z}))$$

has a holomorphic inverse $\psi(\tilde{z})=\vec{z}$ *such that if* $\tilde{z}\in\Delta_\sigma$, *then* $\psi(\tilde{z})\in\Delta_\rho$.

Proof. By Corollary 5.8,

$$\max_{j=1,\ldots,n}\left\{\sum_{k=1}^n\left\|\frac{dp_k}{dz_j}\right\|_\rho\right\}\le\frac{c_1|\vec{e}|_r}{(r-\rho)^3}\le\frac{c_1\delta_2(r-\sigma)^3}{(r-\rho)^3}=8\delta_2c_1\le\frac{1}{2}=K,$$

if $\delta_2 \leq \frac{1}{16c_1}$. Also by Corollary 5.8,

$$\sum_{k=1}^{n} \|p_k\|_\sigma \leq \frac{nc_1|\vec{e}|_r}{(r-\sigma)^2} \leq nc_1\delta_2(r-\sigma) \leq (1-K)(\rho-\sigma),$$

if $\delta_2 \leq \frac{1}{4nc_1}$. The hypotheses of Lemma 6.3 are satisfied with $\Delta_\sigma \subseteq \Delta_\rho$, and $R_{1,k} - R_{2,k} \geq \rho - \sigma$, so given $\tilde{z} \in \Delta_\sigma$, there exists a unique $\vec{z} \in \Delta_\rho$ such that $\tilde{z} = (z_1 + p_1(\vec{z}), \ldots, z_n + p_n(\vec{z}))$. This defines ψ so that $\Psi \circ \psi$ is the identity map on Δ_σ. ∎

For $(z_1, \zeta, x) \in D_{R_1} \subseteq \mathbb{C}^m$, define $z^c \in \mathbb{C}^n$ by:

$$\begin{aligned} z^c &= (z_1,\ x_2 + iE_2(z_1,\zeta,x), \ldots, x_{m-1} + iE_{m-1}(z_1,\zeta,x), \\ &\quad \ldots, \zeta(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) + e_t(z_1,\zeta,x), \ldots, \\ &\quad \zeta^2 + e_{n-1}(z_1,\zeta,x), \zeta(z_1 + x_2 + ix_3) + e_n(z_1,\zeta,x)), \end{aligned}$$

and define a map $\tau : D_{R_1} \to \mathbb{C}^m$ by

$$\begin{aligned} \tau(z_1,\zeta,x) &= (\tau_1(z_1,\zeta,x), \ldots, \tau_m(z_1,\zeta,x)) \\ &= (z_1 + p_1(z^c), \zeta + (p_1(z^c))', \\ &\quad x_2 + \frac{1}{2}(p_2(z^c) + (p_2(z^c))'), \ldots, \\ &\quad x_{m-1} + \frac{1}{2}(p_{m-1}(z^c) + (p_{m-1}(z^c))')). \end{aligned}$$

Theorem 6.5. *There is some constant $\delta_3 > 0$ (depending on m, n) so that for any radius lengths $\frac{1}{2} < r' < r \leq 1$, and $\vec{e}$, $\vec{p}$ as in Corollary 5.7, with $|\vec{e}|_r \leq \delta_3(r-r')^3$, and $\sigma = r' + \frac{1}{3}(r-r')$, the transformation $\tau : (z_1,\zeta,x) \mapsto (\tilde{z}_1, \tilde{\zeta}, \tilde{x})$ has a holomorphic inverse $\phi(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) = (z_1,\zeta,x)$ such that if $(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) \in D_{r'}$, then $\phi(\tilde{z}_1, \tilde{\zeta}, \tilde{x}) \in D_\sigma$.*

Proof. Let $\rho = r' + \frac{2}{3}(r-r')$, so $\sigma - r' = \rho - \sigma = r - \rho = \frac{1}{3}(r-r') < \frac{1}{6}$, and let $\bar{r} = \frac{1}{2}(r+r')$, so $\frac{1}{2} < r' < \sigma < \bar{r} < \rho < r \leq 1$. If $(z_1,\zeta,x) \in D_{\bar{r}}$, and $\delta_3 \leq \frac{2}{3}$, then $|E_2(z_1,\zeta,x)| \leq \delta_3(r-r')^3 = 216\delta_3(\rho-\bar{r})^3 < \frac{216}{12^2}\delta_3(\rho-\bar{r}) \leq \rho - \bar{r}$, and similarly $|e_{n-1}(z_1,\zeta,x)| < \rho^2 - \bar{r}^2$, etc., so $z^c \in \Delta_\rho$, and $\vec{p}(z^c)$ and τ are well-defined and holomorphic on $D_{\bar{r}}$. Using Cauchy's estimate as in Corollary 5.8, for $(z_1,\zeta,x) \in D_\sigma$,

$$\left|\frac{d}{dz_1}p_2(z^c)\right| \leq \frac{|p_2(z^c)|_{\bar{r}}}{\bar{r}-\sigma} \leq \frac{\|p_2\|_\rho}{\frac{1}{2}(\rho-\sigma)} \leq \frac{2c_1|\vec{e}|_r}{(\rho-\sigma)(r-\rho)^2} = \frac{54c_1|\vec{e}|_r}{(r-r')^3}.$$

Similarly, the derivative of each term, $p_1(z^c)$, $p_s(z^c)$, $(p_1(z^c))'$, $(p_s(z^c))'$, with respect to each variable z_1, ζ, x_s, is bounded by a comparable quantity, so there is some constant $c_3 > 0$ (depending on m, n) so that

$$\max_{j=2,\dots,m-1}\left\{\left|\frac{dp_1(z^c)}{dz_1}\right|_\sigma + \left|\frac{d((p_1(z^c))')}{dz_1}\right|_\sigma + \sum_{s=2}^{m-1}\left|\frac{d(\frac{1}{2}(p_s(z^c)+(p_s(z^c))'))}{dz_1}\right|_\sigma,\right.$$
$$\left|\frac{dp_1(z^c)}{d\zeta}\right|_\sigma + \left|\frac{d((p_1(z^c))')}{d\zeta}\right|_\sigma + \sum_{s=2}^{m-1}\left|\frac{d(\frac{1}{2}(p_s(z^c)+(p_s(z^c))'))}{d\zeta}\right|_\sigma,$$
$$\left.\left|\frac{dp_1(z^c)}{dx_j}\right|_\sigma + \left|\frac{d((p_1(z^c))')}{dx_j}\right|_\sigma + \sum_{s=2}^{m-1}\left|\frac{d(\frac{1}{2}(p_s(z^c)+(p_s(z^c))'))}{dx_j}\right|_\sigma\right\}$$
$$\leq \frac{c_3|\vec{e}|_r}{(r-r')^3} \leq c_3\delta_3 \leq \frac{1}{2},$$

if $\delta_3 \leq \frac{1}{2c_3}$. It also follows from Corollary 5.8 that

$$\begin{aligned}
&|p_1(z^c)|_{r'} + |(p_1(z^c))'|_{r'} + \sum_{s=2}^{m-1}\left|\frac{1}{2}(p_s(z^c)+(p_s(z^c))')\right|_{r'} \\
\leq\;& 2\|p_1\|_\rho + \sum_{s=2}^{m-1}\|p_s\|_\rho \\
\leq\;& \frac{mc_1|\vec{e}|_r}{(r-\rho)^2} \leq \frac{mc_1\delta_3(r-r')^3}{(r-\rho)^2} = 9mc_1\delta_3(r-r') \leq \frac{1}{2}(\sigma-r'),
\end{aligned}$$

if $\delta_3 \leq \frac{1}{54mc_1}$. So, by Lemma 6.3, given $(\tilde{z}_1,\tilde{\zeta},\tilde{x}) \in D_{r'}$, there exists a unique $(z_1,\zeta,x) \in D_\sigma$ such that $(\tilde{z}_1,\tilde{\zeta},\tilde{x}) = \tau(z_1,\zeta,x)$. ∎

By inspection of the form of τ, if $(z_1,\zeta,x) \in D_\sigma$, and $\tau(z_1,\zeta,x) = (\tilde{z}_1,\tilde{\zeta},\tilde{x})$, then $\tau(\bar{\zeta},\bar{z}_1,\bar{x}) = (\bar{\tilde{\zeta}},\bar{\tilde{z}}_1,\bar{\tilde{x}})$. If, further, $(\tilde{z}_1,\tilde{\zeta},\tilde{x}) = (\bar{\tilde{\zeta}},\bar{\tilde{z}}_1,\bar{\tilde{x}}) \in D_{r'}$, then $(z_1,\zeta,x) = (\bar{\zeta},\bar{z}_1,\bar{x})$ by uniqueness of the inverse. In particular, if $|\tilde{z}_1| < r'$ and for $s = 2,\dots,m-1$, $\tilde{x}_s$ is real and $|\tilde{x}_s| < r'$, then $\phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x})$ is of the form $(z_1,\bar{z}_1,x)$ for some z_1 with $|z_1| < \sigma$ and x real with $|x_s| < \sigma$. Such (z_1,x) is unique, given $(\tilde{z}_1,\tilde{x})$: suppose there were (z_1^0,x^0) with $|z_1^0| < \sigma$, $|x_s^0| < \sigma$, x^0 real, such that

$$\begin{aligned}
\tilde{z}_1 &= \tau_1(z_1^0,\overline{z_1^0},x^0) \\
\tilde{x}_s &= \tau_{s+1}(z_1^0,\overline{z_1^0},x^0),\ s = 2,\dots,m-1.
\end{aligned}$$

Then the second component $\tau_2(z_1^0,\overline{z_1^0},x^0)$ can be calculated to have some value $\tilde{\zeta}$, so $\tau(z_1^0,\overline{z_1^0},x^0) = (\tilde{z}_1,\tilde{\zeta},\tilde{x})$. By the formula for τ, $\tilde{\zeta} = \bar{\tilde{z}}_1$, so $(\tilde{z}_1,\tilde{\zeta},\tilde{x}) \in D_{r'}$ and $(z_1^0,\overline{z_1^0},x^0) = \phi(\tilde{z}_1,\tilde{\zeta},\tilde{x}) = \phi(\tilde{z}_1,\bar{\tilde{z}}_1,\tilde{x}) = (z_1,\bar{z}_1,x)$, so we can conclude from the uniqueness of Lemma 6.3 that $z_1^0 = z_1$ and $x^0 = x$.

Theorem 6.6. *There exist constants $c_4 > 0$ and $\delta_4 > 0$ (depending on m, n) such that for any $\frac{1}{2} < r' < r \leq 1$ (with σ, ρ as in the previous Theorem), and* *any $\vec{e}$ as in Corollary 5.7 with $|\vec{e}|_r \le \delta_4(r-r')^3$, there exist a holomorphic map $\Psi : \Delta_\rho \to \mathbb{C}^n : (z_1, \ldots, z_n) \mapsto (\tilde{z}_1, \ldots, \tilde{z}_n)$, with a holomorphic inverse $\psi : \Delta_\sigma \to \Delta_\rho$, and a holomorphic map $\tilde{e} = (\tilde{E}_2, \ldots, \tilde{e}_n) : D_{r'} \to \mathbb{C}^{n-1}$, such that the defining equations for M are*

$$\begin{aligned}
\tilde{y}_s &= \tilde{E}_s(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}) \\
\tilde{z}_t &= \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) + \tilde{e}_t(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}) \\
\tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 + \tilde{e}_{n-1}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}) \\
\tilde{z}_n &= \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) + \tilde{e}_n(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}),
\end{aligned}$$

for $|\tilde{z}_1| < r'$, $|\tilde{x}_s| < r'$. Further, the degree of $\tilde{e}$ is at least $2d-2$, and

$$|\tilde{e}|_{r'} \le \frac{c_4|\vec{e}|_r^2}{(r-r')^3}.$$

Proof. Initially, choose $\delta_4 \le \min\{\frac{8}{3}\delta_1, \frac{8}{27}\delta_2, \delta_3\}$, so that Theorems 6.2, 6.4, 6.5 apply, and define Ψ, ψ, $\vec{q}$, and ϕ in terms of the given $\vec{e}$ and the functions $\vec{p}$ constructed in Corollary 5.7. Define $\tilde{e}$ to be the composite of holomorphic maps $\vec{q} \circ \phi : D_{r'} \to \mathbb{C}^{n-1}$, so that by Theorem 6.2,

$$|\tilde{e}|_{r'} \le |\vec{q}|_\sigma \le \frac{c_2|\vec{e}|_r^2}{(r-\sigma)^3} = \frac{c_2|\vec{e}|_r^2}{(\frac{2}{3}(r-r'))^3}.$$

Since $\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})$ has no constant terms, and $\vec{q}$ has degree $\ge 2d-2$ by construction, $\tilde{e}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})$ also has degree at least $2d-2$.

Given $\tilde{z}_1$, $\tilde{x}$ such that $|\tilde{z}_1| < r'$, and $\tilde{x}$ is real with $|\tilde{x}_s| < r'$, define quantities $\tilde{z}_2, \ldots, \tilde{z}_n$ by:

$$\begin{aligned}
\tilde{z}_s &= \tilde{x}_s + i\tilde{E}_s(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}) \\
\tilde{z}_t &= \bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) + \tilde{e}_t(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}) \\
\tilde{z}_{n-1} &= \bar{\tilde{z}}_1^2 + \tilde{e}_{n-1}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}) \\
\tilde{z}_n &= \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) + \tilde{e}_n(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}),
\end{aligned} \tag{32}$$

and denote $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)$. The claim of the Theorem is that $\psi(\tilde{z}) \in M$.

If $(\delta_4)^2 \le \frac{32}{81c_2}$, then

$$|\tilde{e}|_{r'} \le \frac{c_2(\delta_4(r-r')^3)^2}{(r-\sigma)^3} = c_2(\delta_4)^2\frac{3^6}{2^3}(\sigma-r')^3 \le c_2(\delta_4)^2\frac{3^6}{2^3 6^2}(\sigma-r') \le \sigma - r',$$

so $\tilde{z} \in \Delta_\sigma$, the domain of ψ.

By Theorem 6.5, there exists a unique (z_1, x) (the first and last components of $(z_1, \bar{z}_1, x) = \phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})$) such that $|z_1| < \sigma$, x is real with $|x_s| < \sigma$, and

$$\begin{aligned}
&\tilde{z}_1 = z_1 + p_1(z_1, x_2 + iE_2(z_1, \bar{z}_1, x), \ldots, \bar{z}_1(z_1 + x_2 + ix_3) + e_n(z_1, \bar{z}_1, x)) \\
&\tilde{x}_s = x_s + \mathrm{Re}(p_s(z_1, x_2 + iE_2(z_1, \bar{z}_1, x), \ldots, \bar{z}_1(z_1 + x_2 + ix_3) + e_n(z_1, \bar{z}_1, x))).
\end{aligned}$$

Then, define quantities $z_2, \dots, z_n$ by:

$$\begin{aligned}
z_s &= x_s + iE_s(z_1, \bar{z}_1, x)\\
z_t &= \bar{z}_1(x_{2(t-m+2)} + ix_{2(t-m+2)+1}) + e_t(z_1, \bar{z}_1, x)\\
z_{n-1} &= \bar{z}_1^2 + e_{n-1}(z_1, \bar{z}_1, x)\\
z_n &= \bar{z}_1(z_1 + x_2 + ix_3) + e_n(z_1, \bar{z}_1, x),
\end{aligned}$$

and denote, as in (11) and (26), $\vec{z} = (z_1, x, \dots, \bar{z}_1(z_1 + x_2 + ix_3))$ and $\vec{z} + \vec{e} = (z_1, z_2, \dots, z_n)$ as in (8) and (27). Since $|z_1| < \sigma < r$ and $|x_s| < \sigma < r$, $\vec{z} + \vec{e} \in M$, and if $\delta_4 \leq \frac{4}{3}$, then

$$|\vec{e}|_\sigma \leq |\vec{e}|_r \leq \delta_4(r - r')^3 = \delta_4 \cdot 27(\rho - \sigma)^3 < \delta_4 \frac{27}{6^2}(\rho - \sigma) \leq (\rho - \sigma),$$

so $\vec{z} + \vec{e} \in \Delta_\rho$, which is contained in the domain of $\vec{p}$.

$$\begin{aligned}
&\Psi(\vec{z} + \vec{e})\\
= \ &(z_1 + p_1(\vec{z} + \vec{e}), \dots, z_n + p_n(\vec{z} + \vec{e})),\\
= \ &(\tilde{z}_1, \dots, \tilde{x}_s + iE_s(z_1, \bar{z}_1, x) + i\mathrm{Im}(p_s(\vec{z} + \vec{e})), \dots,\\
&\overline{(\tilde{z}_1 - p_1(\vec{z} + \vec{e}))}(\tilde{x}_{2(t-m+2)} - \mathrm{Re}(p_{2(t-m+2)}(\vec{z} + \vec{e})))\\
&\quad + i\overline{(\tilde{z}_1 - p_1(\vec{z} + \vec{e}))}(\tilde{x}_{2(t-m+2)+1} - \mathrm{Re}(p_{2(t-m+2)+1}(\vec{z} + \vec{e})))\\
&\quad + e_t(z_1, \bar{z}_1, x) + p_t(\vec{z} + \vec{e}), \dots\\
&\overline{(\tilde{z}_1 - p_1(\vec{z} + \vec{e}))}^2 + e_{n-1}(z_1, \bar{z}_1, x) + p_{n-1}(\vec{z} + \vec{e}),\\
&\overline{(\tilde{z}_1 - p_1(\vec{z} + \vec{e}))}(\tilde{z}_1 - p_1(\vec{z} + \vec{e}) + \tilde{x}_2 - \mathrm{Re}(p_2(\vec{z} + \vec{e})) + i\tilde{x}_3 - i\mathrm{Re}(p_3(\vec{z} + \vec{e})))\\
&\quad + e_n(z_1, \bar{z}_1, x) + p_n(\vec{z} + \vec{e})),\\
= \ &(\tilde{z}_1, \dots, \tilde{x}_s + iQ_s(z_1, \bar{z}_1, x), \dots,\\
&\bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) + q_t(z_1, \bar{z}_1, x), \dots,\\
&\bar{\tilde{z}}_1^2 + q_{n-1}(z_1, \bar{z}_1, x), \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) + q_n(z_1, \bar{z}_1, x))\\
= \ &(\tilde{z}_1, \dots, \tilde{x}_s + iQ_s(\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})), \dots,\\
&\bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) + q_t(\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})), \dots,\\
&\bar{\tilde{z}}_1^2 + q_{n-1}(\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})), \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) + q_n(\phi(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})))\\
= \ &(\tilde{z}_1, \dots, \tilde{x}_s + i\tilde{E}_s(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}), \dots,\\
&\bar{\tilde{z}}_1(\tilde{x}_{2(t-m+2)} + i\tilde{x}_{2(t-m+2)+1}) + \tilde{e}_t(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}), \dots,\\
&\bar{\tilde{z}}_1^2 + \tilde{e}_{n-1}(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x}), \bar{\tilde{z}}_1(\tilde{z}_1 + \tilde{x}_2 + i\tilde{x}_3) + \tilde{e}_n(\tilde{z}_1, \bar{\tilde{z}}_1, \tilde{x})) = \tilde{z},
\end{aligned}$$

by construction of $\vec{q}$, $\tilde{e}$, and $\tilde{z}$, and using the fact that $\vec{p}$ is a solution of (10–11). By the uniqueness of Theorem 6.4, $\psi(\tilde{z}) = \vec{z} + \vec{e} \in M$. ∎

7. Composition of approximate solutions

The previous Theorem's quadratic estimate on the size of $\tilde{e}$ in terms of $\vec{e}$ allows for the rapid convergence of a sequence of approximations. A couple technical Lemmas will be needed to measure the behavior of composite mappings. Theorem 7.7, which is the last step in proving Proposition 3.3, uses these Lemmas and the estimates of the previous Section to prove convergence of a sequence of transformations, following the ideas of [**M**].

Notation 7.1. For $R_1 > 0$ and a $n \times n$ matrix of complex valued functions $F = (F_{kj}(\vec{z}))$ on Δ_{R_1}, define

$$|||F|||_{R_1} = \max_{j=1,\dots,n} \left\{ \sum_{k=1}^{n} \sup_{\vec{z} \in \Delta_{R_1}} |F_{kj}(\vec{z})| \right\}.$$

This "maximum column sum" norm appeared already, in Corollary 5.8 and Lemmas 6.1, 6.3, in the case where $F = \mathrm{D}f = \mathrm{D}_{\vec{z}}f$, the Jacobian matrix of some map $f : \Delta_{R_1} \to \mathbb{C}^n$ at $\vec{z} \in \Delta_{R_1}$. The 3×3 case of the following Lemma was proved in [$\mathbf{C}_4$].

Lemma 7.2. *If $|||A|||_{R_1} < 1$, then $\mathbb{I} + A$ is invertible (where $\mathbb{I}$ is the $n \times n$ identity matrix), and*

$$|||(\mathbb{I} + A)^{-1}|||_{R_1} \leq \frac{1}{1 - |||A|||_{R_1}}.$$

∎

Also, the following elementary fact from the calculus of one real variable will be used.

Lemma 7.3. *If μ_k is a sequence such that $0 \leq \mu_k < 1$ and $\sum_{k=0}^{\infty} \mu_k$ is a convergent series, then the sequence of partial products*

$$\prod_{k=0}^{N} \frac{1}{1 - \mu_k}$$

is bounded above by some positive limit. ∎

Notation 7.4. For $\nu = 0, 1, 2, \dots$, define a sequence $\{1, \frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \dots\}$ by the formula

$$r_\nu = \frac{1}{2}\left(1 + \frac{1}{\nu + 1}\right).$$

Note that $\frac{1}{2} < r_\nu \leq 1$, and the sequence is decreasing, with

$$\begin{aligned} r_\nu - r_{\nu+1} &= \frac{1}{2(\nu+1)(\nu+2)}, \\ \frac{r_{\nu+1} - r_{\nu+2}}{r_\nu - r_{\nu+1}} &= \frac{\nu+1}{\nu+3} \geq \frac{1}{3}. \end{aligned}$$

Notation 7.5. Define $\sigma_\nu = r_{\nu+1} + \frac{1}{3}(r_\nu - r_{\nu+1})$, $\rho_\nu = r_{\nu+1} + \frac{2}{3}(r_\nu - r_{\nu+1})$, as in Theorem 6.5.

Recall that given $\eta > 0$, there is some scaling transformation so that $M \cap \Delta_1$ is defined by (5), with $\vec{e}$ holomorphic on D_1, degree $d \geq 3$, and $|\vec{e}|_1 \leq \eta$.

Notation 7.6. Denote $\vec{e}_0 = \vec{e}$ (so $|\vec{e}_0|_{r_0} = |\vec{e}|_1 \leq \eta$), and inductively define the formal series $\vec{e}_{\nu+1}(z_1, \zeta, x)$ in terms of $\vec{e}_\nu(z_1, \zeta, x)$, by the $\vec{e} \mapsto \tilde{e}$ procedure of Theorem 6.6, with $r = r_\nu$, $r' = r_{\nu+1}$. Each $\vec{e}_\nu$ defines, as in the previous Theorems, functions $\vec{p}_\nu$, $\vec{q}_\nu$, Ψ_ν, ψ_ν, ϕ_ν, and the degree of $\vec{e}_\nu$ is denoted d_ν.

Also recall that the degree $d_{\nu+1}$ of $\vec{e}_{\nu+1}$ is at least $2d_\nu - 2$; it can be checked that this, together with $d_0 = d \geq 3$, implies $d_\nu \geq 2^\nu + 2$.

The plan is to show that the bound for $\vec{e}_\nu$ in the hypothesis of Theorem 6.6 holds for all ν, to get a sequence of transformations $\psi_\nu : \Delta_{\sigma_\nu} \to \Delta_{\rho_\nu}$, so that the composition $\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu : \Delta_{\sigma_\nu} \to \Delta_{\rho_0}$ is well-defined, $\vec{e}_\nu$ is holomorphic on D_{r_ν}, and $\lim_{\nu\to\infty} |\vec{e}_\nu|_{r_\nu} = 0$.

Theorem 7.7. *There exists $\eta > 0$ (depending on m, n) so that if $\vec{e}_0$ and M are as described above, then there exists a holomorphic transformation $\psi : \Delta_{\frac{1}{2}} \to \mathbb{C}^n$, with a holomorphic inverse Ψ, and such that if $\tilde{z} \in \widetilde{M} \cap \Delta_{\frac{1}{2}}$, then $\psi(\tilde{z}) \in M$.*

Proof. Let $\delta_5 = \min\{\delta_4, \frac{1}{27c_4}\}$, and choose $0 < \eta < \min\{\frac{\delta_5}{64}, \frac{1}{1728c_1}\}$. It will be shown that

$$|\vec{e}_\nu|_{r_\nu} \leq \delta_5(r_\nu - r_{\nu+1})^3 \implies |\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \delta_5(r_{\nu+1} - r_{\nu+2})^3.$$

By Theorem 6.6, $|\vec{e}_\nu|_{r_\nu} \leq \delta_4(r_\nu - r_{\nu+1})^3$ and $|\vec{e}_\nu|_{r_\nu} \leq \frac{1}{27c_4}(r_\nu - r_{\nu+1})^3$ imply

$$|\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \frac{c_4|\vec{e}_\nu|^2_{r_\nu}}{(r_\nu - r_{\nu+1})^3} \leq \frac{1}{27}|\vec{e}_\nu|_{r_\nu};$$

this already suggests a geometric decrease in the sequence of norms. Then, using the properties of the sequence r_ν,

$$\frac{1}{27}|\vec{e}_\nu|_{r_\nu} \leq \frac{1}{27}\delta_5(r_\nu - r_{\nu+1})^3 \leq \delta_5(r_{\nu+1} - r_{\nu+2})^3,$$

which proves the claimed implication. Using this as an inductive step, and starting the induction with $|\vec{e}_0|_{r_0} \leq \eta < \frac{1}{64}\delta_5 = \delta_5(r_0 - r_1)^3$, the hypothesis of Theorem 6.6 is satisfied for all ν. The first of three conclusions from

Theorem 6.6 is that $\vec{e}_\nu$ is holomorphic on D_{r_ν}, with degree $d_\nu \geq 2^\nu + 2$, and $|\vec{e}_\nu|_{r_\nu} \leq 27^{-\nu}\eta$. Secondly, $\psi_0 \circ \ldots \circ \psi_\nu$ is a well-defined holomorphic map $\Delta_{\sigma_\nu} \to \Delta_{\rho_0}$, and $\Psi_\nu \circ \ldots \circ \Psi_0$ is well-defined and holomorphic on the image $(\psi_0 \circ \ldots \circ \psi_\nu)(\Delta_{\sigma_\nu})$, so that $\Psi_\nu \circ \ldots \circ \Psi_0 \circ \psi_0 \circ \ldots \circ \psi_\nu$ is the identity on Δ_{σ_ν}. The third conclusion is that if $|\tilde{z}_1| < r_{\nu+1}$ and $|\tilde{x}_s| < r_{\nu+1}$, and $\tilde{z}$ is defined as in (32) with $\tilde{e} = \vec{e}_{\nu+1}$, then $(\psi_0 \circ \ldots \circ \psi_\nu)(\tilde{z}) \in M$. For any $\vec{z} = (z_1, \ldots, z_n) \in \Delta_{\frac{1}{2}}$, the sequence (depending on ν) $(\psi_0 \circ \ldots \circ \psi_{\nu-1} \circ \psi_\nu)(\vec{z})$ is contained in $\Delta_{\rho_0} = \Delta_{11/12}$. The following argument, beginning with several applications of Lemma 6.1, shows this sequence is a Cauchy sequence, and converges to some value $\psi(\vec{z})$.

$$\begin{aligned}
& \sum_{k=1}^{n} |(\psi_0 \circ \ldots \circ \psi_{\nu+1})_k(\vec{z}) - (\psi_0 \circ \ldots \circ \psi_\nu)_k(\vec{z})| \\
= \; & \sum_{k=1}^{n} |(\psi_0)_k((\psi_1 \circ \ldots \circ \psi_{\nu+1})(\vec{z})) - (\psi_0)_k((\psi_1 \circ \ldots \circ \psi_\nu)(\vec{z}))| \\
\leq \; & |||\mathrm{D}\psi_0|||_{\rho_1} \cdot \sum_{j=1}^{n} |(\psi_1 \circ \ldots \circ \psi_{\nu+1})_j(\vec{z}) - (\psi_1 \circ \ldots \circ \psi_\nu)_j(\vec{z})| \\
\leq \; & \left(\prod_{\ell=0}^{\nu} |||\mathrm{D}\psi_\ell|||_{\rho_{\ell+1}}\right) \cdot \sum_{j=1}^{n} |(\psi_{\nu+1})_j(\vec{z}) - z_j|. \qquad (33)
\end{aligned}$$

By the estimate from Lemma 6.3, with $f = \vec{p}_{\nu+1}$ and $K = \frac{1}{2}$ from the Proof of Theorem 6.4, and then using the bound for $\vec{p}$ from Corollary 5.8,

$$\begin{aligned}
& \sum_{j=1}^{n} |(\psi_{\nu+1})_j(\vec{z}) - z_j| \leq \frac{1}{1 - \frac{1}{2}} \sum_{j=1}^{n} |(\vec{p}_{\nu+1})_j(\vec{z})| \leq 2 \sum_{j=1}^{n} \|(\vec{p}_{\nu+1})_j\|_{\frac{1}{2}} \\
\leq \; & 2 \sum_{j=1}^{n} \|(\vec{p}_{\nu+1})_j\|_{\rho_{\nu+1}} \leq 2n \frac{c_1 |\vec{e}_{\nu+1}|_{r_{\nu+1}}}{(r_{\nu+1} - \rho_{\nu+1})^2} = 18n \frac{c_1 |\vec{e}_{\nu+1}|_{r_{\nu+1}}}{(r_{\nu+1} - r_{\nu+2})^2} \\
= \; & 72 n c_1 (\nu+2)^2 (\nu+3)^2 |\vec{e}_{\nu+1}|_{r_{\nu+1}} \leq \frac{72 n c_1 (\nu+2)^2 (\nu+3)^2 \eta}{27^{\nu+1}}.
\end{aligned}$$

It follows from $\mathrm{D}_{\vec{z}}\psi_\ell = (\mathbb{I} + \mathrm{D}_{\psi_\ell(\vec{z})}\vec{p}_\ell)^{-1}$ and Lemma 7.2 that:

$$|||\mathrm{D}\psi_\ell|||_{\rho_{\ell+1}} = |||(\mathbb{I} + \mathrm{D}_{\psi_\ell(\vec{z})}\vec{p}_\ell)^{-1}|||_{\rho_{\ell+1}} \leq |||(\mathbb{I} + \mathrm{D}\vec{p}_\ell)^{-1}|||_{\rho_\ell} \leq \frac{1}{1 - |||\mathrm{D}\vec{p}_\ell|||_{\rho_\ell}}.$$

Then, by Lemma 7.3, the product from (33) is bounded above by some constant $c_5 > 0$, since by Corollary 5.8,

$$\begin{aligned}
\sum_{\ell=0}^{\infty} |||\mathrm{D}\vec{p}_\ell|||_{\rho_\ell} &\leq \sum_{\ell=0}^{\infty} \frac{c_1|\vec{e}_\ell|_{r_\ell}}{(r_\ell-\rho_\ell)^3} = \sum_{\ell=0}^{\infty} \frac{27c_1|\vec{e}_\ell|_{r_\ell}}{(r_\ell-r_{\ell+1})^3}\\
= \ \sum_{\ell=0}^{\infty} 216(\ell+1)^3(\ell+2)^3 c_1|\vec{e}_\ell|_{r_\ell} &\leq \sum_{\ell=0}^{\infty} \frac{216(\ell+1)^3(\ell+2)^3 c_1\eta}{27^\ell},
\end{aligned}$$

a convergent infinite series with terms < 1.

The inequality

$$\sum_{k=1}^{n} |(\psi_0 \circ \ldots \circ \psi_{\nu+1})_k(\vec{z}) - (\psi_0 \circ \ldots \circ \psi_\nu)_k(\vec{z})| \leq \frac{72nc_1c_5(\nu+2)^2(\nu+3)^2\eta}{27^{\nu+1}}$$

is enough to show that the sequence of composite functions converges pointwise and uniformly to a function ψ on $\Delta_{\frac{1}{2}}$. ∎

Remark. Although some details remain to be checked, it seems plausible that a similar rapid convergence argument could be used to prove an analogous analytic stability property for a nondegenerate CR singularity of a real 3-manifold in $\mathbb{C}^4$, as conjectured in [$\mathbf{C}_5$].

8. Analogy with singularity theory

To continue with the theme of analogies between the normal form result and the properties of Whitney's cross-cap singularity, we briefly consider the notion of complexification. If the defining equations of a real m-submanifold M in $\mathbb{C}^n$ with a CR singularity at $\vec{0}$ are given as a graph over the tangent space as in (1), then M can also be considered as the image of a real analytic parametrization $\pi : \mathbb{R}^m \to \mathbb{R}^{2n} : (z,x) \mapsto (z,x,H_s(z,x),h_u(z,x))$. Then the spaces $\mathbb{R}^m$, $\mathbb{R}^{2n}$ can be embedded as totally real subspaces of $\mathbb{C}^m$, $\mathbb{C}^{2n}$, and there is a a complex analytic map $\pi_c : \mathbb{C}^m \to \mathbb{C}^{2n}$ which restricts to π on the totally real subspaces. In the following examples, composing with a projection $P : \mathbb{C}^{2n} \to \mathbb{C}^n$ gives a holomorphic map $P \circ \pi_c$ which restricts to π on the totally real $\mathbb{R}^m$ subspace, and its image is a complex subvariety of $\mathbb{C}^n$ containing M. Even though π_c is an embedding, the composite $P \circ \pi_c$ can be singular, and the image of its critical point set contains the CR singular locus of M. For details and more examples of this construction, see [**Webster**], [$\mathbf{C}_2$], [$\mathbf{C}_3$], and to be more precise, these maps should be considered only in some neighborhood of the origin in the domain and target.

Example 8.1. In the $m = n = 2$ case ([**Bishop**]), the local defining equation of a real surface with a nondegenerate CR singularity in $\mathbb{C}^2$ can be normalized to $z_2 = \beta(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1 + O(3)$, where the coefficient $\beta \geq 0$ is a biholomorphic invariant. Considering the real embedding's quadratic part,

$$\pi : (z_1, \bar{z}_1) \mapsto (z_1, \bar{z}_1, z_2 = \beta(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1, \bar{z}_2 = \overline{\beta(z_1^2 + \bar{z}_1^2) + z_1\bar{z}_1})$$

is a real analytic map from the totally real subspace $\{(z_1, w_1) : w_1 = \bar{z}_1\}$ of $\mathbb{C}^2$ to the totally real subspace $\{(z_1, w_1, z_2, w_2) : w_1 = \bar{z}_1, w_2 = \bar{z}_2\}$ of $\mathbb{C}^4$, which extends to a complex analytic embedding $\pi_c : (z_1, w_1) \mapsto (z_1, w_1, \beta(z_1^2 + w_1^2) + z_1w_1, \beta(w_1^2 + z_1^2) + w_1z_1)$. Then composing with the projection $P : \mathbb{C}^4 \to \mathbb{C}^2$ that forgets the w_1, w_2 variables in the target gives a map $P \circ \pi_c : (z, w) \mapsto (z, \beta(z^2 + w^2) + zw)$. For $\beta > 0$, this is a ramified two-to-one map onto $\mathbb{C}^2$ ([**MW**], [**Webster**]), and is analogous to Whitney's "fold" singularity $(x, y) \mapsto (x, y^2)$.

Example 8.2. An example of a cubic normal form for a CR singular surface in $\mathbb{C}^2$ in the $\beta = 0$ case is $z_2 = z_1\bar{z}_1 + \bar{z}_1^3$ ([**M**]). The map $P \circ \pi_c : (z_1, w_1) \mapsto (z_1, z_1w_1 + w_1^3)$ is analogous to Whitney's "cusp", $(x, y) \mapsto (x, xy + y^3)$.

Example 8.3. An example of a surface M in $\mathbb{C}^3$ with a topologically unstable CR singularity, considered by [$\mathbf{C}_4$], has real equations $z_2 = \bar{z}_1^2$, $z_3 = z_1\bar{z}_1$, which complexify to $P\circ\pi_c : (z_1, w_1) \mapsto (z_1, w_1^2, z_1w_1)$, exactly Whitney's normal form for the parametrization of the cross-cap singularity. The image of $P \circ \pi_c$ in $\mathbb{C}^3$ is $\{z_1^2z_2 - z_3^2 = 0\}$, a singular complex hypersurface (Whitney's "umbrella" surface), and the smallest complex variety containing M.

Example 8.4. For the normal form variety $\widetilde{M}^{4,5}$, a parametrization $\mathbb{C}^4 \to \mathbb{C}^{10} \to \mathbb{C}^5$ of the complexification is

$$(z_1, w_1, z_2, z_3) \mapsto (z_1, z_2, z_3, w_1^2, w_1(z_1 + z_2 + iz_3)).$$

The real manifold $\widetilde{M}^{4,5}$ is the image of the restriction of this map to the totally real subspace $\{w_1 = \bar{z}_1, z_2 = \bar{z}_2, z_3 = \bar{z}_3\}$ in the domain. The holomorphic map $\mathbb{C}^4 \to \mathbb{C}^5$ parametrizes a singular complex hypersurface $\mathcal{H}$, which is the product of Whitney's cross-cap surface and a complex 2-plane, and the image $\{(z_1 + z_2 + iz_3)^2z_4 - z_5^2 = 0\}$ is the smallest complex variety in $\mathbb{C}^5$ containing $\widetilde{M}^{4,5}$; a similar expression appeared in (17). The geometry of $\widetilde{M}^{4,5} \subseteq \mathcal{H}$ is considered in [$\mathbf{C}_3$] §8, but with a different expression for the quadratic normal form.

Example 8.5. In general, the real variety $\widetilde{M}^{m,n}$ is contained in a singular subvariety of complex dimension m in $\mathbb{C}^n$, the defining ideal of which contains, for example, $(z_1 + z_2 + iz_3)^2z_{n-1} - z_n^2$. As a consequence of Proposition 3.3, any real analytic M is not a local uniqueness set for holomorphic functions in a neighborhood of a nondegenerate CR singularity (cf [$\mathbf{H}_2$]).

For surfaces in $\mathbb{C}^2$, the two-to-one nature of the complexification $\mathbb{C}^2 \to \mathbb{C}^2$ as in Example 8.1 was used by [**MW**] to solve a normal form problem in the $0 < \beta < \frac{1}{2}$ case. Their methods are different from that of this paper;

for example in the $(m,n) = (4,5)$ case, the map $\mathbb{C}^4 \to \mathbb{C}^5$ from Example 8.4 is generally one-to-one, the two-to-one locus being contained in a complex subvariety in the domain as shown in [**C**$_3$].

Normal forms for the complexifications that look more like Whitney's monomial normal forms would be possible using a larger group, where the z and w variables could be transformed independently. Under the subgroup used to normalize the CR singularity, one expects equivalence classes of maps to be smaller, and continuous parameters ("moduli") to appear sooner (for more and for lower-order terms). However, invariants which distinguish maps under the larger group will still distinguish them under the smaller group. One may speculate that invariants of the complexification, such as the intrinsic derivative, the Boardman sequence, Jacobian extensions, etc., could provide a coarse but general beginning to the development of a CR singularity theory analogous to the singularity theory of maps ([**GG**], [**P**]).

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