ADDENDUM TO: WEIGHTED PROJECTIVE SPACES AND A GENERALIZATION OF EVES' THEOREM

ADAM COFFMAN

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MSC 2010: Primary 51N15; Secondary 05B30, 14E05, 14N05, 51A20, 51M25, 51N05, 51N35, 68T45

7. Updates

A version of the Remainder Theorem used in the Proof of Theorem 16, Section 3, is more specifically attributed to Sun Zi, as in [LA].

There are reviews in MR and Zbl for this article: [C].
An older version of [C] is on the arXiv: `arxiv.org:1204.1686`
My contact information has changed; my current home page can be found at:
`http://users.pfw.edu/CoffmanA/`

8. More Examples for Section 2.2

Example 16. The monomial map $\mathbf{f} : \mathbb{R}^2_* \to \mathbb{R}^2 : (z_0, z_1) \mapsto (z_0^2, z_1)$ induces a well-defined map $f : \mathbb{R}P(2, 2) \to \mathbb{R}P(2, 1)$ as in Lemma 8, but the induced map is not onto. The point $[-1 : 1]_{\mathbf{q}}$ is not in the image of f; there is no $(z_0, z_1) \in \mathbb{R}^2_*$ such that $(z_0^2, z_1) \sim_{\mathbf{q}} (-1, 1)$.

Example 17. For $m \in \mathbb{N}$ and two weights:

$$\begin{aligned} \mathbf{q} &= (q_0, q_1, q_2, \dots, q_n), \\ \mathbf{p} &= (q_0, mq_1, mq_2, \dots, mq_n), \end{aligned}$$

another situation in which the map

$$\begin{aligned} \mathbf{f} \ &: \ \mathbb{K}^{n+1}_* \to \mathbb{K}^{n+1} \\ &: \ (z_0, z_1, z_2, \dots, z_n) \mapsto (z_0^m, z_1, z_2, \dots, z_n), \end{aligned}$$

as in Lemma 8, defines an onto map $f : \mathbb{K}P(q_0, mq_1, \dots, mq_n) \to \mathbb{K}P(q_0, \dots, q_n)$ is the case where $\mathbb{K} = \mathbb{R}$ and q_0 is odd. For $w_0 \geq 0$, make the same choices mentioned in the Proof of Lemma 8, and for $w_0 < 0$, choose $\lambda = -1$, any z_0 with $z_0^m = (-1)^{q_0} w_0 = |w_0|$, and $z_k = w_k/(-1)^{q_k}$ for $k = 1, \dots, n$.

Example 18. Let $\mathbb{K} = \mathbb{R}$, and consider the weights $\mathbf{p}$ and $\mathbf{q}$ as in Lemmas 8 and 10 and Example 17. Here we assume m is odd but make no assumption on q_0. Then the map

$$\mathbf{f}(z_0, z_1, z_2, \dots, z_n) = (z_0^m, z_1, z_2, \dots, z_n)$$

from Lemma 8 induces a well-defined, onto map

$$f : \mathbb{R}P(\mathbf{p}) \to \mathbb{R}P(\mathbf{q}).$$

It is also one-to-one: the algebra problem is to solve the same equations (5), (6) from the Proof of Lemma 10, for a real μ in terms of real $\mathbf{z}$, $\mathbf{z}'$, λ. Given $\lambda \neq 0$, let μ be the unique real solution of $\mu^m = \lambda$. Then, for $j = 1, \ldots, n$, $\mu^{mq_j} z_j = \lambda^{q_j} z_j = z'_j$, and $(\mu^{q_0} z_0)^m = \lambda^{q_0} z_0^m = (z'_0)^m \implies \mu^{q_0} z_0 = z'_0$.

Example 19. Let $\mathbb{K} = \mathbb{R}$, and consider the weights $\mathbf{p}$ and $\mathbf{q}$ as in Lemmas 8 and 10 and Example 17. Here we assume m is even, q_0 is odd, and all $q_1, \ldots, q_n$ are even. Then the map

$$\mathbf{f}(z_0, z_1, z_2, \ldots, z_n) = (z_0^m, z_1, z_2, \ldots, z_n)$$

from Lemma 8 induces a well-defined, onto map

$$f : \mathbb{R}P(\mathbf{p}) \to \mathbb{R}P(\mathbf{q}).$$

It is also one-to-one: the algebra problem is to solve (5), (6), for a real μ in terms of real $\mathbf{z}$, $\mathbf{z}'$, λ. Given $\lambda \neq 0$, the equation $\mu^m = |\lambda|$ has exactly two real solutions, $\{\mu_1 = |\lambda|^{1/m}, \mu_2 = -|\lambda|^{1/m}\}$. Then, for $k = 1, 2$, $j = 1, \ldots, n$,

$$\mu_k^{mq_j} z_j = |\lambda|^{q_j} z_j = \lambda^{q_j} z_j = z'_j.$$

For $k = 1, 2$, $(\mu_k^{q_0} z_0)^m = |\lambda|^{q_0} z_0^m = |\lambda^{q_0} z_0^m| = (z'_0)^m$, so the set

$$\{\mu_1^{q_0} z_0, \mu_2^{q_0} z_0 = -\mu_1^{q_0} z_0\}$$

is contained in the set $\{z'_0, -z'_0\}$, and one of the two roots is the required μ satisfying $\mu^{q_0} z_0 = z'_0$.

Example 20. For an even number p_1, the function

$$\mathbf{f}(z_0, z_1) = (z_0^{p_1}, z_1)$$

induces a well-defined, onto map

$$f : \mathbb{R}P(1, p_1) \to \mathbb{R}P(1, 1)$$

as in Lemma 8. The induced map is not one-to-one:

$$\mathbf{f}(0, 1) = (0, 1) \sim_{\mathbf{q}} \mathbf{f}(0, -1) = (0, -1),$$

but $(0, 1) \not\sim_{\mathbf{p}} (0, -1)$.

9. More Examples for Section 3

Example 21. Example 6 shows that the space $\mathbb{R}P(1, p_1)$ is reconstructible. Even though the map $h_{01}([z_0 : z_1]_{\mathbf{p}}) = [z_0^{p_1} : z_1]$ is not globally one-to-one when p_1 is even, as shown in Example 20, it is one-to-one when restricted to $D_{\mathbf{p}}$.

The following two examples are special cases of Theorem 17, on real weighted projective spaces.

Example 22. If one of the numbers p_0, p_1 is odd, then the space $\mathbb{R}P(p_0, p_1)$ is reconstructible. WLOG, let p_0 be odd. For the axis projection $c_{01}([z_0 : z_1]_{\mathbf{p}}) = [z_0^{p_1} : z_1^{p_0}]$, the following diagram is commutative. The label on the left arrow means that the indicated map is induced by the polynomial map $\mathbb{R}^2_* \to \mathbb{R}^2 : (z_0, z_1) \mapsto (z_0, z_1^{p_0})$.

$$\begin{tikzcd}
\mathbb{R}P(p_0, p_1) \arrow[r, "c_{01}"] \arrow[d, "{(z_0, z_1^{p_0})}"'] & \mathbb{R}P^1 \\
\mathbb{R}P(1, p_1) \arrow[ur, "{(z_0^{p_1}, z_1)}"'] &
\end{tikzcd}$$

The map on the left is globally one-to-one as in Example 18, and takes $D_{(p_0,p_1)}$ to $D_{(1,p_1)}$. The lower right map is one-to-one on $D_{(1,p_1)}$: either by Example 18 for odd p_1, or by Example 21 for even p_1.

Example 23. If both p_0 and p_1 are even, then $\mathbb{R}P(p_0,p_1)$ is not reconstructible. Consider an axis projection induced by $\mathbf{c}_{01}(z_0,z_1)=(z_0^a,z_1^b)$. By Lemma 15 we may assume that a and b are not both even. If a and b are both odd, then

$$\mathbf{c}_{01}(1,1)=(1,1)\sim_{(1,1)}\mathbf{c}_{01}(-1,-1)=(-1,-1),$$

but $(1,1)\not\sim_{\mathbf{p}}(-1,-1)$, so c_{01} is not one-to-one. If a is even and b is odd (the remaining case being similar), then

$$\mathbf{c}_{01}(1,-1)=(1,-1)\sim_{(1,1)}\mathbf{c}_{01}(-1,-1)=(1,-1),$$

but $(1,-1)\not\sim_{\mathbf{p}}(-1,-1)$, so again c_{01} is not one-to-one.

References

[C] A. Coffman, *Weighted projective spaces and a generalization of Eves' Theorem*, Journal of Mathematical Imaging and Vision (3) **48** (2014), 432–450; together with a 2-page online-only version of this addendum. MR 3171423, Zbl 06316454.

[LA] L. Y. Lam and T. S. Ang, *Fleeting Footsteps. Tracing the Conception of Arithmetic and Algebra in Ancient China*. Revised ed. World Scientific Publishing Co., Inc., River Edge, NJ, 2004. MR2092881 (2005d:01005), Zbl 1062.01005.

Department of Mathematical Sciences, Purdue University Fort Wayne, 2101 E. Coliseum Blvd., Fort Wayne, IN 46805-1499

Email address: CoffmanA@pfw.edu