Notes on sequences and series in the calculus of one variable

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These are notes on sequences and series, at a first-year calculus level. The goal is to see how complex numbers and infinite series are related. I'll start with a few comments about the "Axiom of Completeness," which does not apply to the complex numbers. In fact, $\mathbb{C}$ will not appear until page 5.

1.1 Sets of real numbers

Recall that $\mathbb{R}$ denotes the set of real numbers, which contains zero and also all positive or negative decimal expansions. The rational numbers, defined as $\mathbb{Q} = \{\frac{p}{q} : p, q \text{ integers}, q \neq 0\}$, have repeating decimal expansions (including the terminating decimals, where 0 repeats). So, $\mathbb{Q}$ is a subset of $\mathbb{R}$, but the two sets are definitely not equal. $\sqrt{2}$, π, and e are all examples of "irrationals," real numbers which are not quotients of integers, and whose decimal expansions do not repeat.

The two number systems $\mathbb{R}$ and $\mathbb{Q}$ have many things in common, including all the laws of arithmetic (the commutative, associative, distributive laws, etc., for $+$, $-$, $\times$, and $\div$, with division allowed for any numbers except zero in the denominator.) and the rules for inequalities. The set $\mathbb{Z}$ of integers (positive, 0, and negative) obeys the same rules except that dividing two integers does not always give an integer. In $\mathbb{N}$, the set of natural numbers $\{1, 2, 3, 4, \ldots\}$, the sum and product of natural numbers is in $\mathbb{N}$, but not always the difference or quotient.

Let S be a subset of either $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$ or $\mathbb{N}$.

Definition 1.1. A set S is "bounded above," with an "upper bound U," if every element $x \in S$ satisfies $x \leq U$.

Definition 1.2. A set S is "bounded below," with a "lower bound B," if every element $x \in S$ satisfies $x \geq B$.

Not every set S has an upper bound, for example, $S = [0, \infty)$ has a lower bound, but not an upper bound. Upper bounds are never "unique," either, since if U is an upper bound of S, then so is $U + 1$.

Definition 1.3. A set S is "bounded," with a "bound M," if every element $x \in S$ satisfies $|x| \leq M$.

Theorem 1.4. *A set $S \subseteq \mathbb{R}$ is bounded if and only if it is bounded above and bounded below.*

Proof. First, assume S is bounded, with bound M. It follows that S has an upper bound: $x \le |x| \le M$, and a lower bound: $-x \le |x| \le M \implies x \ge -M$ (so, $U = M$, and $B = -M$).

Second, for the converse, if S has both an upper bound U and a lower bound L, then S is bounded: $L \le x \implies -x \le -L$, and since $|x|$ is either x or $-x$, one of these must hold: $|x| \le U$, or $|x| \le -L$. Define the bound to be $M = \max\{U, -L\}$, so $|x| \le M$. ∎

For the next two statements, let A and B be subsets of $\mathbb{R}$ or $\mathbb{Q}$ or $\mathbb{Z}$ or $\mathbb{N}$.

Theorem 1.5. *If $A \subseteq B$, and B has an upper bound U, then U is also an upper bound of A.*

Proof. If $x \in A$, then $x \in B$ (this is the definition of $\subseteq$), so $x \le U$ (this is the definition of upper bound of B). Since $x \le U$ for all $x \in A$, U is, by definition, an upper bound of A. ∎

The following is just the logical contrapositive of the previous Theorem.

Corollary 1.6. *If $A \subseteq B$, and A has no upper bound, then B has no upper bound.* ∎

1.2 The Completeness Property

Definition 1.7. A number b is a "least upper bound of S" means that both of the following are true:

1. b is an upper bound of S. (so if $x \in S$, then $x \le b$.)
2. if U is any upper bound of S, then $b \le U$.

The least upper bound b can be abbreviated $b = \text{lub}S$. There's a similar definition of "greatest lower bound" $g = \text{glb}S$:

Definition 1.8. A number g is a "greatest lower bound of S" means that both of the following are true:

1. g is a lower bound of S. (so if $x \in S$, then $x \ge g$.)
2. if L is any lower bound of S, then $g \ge L$.

For example, if $S = [0, 1)$, then $\text{glb}S = 0$ and $\text{lub}S = 1$. A set that has no upper bound will also have no least upper bound. In general, the lub of a set might or might not be an element of the set. However, the following Theorem says that the set always has an element close to the lub (close meaning "within ϵ").

Theorem 1.9. *Suppose S has a least upper bound b. Then, for any $\epsilon > 0$, there exists $x \in S$ such that*

$$b - \epsilon < x \le b.$$

Proof. Suppose, toward a contradiction, that there is some ϵ for which there does not exist such an element x. Then, any $x \in S$ satisfies either $x \le b - \epsilon$ or $x > b$. The $x > b$ case is impossible since b is an upper bound, but then $x \le b - \epsilon$ for all x would mean than $b - \epsilon$ is an upper bound less than the least upper bound b. This contradiction proves the Theorem. ∎

Definition 1.10. A number system $\mathbb{F}$ has the "completeness property" if every non-empty subset $S \subseteq \mathbb{F}$ that has an upper bound also has a least upper bound lub$S \in \mathbb{F}$.

The rational numbers $\mathbb{Q}$ do not have the completeness property. The sequence

$$\{1, 1.4, 1.414, 1.4142, 1.41421, 1.414213, 1.4142136\ldots\}$$

of terminating decimal approximations of $\sqrt{2}$ is a sequence of rational numbers, with an upper bound ($U = \frac{3}{2} \in \mathbb{Q}$ for example), but it does not have a least upper bound *in the set of rational numbers.*

The real numbers $\mathbb{R}$ do have the completeness property. (I am not going to prove this, we will take it for granted as a geometric property of the real number line.) For example, any postive real number x has a decimal expansion, repeating or not, and the set of numbers formed by its terminating decimal approximations forms a sequence with least upper bound x.

The set of integers $\mathbb{Z}$ also has the completeness property. Any bounded set S of integers must be finite, and the lub of S is just the largest element of S. Similarly, $\mathbb{N}$ also has the completeness property.

Theorem 1.11. *Let $A \subseteq B \subseteq \mathbb{F}$, where $\mathbb{F}$ has the completeness property. If A is non-empty, and B has a least upper bound β, then A has a least upper bound $\alpha \in \mathbb{F}$, and $\alpha \le \beta$.*

Proof. By definition of lub, β is an upper bound of B, so by Theorem 1.5, β is also an upper bound of A. Then by the completeness property, there exists a least upper bound α of A, which, by definition of least upper bound, is less than or equal to any upper bound: $\alpha \le \beta$. ∎

The "completeness property" doesn't say anything about lower bounds. However, the following Theorem shows how to find a lower bound, or a glb, just by switching some $\pm$ signs and then using an upper bound.

Theorem 1.12. *Suppose the number system $\mathbb{F}$ has the completeness property, and also the property that for every number $x \in \mathbb{F}$, the system contains an opposite number: $-x \in \mathbb{F}$. Then, any non-empty subset $S \subseteq \mathbb{F}$ that has a lower bound $B \in \mathbb{F}$, also has a greatest lower bound glb$S \in \mathbb{F}$.*

Proof. Let $-S$ denote the set of numbers opposite to the numbers in S (by hypothesis, every element of $-S$ is in $\mathbb{F}$, and if S is non-empty, then so is $-S$). So, let y be any element of $-S$; y must be the opposite of something in S, so $y = -x$, for $x \in S$, and $x \ge B$, so $-y \ge B$, and $y \le -B$. This shows that any element of $-S$ is less than or equal to the number $-B$ (again, by hypothesis, $B \in \mathbb{F} \implies -B \in \mathbb{F}$). So, $-B$ is, by definition, an upper bound for the set $-S$. By the completeness property, there exists a least upper bound $b \in \mathbb{F}$ of $-S$. It turns out that $-b$ is a greatest lower bound of the original set S.

To see this, the first thing to check is that $-b$ is a lower bound of S. So, let x be any element of S. Since b is an upper bound of $-S$, and $-x \in -S$, $-x \le b$ by definition of upper bound, and it follows that $x \ge -b$. The other thing to check is that $-b$ is greater than any other lower bound of S. Suppose B' is a lower bound for S. Then, just as previously, $-B'$ is an upper bound for the set $-S$, which must be bigger than the least upper bound, so $-B' \ge b$. It follows that $B' \le -b$, so $-b$ is greater than the lower bound B'. ∎

For example, $\mathbb{Z}$ and $\mathbb{R}$ satisfy the conditions of the Theorem. The set $\mathbb{N}$ doesn't have the "opposite" property, so this Theorem doesn't apply to $\mathbb{F} = \mathbb{N}$. However, every non-empty subset S of $\mathbb{N}$ has a lower bound, anyway (just use $1 \in \mathbb{N}$), and has a greatest lower bound (the smallest element of S).

1.3 Real Sequences

Definition 1.13. A function $f(x)$ is called "bounded on the domain D" if the set of values $\{f(x) : x \in D\}$ is a bounded set. This means that there's a number M so that for every number $x \in D$, the inequality $|f(x)| \leq M$ holds.

An example of this is $D = \mathbb{N}$, so the function is a sequence:

Definition 1.14. A sequence $a = (a_1, a_2, \ldots, a_n, \ldots)$ is called "bounded" if the set of values $\{a_1, a_2, \ldots, a_n, \ldots\}$ is a bounded set. This means that there is a number M so that for every natural number $n \in \mathbb{N}$, the inequality $|a_n| \leq M$ holds.

The definition of "increasing" is the same for sequences as it is for functions. (because sequences are just a type of function)

Definition 1.15. A sequence a is "increasing" means: $\underline{\text{if } p < q, \text{ then } a_p < a_q}$.

Definition 1.16. A sequence a is "decreasing" means: $\underline{\text{if } p < q, \text{ then } a_q < a_p}$.

Definition 1.17. A sequence a is "monotonic" (or "monotone") if one of the previous two definitions holds.

Definition 1.18. A sequence could be called "weakly increasing," "weakly decreasing," or "weakly monotonic" if the $<$ symbols in the previous three definitions are replaced by $\leq$.

Definition 1.19. A sequence a is "convergent", with "limit L," if for any $\epsilon > 0$, there is some cutoff N with the following property: if $n \geq N$, then $|a_n - L| < \epsilon$. The fact that a is convergent with limit L can be abbreviated:

$$\lim_{n\to\infty} a_n = L.$$

Theorem 1.20. *Suppose b_n is a weakly decreasing sequence, and $\lim_{n\to\infty} b_n = L$. Then $b_n \geq L$ for all n.*

Proof. Suppose, toward a contradiction, that there is some index k so that $b_k < L$. Then let $\epsilon = L - b_k > 0$, so that by the definition of limit, there's some N so that if $n \geq N$, then $|b_n - L| < \epsilon$. This implies $-\epsilon < b_n - L < \epsilon$, and so $-(L - b_k) < b_n - L < L - b_k$ for all $n \geq N$, including some n which are bigger than k. Adding L to the inequality gives $b_k < b_n < 2L - b_k$, for some $n > k$. However, having both $n > k$ and $b_k < b_n$ contradicts the "weakly decreasing" hypothesis, which says that if $k < n$, then $b_k \geq b_n$. ∎

Theorem 1.21. *Suppose b_n is a weakly increasing sequence, and $\lim_{n\to\infty} b_n = L$. Then $b_n \leq L$ for all n.*

Proof. The proof would be very similar to the previous proof. ∎

The previous Theorems assumed a monotonic (decreasing or increasing) sequence was convergent, and proved there was a (lower or upper) bound. The next Theorem assumes a monotonic sequence is bounded, and proves that it is convergent.

Theorem 1.22. *In a number system $\mathbb{F}$ which has the completeness property, let $a = (a_1, \ldots, a_n, \ldots)$ be a sequence. If the sequence is weakly increasing, and bounded above, with upper bound $U \in \mathbb{F}$, then a is a convergent sequence, and its limit is some number $L \in \mathbb{F}$, such that $L \leq U$.*

Proof. Since the set of values of a is non-empty, and has an upper bound, it has a "least upper bound" $\text{lub}\{a_1, a_2, \ldots, a_n, \ldots\} = L \in \mathbb{R}$, $L \leq U$. Given any $\epsilon > 0$, there exists some sequence element $a_N \in (L - \epsilon, L]$ by Theorem 1.9. Because a is weakly increasing, the inequality $L - \epsilon < a_N \leq a_n$ holds for any $n \geq N$, and $a_n \leq L$ by definition of lub, so $L - \epsilon < a_n \leq L < L + \epsilon$. It follows that for any $\epsilon > 0$, there is some N so that $|a_n - L| < \epsilon$ for all $n \geq N$, which is the definition of "a converges to L." ∎

A similar Theorem, proved in a similar way using glb, states that a weakly decreasing sequence with lower bound is convergent, with limit equal to its greatest lower bound. These two Theorems together are called the "Monotonic Sequence Theorem," and the MST is usually used with $\mathbb{F} = \mathbb{R}$. The MST is false for $\mathbb{F} = \mathbb{Q}$, which does not have the completeness property: the earlier example showed that a bounded, increasing sequence of rational numbers need not converge to a rational number. MST is true for $\mathbb{Z}$: every weakly increasing sequence of integers which is bounded above by an integer U converges to some integer $L \leq U$.

1.4 Complex Sequences

Note that many of the above ideas don't apply to the complex number system $\mathbb{C}$. This is because there's no way to work with inequalities. The definitions of "lower bound," "upper bound," "lub," "glb," "increasing," "decreasing," and the Monotonic Sequence Theorem don't apply to complex sequences. However, it is still possible to have a <u>sequence</u> of complex numbers (a function with domain $\mathbb{N}$ where the output values are complex <u>numbers</u>). Also, the definition of <u>bounded</u> makes sense for subsets $S \subseteq \mathbb{C}$. Definition 1.3 uses the absolute value, so $|z| \leq M$ is a comparison of real numbers. Geometrically, a bounded set S is contained in some large disc (radius M) centered at the origin of the complex number plane. This also leads to the definition of bounded functions with complex values, and bounded sequences with complex values: the values have to form a bounded subset of the target set $\mathbb{C}$. The definition of <u>convergent</u> also still works for complex sequences (Definition 1.19), since it also uses absolute <u>values</u>.

Here's a Theorem about two complex sequences, a_n and b_n:

Theorem 1.23. *If $\lim_{n\to\infty} a_n = L$, and $|a_n| = |b_n|$, then b_n is a bounded sequence.*

Proof. The definition of "bounded" says we have to show there's some number M so that $|b_n| \leq M$. Given any $\epsilon > 0$, by the definition of limit, there's some N so that $|a_n - L| < \epsilon$ for $n \geq N$. By the triangle inequality, $|b_n| = |a_n| = |a_n - L + L| \leq |a_n - L| + |L|$. This implies $|b_n| < \epsilon + |L|$ for $n \geq N$, so $|L| + \epsilon$ is a bound for all the elements in the b sequence except the first N. A bound for the whole b sequence is the following maximum of $N + 1$ nonnegative numbers: $M = \max\{|b_1|, |b_2|, \ldots, |b_N|, |L| + \epsilon\}$. ∎

For example, let a be the constant sequence $a_n = 1$, and then b_n can be any sequence of 1's and -1's. The Theorem doesn't say that b must converge, but only that the convergence of a implies the boundedness of b. As another example, with $b_n = a_n$, we get the following:

Corollary 1.24. *If a_n is a convergent sequence, then a_n is bounded.* ∎

Theorem 1.23 can also be rephrased as its logically equivalent contrapositive:

Corollary 1.25. *If b_n is not a bounded sequence, and $|a_n| = |b_n|$, then $\lim_{n\to\infty} a_n$ does not exist.* ∎

1.5 Complex series

Here are a few Theorems that apply to series formed by sequences of complex numbers. The first Theorem could be proved using the rules for limits, but here's a proof that uses only the definition of limit.

Theorem 1.26. *If $\sum_{k=1}^{\infty} a_k$ is a convergent series, then $\lim_{k\to\infty} a_k = 0$.*

Proof. $\lim_{k\to\infty} a_k = 0$ means that for any $\epsilon > 0$, there's some K so that if $k > K$, then $|a_k - 0| < \epsilon$. So, given $\epsilon > 0$, we need to find some cutoff K, past which a_k will be closer to 0 than ϵ.

Denote the partial sums of the series by $s_n = \sum_{k=1}^{n} a_k$. The definition of "convergent series" is that s_n gets close to some limit L: $\lim_{n\to\infty} s_n = L$. In fact, s_n gets within $\epsilon/2$ of L for large enough n: by definition of limit, there's some number N so that if $n > N$, then $|s_n - L| < \epsilon/2$.

Let $K = N + 1$. Then, $k > K = N + 1 \implies k > k - 1 > N$, so we can plug both k and $k - 1$ into the n from the previous paragraph to get $|s_k - L| < \epsilon/2$ and $|s_{k-1} - L| < \epsilon/2$. The triangle inequality (which is still true for complex numbers!) shows how small a_k must be:

$$|a_k| = |s_k - s_{k-1}| = |(s_k - L) + (L - s_{k-1})| \leq |s_k - L| + |s_{k-1} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

∎

The contrapositive follows immediately, and is also useful:

Corollary 1.27 (The "Divergence Test"). *If $\lim_{k\to\infty} a_k \neq 0$ (it approaches some other limit, or the limit does not exist), then $\sum_{k=1}^{\infty} a_k$ is a divergent series.* ∎

Theorem 1.28 (Geometric Sequence). *Given a number $r \in \mathbb{C}$, the sequence r^n diverges if $|r| \geq 1$ and $r \neq 1$. If $|r| < 1$, the sequence converges to 0, and if $r = 1$, the sequence 1^n converges to 1.*

Proof. If $r = 1$, $1^n = 1$ is a constant sequence with limit 1.

If $|r| < 1$, to show $r^n \to 0$, we need to find how big n must be ($n > N$) to get $|r^n - 0| < \epsilon$. If $r = 0$, the sequence is constant, with limit 0. If $r \neq 0$, use the increasing function ln on the inequality $|r|^n < \epsilon$ to get $n\ln(|r|) < \ln(\epsilon)$. Since $0 < |r| < 1$, $\ln(|r|) < 0$, so dividing by $\ln(|r|)$ reverses the inequality: $n > \ln(\epsilon)/\ln(|r|)$. Let N be the number $\ln(\epsilon)/\ln(|r|)$, which is the cutoff guaranteeing $|r^n| < \epsilon$.

If $|r| > 1$, we can show even more than $\lim r^n$ DNE: it will follow from Corollary 1.25 that $\lim_{n\to\infty} r^n$ does not exist, because r^n is not a bounded sequence. To show that r^n is not bounded, we need to find how big n must be ($n > N$) to get $|r^n| > M$ (so any number $M > 0$ is not a bound for r^n). Again, using ln on both sides of $|r|^n > M$ gives $n\ln(|r|) > \ln(M)$, and since $|r| > 1$, $\ln(|r|)$ is positive and we can divide without switching the inequality: $n > \ln(M)/\ln(|r|) = N$.

If $|r| = 1$ (so r is on the unit circle) but $r \neq 1$, then $|1 - r|$ is a positive number, so we can use $\frac{1}{2}|1 - r|$ as an output tolerance. The sequence r^n is bounded: $|r^n| = |r|^n = 1^n = 1$, but not convergent. Suppose, toward a contradiction, that r^n has some limit L, which means there's some cutoff N so that for all $n \geq N$, $|r^n - L| < \frac{1}{2}|1 - r|$. In particular, for some $n \geq N$, $n + 1$ is also $> N$, so $|r^n - L|$ and $|r^{n+1} - L|$ are both $< \frac{1}{2}|1 - r|$. However, then

$$|r^n - r^{n+1}| = |r^n(1 - r)| = |r^n||1 - r| = 1|1 - r| = |1 - r|,$$

and also (by the triangle inequality for complex number addition)

$$|r^n - r^{n+1}| = |(r^n - L) - (r^{n+1} - L)| \leq |r^n - L| + |r^{n+1} - L| < \frac{1}{2}|1 - r| + \frac{1}{2}|1 - r|,$$

but this implies $|1 - r| < |1 - r|$, which is false, so the contradiction shows that there is no such limit L. ∎

Theorem 1.29 (Geometric Series). *Given a number $r \in \mathbb{C}$, the series $\sum_{n=1}^{\infty} r^n$ diverges if $|r| \geq 1$. If $|r| < 1$, the series converges to $\frac{r}{1-r}$.*

Proof. The first case to check is $|r| = 1$ (so r is on the unit circle inside the complex number plane). Then, $\lim_{n\to\infty} r^n$ does not converge to 0 (the limit is either 1 or DNE by the previous Theorem), so the Divergence Test shows the series diverges.

For the rest of the Proof, assume $|r| \neq 1$. The partial sum $s_n = r^1 + r^2 + \cdots + r^n$ satisfies $s_n - r \cdot s_n = (r^1 + r^2 + \cdots + r^n) - r^1 \cdot (r^1 + r^2 + \cdots + r^n) = r - r^{n+1}$. Solving for s_n, (dividing by $1 - r$, which is non-zero because we're assuming $r \neq 1$) gives $s_n = \frac{r - r^{n+1}}{1-r} = \frac{r(1-r^n)}{1-r}$. The infinite sum is the limit of the partial sums:

$$\sum_{n=1}^{\infty} r^n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{r(1 - r^n)}{1 - r} = \frac{r(1 - \left(\lim_{n\to\infty} r^n\right))}{1 - r}.$$

If $|r| < 1$, the limit of r^n is 0, which proves the limit of the s_n is equal to the formula stated by the Theorem.

If $|r| > 1$, then the limit of r^n does not exist, so the series diverges (the Divergence Test again, or using the above formula for the partial sums). ∎

1.6 Real series

Here are some important Theorems about <u>real</u> series. The proofs depend on the Monotonic Sequence Theorem.

Theorem 1.30 (The "Comparison Test" for convergence). *If sequences a_n and b_n satisfy:*
(1) $0 \leq a_n \leq b_n$, and

(2) $\sum_{n=1}^{\infty} b_n$ is a convergent series,

then $\sum_{n=1}^{\infty} a_n$ is a convergent series.

Proof. Condition (2), by definition of "convergent series," means that the sequence of partial sums of b_n, $S_n = \sum_{k=1}^{n} b_k$, is a convergent sequence, with limit $L = \lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} b_k$. Also, S_n is a weakly increasing sequence:

$$S_{n+1} - S_n = \sum_{k=1}^{n+1} b_k - \sum_{k=1}^{n} b_k = b_{n+1} \geq 0 \implies S_n \leq S_{n+1}.$$

By Theorem 1.21, $S_n \leq L$ for all n.

The sequence of partial sums of a_n, $s_n = \sum_{k=1}^{n} a_k$, is weakly increasing (again, $s_{n+1} - s_n = a_n \geq 0 \implies s_n \geq s_{n-1}$), and bounded above:

$$s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k = S_n \leq L.$$

The first inequality follows from (1) (each a_k is less than b_k, so the partial sums of the a_k are also less than the partial sums of the b_k). By the Monotonic Sequence Theorem, s_n converges to some real number less than or equal to the upper bound L:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k \leq L = \sum_{k=1}^{\infty} b_k.$$

∎

If hypothesis (1), $0 \leq a_n \leq b_n$, holds only for large n (that is, if there is some cutoff N so that $n \geq N \implies 0 \leq a_n \leq b_n$), then the Comparison test can still be used to determine convergence, but we no longer get the estimate $\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$.

The contrapositive of the Theorem is called the "Comparison test for divergence":

Corollary 1.31. *If sequences a_n and b_n satisfy:*

(1) $0 \leq a_n \leq b_n$, and

(2) $\sum_{n=1}^{\infty} a_n$ is a divergent series,

then $\sum_{n=1}^{\infty} b_n$ is a divergent series. ∎

So, to determine if a positive series $\sum a_n$ is convergent, the Comparison test is useful if you can find some other series b_n with $a_n \leq b_n$. The next Theorem is more useful if you can find a series b_n with $a_n \approx b_n$ (b_n is approximately the same as a_n, but not necessarily always greater than a_n).

Theorem 1.32 (The "Limit Comparison Test" for convergence). *If sequences a_n and b_n satisfy:*

(1) $a_n \geq 0$ and $b_n > 0$,

(2) $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, and

(3) $\sum_{n=1}^{\infty} b_n$ is a convergent series,

Then $\sum_{n=1}^{\infty} a_n$ is a convergent series.

Proof. Formula (2) means that for any $\epsilon > 0$, there's some N so that for $n \geq N$, $|\frac{a_n}{b_n} - c| < \epsilon$. This implies $\frac{a_n}{b_n} - c < \epsilon$, and since $b_n > 0$, we can multiply both sides by b_n (without reversing the inequality!) to get $a_n - b_n \cdot c < b_n \cdot \epsilon$. So, $a_n < b_n(c + \epsilon)$ for $n \geq N$. Since $\sum_{n=1}^{\infty} b_n$ is a convergent series, and $c + \epsilon$ is just a scalar, $\sum_{n=1}^{\infty} (b_n \cdot (c + \epsilon))$ is a convergent series, and, using the previous Theorem, $\sum_{n=N}^{\infty} a_n$ is convergent by comparison to $\sum_{n=N}^{\infty} (b_n \cdot (c + \epsilon))$. This shows the a_n series is convergent, since we can just add the first $N - 1$ terms without changing the fact that it converges: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n$. ∎

Note that c can be any constant, and the idea of the limit (2) is that $a_n \approx c \cdot b_n$, i.e., the a sequence is approximately just a scalar multiple of the b sequence, and the above proof just stated this approximation more precisely. Unlike the Comparison test, we do not get any estimate for the sum of the a_n series. If $\lim_{n\to\infty} \frac{a_n}{b_n}$ does not exist (for example, if it's $+\infty$), then this Theorem does not apply.

Theorem 1.33 (The "Limit Comparison Test" for divergence). *If sequences a_n and b_n satisfy:*

(1) $a_n \geq 0$ and $b_n > 0$,

(2) $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, and

(3) $\sum\limits_{n=1}^{\infty} b_n$ is a divergent series,

Then $\sum\limits_{n=1}^{\infty} a_n$ is a divergent series.

Proof. Note the new condition in (2): the limit c has to be positive! We can use this positivity to let $\epsilon = \frac{1}{2}c > 0$, so that the definition of the limit in (2) means there's some N so that if $n \geq N$, then $|\frac{a_n}{b_n} - c| < \frac{1}{2}c$, which implies $-\frac{c}{2} < \frac{a_n}{b_n} - c < \frac{c}{2}$, and adding c to the inequality, and then multiplying by $b_n > 0$, gives $\frac{c}{2} \cdot b_n < a_n$. By (3), $\sum\limits_{n=1}^{\infty} \frac{c}{2} b_n$ is a divergent series (using $c > 0$ again, we get a positive scalar multiple of a divergent series), so by the Comparison test for divergence, $\sum\limits_{n=1}^{\infty} a_n$ is a divergent series. ∎

The new condition $c > 0$ is necessary. If $c = 0$, and b_n forms a divergent series, the Limit Comparison test is <u>inconclusive</u>. The approximation $a_n/b_n \approx c \implies a_n \approx c \cdot b_n$ isn't very useful if $c = 0$: it just says $a_n \approx 0$, which is not enough to establish convergence or divergence of the series. For example, $a_n = 1/n^2$ and $b_n = 1/n$ are positive sequences, with $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$. The b series diverges, but the a series converges.

Theorem 1.34 (The "Alternating Series Test"). *If the sequence b_n satisfies*

(1) $b_n \geq b_{n+1}$ ("weakly decreasing"), and

(2) $\lim\limits_{n\to\infty} b_n = 0$,

then the series $\sum\limits_{n=1}^{\infty} (-1)^{n-1} b_n$ is convergent.

Proof. First of all, notice the sequence b_n must be non-negative. If it's (1) weakly decreasing, and (2) has limit zero, then by Theorem 1.20, $b_n \geq 0$.

Define the "even partial sum sequence" $e_n = \sum\limits_{i=1}^{2n} (-1)^{i-1} b_i = b_1 - b_2 + b_3 - b_4 + \ldots - b_{2n}$. Then $e_{n+1} - e_n = b_{2n+2} - b_{2n+1} \geq 0$, by (1), so e_n is weakly increasing.

Define the "odd partial sum sequence" $d_n = \sum\limits_{i=1}^{2n-1} (-1)^{i-1} b_i = b_1 - b_2 + b_3 - b_4 + \ldots + b_{2n-1}$. Then $d_n - d_{n+1} = b_{2n} - b_{2n+1} \geq 0$, by (1), so d_n is weakly decreasing.

Since b is non-negative, $d_n - e_n = b_{2n} \geq 0$, so $e_n \leq d_n$.

Using this inequality, any d_n is an upper bound for all sequence values e_m: if $n \le m$, then $e_m \le d_m \le d_n$, and if $n > m$, then $e_m \le e_n \le d_n$. So, e_n is a bounded, weakly increasing sequence, and the Monotonic Sequence Theorem implies that $\lim_{m\to\infty} e_m = L_1 = \text{lub}\{e_m\} \le d_n$.

Using the inequality $e_n \le d_n$, any e_n is a lower bound for all sequence values d_m: if $n \le m$, then $d_m \ge d_n \ge e_n$, and if $n > m$, then $d_m \ge e_m \ge e_n$. So, d_n is a bounded, weakly decreasing sequence, and the Monotonic Sequence Theorem implies that $\lim_{m\to\infty} d_m = L_2 = \text{glb}\{d_m\} \ge e_n$.

The two inequalities $e_n \le L_1 \le d_n$ and $e_n \le L_2 \le d_{n+1}$ can be subtracted:

$$\begin{aligned} e_n &\le L_1 \le d_n \\ -d_{n+1} &\le -L_2 \le -e_n \\ \implies e_n - d_{n+1} &\le L_1 - L_2 \le d_n - e_n \\ \implies -b_{2n+1} &\le L_1 - L_2 \le b_{2n}. \end{aligned}$$

Then by (2), for any $\epsilon > 0$, there's some N so that $|b_k - 0| < \epsilon$ for $k > N$, so if $2n > N$, then $-\epsilon < -b_{2n+1} \le L_1 - L_2 \le b_{2n} < \epsilon$. This shows $L_1 - L_2$ is smaller than any positive ϵ and greater than any negative number $-\epsilon$, so $L_1 - L_2 = 0$, and $L_1 = L_2$. Let L denote $L_1 = L_2$, so whether k is even ($k = 2n$) or odd ($k = 2n-1$), the inequalities $0 \le L - e_n \le d_{n+1} - e_n = b_{2n+1}$ and $0 \le d_n - L \le d_n - e_n = b_{2n}$ can be summarized as

$$\left| L - \sum_{i=1}^{k} (-1)^{i-1} b_i \right| \le b_{k+1}.$$

This is the "Alternating Series Estimate," and it follows from (2) that if $k > N$, then $b_{k+1} = |b_{k+1} - 0| < \epsilon$, which proves convergence:

$$\left| L - \sum_{i=1}^{k} (-1)^{i-1} b_i \right| < \epsilon \implies L = \lim_{k\to\infty} \sum_{i=1}^{k} (-1)^{i-1} b_i = \sum_{i=1}^{\infty} (-1)^{i-1} b_i.$$

∎

The inequality $e_n \le L \le d_n$ from the Proof provides both a lower and upper bound for the exact value of the alternating infinite sum. The proof of convergence doesn't require that the sequence is weakly decreasing for all n — it is enough to assume b_n is weakly decreasing past some cutoff, M: if $n \ge M$, then $b_n \ge b_{n+1}$. (The first M terms don't affect convergence or divergence.) However, in this case, the Alternating Series Estimate will only hold for $k \ge M$.

1.7 More about complex series

The Theorems in Section 1.6 apply only to real sequences. But if we have a sequence of a_n complex numbers, we can consider the related sequence $|a_n|$, which is real and non-negative. Recall the absolute value of a complex number is defined by $|x + iy| = \sqrt{x^2 + y^2}$, which is its distance to the origin. The following Theorem applies to any complex series a_n.

Theorem 1.35. *If* $\sum_{n=1}^{\infty} |a_n|$ *is a convergent series, then* $\sum_{n=1}^{\infty} a_n$ *is a convergent series.*

Proof. The complex numbers a_n have real and imaginary parts, $a_n = x_n + iy_n$. The x_n and y_n are real numbers, and could be positive or negative. We'll focus on just the x_n sequence, which satisfies the following inequality: $-|x_n| \le x_n \le |x_n|$. In fact, x_n is actually equal to either $|x_n|$, if it's positive, or $-|x_n|$ if it's negative, just by definition of absolute value for real numbers. Adding $|x_n|$ to the inequality gives $0 \le x_n + |x_n| \le 2|x_n|$.

Another inequality which holds for real numbers y is $0 \le y^2$, so $0 \le x^2 \le x^2 + y^2$, and the square root function is increasing on $[0, \infty)$, so $\sqrt{x^2} \le \sqrt{x^2 + y^2}$ for any real x and y, which proves $0 \le |x_n| \le \sqrt{x_n^2 + y_n^2} = |a_n|$. The Comparison test shows that $\sum_{n=1}^{\infty} |x_n|$ is convergent, by comparison with the convergent series $\sum_{n=1}^{\infty} |a_n|$.

Combining the above inequalities gives $0 \le x_n + |x_n| \le 2|x_n| \le 2|a_n|$, and again the Comparison test shows $\sum_{n=1}^{\infty}(x_n + |x_n|)$ is convergent, by comparison with the convergent series $\sum_{n=1}^{\infty} 2|a_n|$. Subtracting two convergent series gives another convergent series:

$$\sum_{n=1}^{\infty}(x_n + |x_n|) - \sum_{n=1}^{\infty} |x_n| = \sum_{n=1}^{\infty}(x_n + |x_n| - |x_n|) = \sum_{n=1}^{\infty} x_n.$$

A similar argument, using the inequality $y_n \le |y_n| \le \sqrt{x_n^2 + y_n^2} = |a_n|$, shows that $\sum_{n=1}^{\infty} y_n$ is convergent. Then, the original series a_n is a sum of convergent series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(x_n + iy_n) = \left(\sum_{n=1}^{\infty} x_n\right) + i\left(\sum_{n=1}^{\infty} y_n\right).$$

∎

The Theorem can be summarized by saying that any absolutely convergent series is a convergent series.

Here are some comments on the "Ratio test," where a_n can be any sequence of complex terms.

Theorem 1.36 (The Ratio Test). *If* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$, *and* $L < 1$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent. If* $L > 1$, *or* $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, *then the series is divergent.*

The Theorem is inconclusive if $L = 1$, or if the limit L does not exist, but also does not satisfy the technical definition of a $+\infty$ limit. The basic idea is an analogy with the geometric series: if $a_n = r^n$, then the ratio is exactly constant: $|(r^{n+1})/r^n| = |r|$, so $L = |r|$, and the convergence or divergence of the series depends on whether $|r| < 1$ or $|r| > 1$. Series satisfying the conditions of the Ratio test don't necessarily have a constant ratio, but it's close to L, so it's approximately constant, and the a_n series is close to a geometric series.

Proof. First, consider $L < 1$, and let $\epsilon = (1-L)/2 > 0$ (so ϵ is half the distance from L to 1). Then, there's some number N so that if $n \geq N$, then $|\left|\frac{a_{n+1}}{a_n}\right| - L| < (1-L)/2$, which implies $\left|\frac{a_{n+1}}{a_n}\right| < L + (1-L)/2 = \frac{1+L}{2}$, and $|a_{n+1}| < (\frac{1+L}{2})|a_n|$. Since this is true for all $n \geq N$, we can start with $n = N$, to get $|a_{N+1}| < (\frac{1}{2} + \frac{L}{2})|a_N|$, and then continue with $n = N+1$, to get

$$|a_{N+2}| < (\frac{1+L}{2}) \cdot |a_{N+1}| < (\frac{1+L}{2}) \cdot \left((\frac{1+L}{2})|a_N| \right) = (\frac{1+L}{2})^2 |a_N|.$$

The pattern continues recursively, so that for any $k \in \mathbb{N}$,

$$|a_{N+k}| < (\frac{1+L}{2})^k |a_N|.$$

The Comparison test applies:

$$\sum_{k=1}^{\infty} |a_{N+k}| \leq \sum_{k=1}^{\infty} \left((\frac{1}{2} + \frac{L}{2})^k |a_N| \right) = |a_N| \sum_{k=1}^{\infty} (\frac{1+L}{2})^k,$$

which, since $\frac{1+L}{2} < \frac{1}{2} + \frac{1}{2} = 1$, is a convergent geometric series. The series starting with $n = 1$ is also convergent: $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{k=1}^{\infty} |a_{N+k}|$.

The second case is $L > 1$. Let $\epsilon = (L-1)/2 > 0$ (again, this is half the distance from L to 1), so there's some number N so that if $n \geq N$, then $|\left|\frac{a_{n+1}}{a_n}\right| - L| < (L-1)/2$. This implies $a_n \neq 0$ for $n \geq N$ (since we're dividing by a_n), and also the two inequalities: $-(L-1)/2 < \left|\frac{a_{n+1}}{a_n}\right| - L < (L-1)/2$. Adding L to the inequality gives $-(L-1)/2 + L < \left|\frac{a_{n+1}}{a_n}\right|$, so $\left|\frac{a_{n+1}}{a_n}\right| > \frac{L+1}{2}$, and $|a_{n+1}| > (\frac{L+1}{2})|a_n|$ for $n \geq N$. It follows recursively as in the previous paragraph that

$$|a_{N+k}| > (\frac{1+L}{2})^k |a_N|.$$

The Proof of the Geometric Sequence Theorem (Theorem 1.28) showed that $\frac{L+1}{2} > \frac{1}{2} + \frac{1}{2} = 1$ implies $(\frac{L+1}{2})^k$ is an unbounded sequence, and since $a_N \neq 0$, $(\frac{1+L}{2})^k |a_N|$ is also unbounded, and so is $|a_n|$. By Corollary 1.25, a_n is not a convergent sequence, so by the Divergence test, $\sum_{n=1}^{\infty} a_n$ is divergent.

The third case is $L = \infty$, which means that for any M, there's some N so that if $n \geq N$, then $|a_{n+1}/a_n| \geq M$. In particular, let $M = 2$, so there's a corresponding cutoff N where $|a_{n+1}/a_n| \geq 2$ for $n \geq N$. Starting at the non-zero term a_N, $|a_{N+1}| \geq 2|a_N|$, and $|a_{N+2}| \geq 2|a_{N+1}| \geq 2 \cdot (2|a_N|)$, so recursively, $|a_{N+k}| \geq 2^k |a_N|$. Similarly to the previous paragraph, the sequence a_n is unbounded, and the series is divergent. ∎

Lemma 1.37. *If r is a complex number such that $|r| < 1$, then $\lim_{n \to \infty} (n \cdot r^n) = 0$.*

Proof. This could be proved by L'Hôpital's Rule, but instead we'll use the Ratio Test and Theorem 1.26. If $r = 0$, then the sequence $n \cdot r^n$ is constant, with limit 0. For $r \neq 0$, consider the following series:

$$\sum_{n=1}^{\infty} n \cdot r^n.$$

The limit from the Ratio test is $L = \lim_{n\to\infty} \left| \frac{(n+1)r^{(n+1)}}{nr^n} \right| = \lim_{n\to\infty} \frac{n+1}{n}|r| = |r|$, so $L < 1$ by hypothesis, the series is absolutely convergent by the Ratio test, and the sequence has limit 0 by Theorem 1.26. ∎

Here are some comments on the "Root Test," which is similar to the Ratio test. a_n can be complex, but the $(1/n)$ power refers to the non-negative n^{th} root of the real number $|a_n|$.

Theorem 1.38 (The Root Test). *If* $\lim_{n\to\infty} |a_n|^{(1/n)} = L$*, and* $L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent. If* $L > 1$*, or* $\lim_{n\to\infty} |a_n|^{(1/n)} = \infty$ *then the series is divergent.*

The Theorem is inconclusive if $L = 1$, or if the limit L does not exist, but also does not satisfy the technical definition of a $+\infty$ limit. The basic idea is that the geometric series r^n has a constant n^{th} root, $|r^n|^{(1/n)} = |r|$, so $L = |r|$, and the convergence and divergence results depend on whether $|r| < 1$ or $|r| > 1$. Series satisfying the conditions of the root test don't necessarily have a constant $(a_n)^{(1/n)}$, but it's close to L, so it's approximately constant, and the a_n series is close to a geometric series.

Proof. First, consider $L < 1$, and let $\epsilon = (1-L)/2 > 0$. Then, there's some number N so that if $n > N$, then $||a_n|^{(1/n)} - L| < (1-L)/2$, which implies $|a_n|^{(1/n)} < L + (1-L)/2 = \frac{1}{2} + \frac{L}{2}$. Applying the function $f(x) = x^n$ to both sides gives $|a_n| < (\frac{1}{2} + \frac{L}{2})^n$ (since f is increasing for $x \geq 0$). Since $\frac{1}{2} + \frac{L}{2} < \frac{1}{2} + \frac{1}{2} = 1$, $\sum_{n=1}^{\infty} (\frac{1}{2} + \frac{L}{2})^n$ is a convergent geometric series, and $0 \leq |a_n| < (\frac{1}{2} + \frac{L}{2})^n$ for $n > N$, so $\sum_{n=1}^{\infty} |a_n|$ is convergent by the Comparison Test.

The second case is $L > 1$. Let $\epsilon = (L-1)/2 > 0$, so there's some number N so that if $n > N$, then $||a_n|^{(1/n)} - L| < (L-1)/2$, which implies $-(L-1)/2 < |a_n|^{(1/n)} - L < (L-1)/2$. Adding L to the inequality gives $-(L-1)/2 + L < |a_n|^{(1/n)}$, so $|a_n|^{(1/n)} > \frac{L}{2} + \frac{1}{2} > 0$, and $|a_n| > (\frac{L}{2} + \frac{1}{2})^n$ for $n > N$. Since $\frac{L}{2} + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = 1$, the Proof of the Geometric Sequence Theorem (Theorem 1.28) showed that $(\frac{L}{2} + \frac{1}{2})^n$ is an unbounded sequence, so $|a_n|$ is also unbounded. By Corollary 1.25, a_n is not a convergent sequence, so by the Divergence Test, $\sum_{n=1}^{\infty} a_n$ is divergent.

The third case is $L = \infty$, which means for any M, there's some N so that if $n \geq N$, then $|a_n|^{(1/n)} \geq M$. In particular, let $M = 1$, so there's some corresponding cutoff N so that $|a_n|^{(1/n)} \geq 1$ for $n \geq N$, which implies (again raising both sides to the n^{th} power) $|a_n| \geq 1^n = 1$. It follows (from the definition of limit) that $\lim_{n\to\infty} a_n \neq 0$, so the series diverges by the Divergence test. ∎

Example 1.39. Define a sequence $a_n = \dfrac{1}{2^{n+(-1)^n}}$. The Root test shows that this forms a convergent series $\sum a_n$, since $(a_n)^{(1/n)} = (2^{-n} \cdot 2^{(-1)^{n+1}})^{(1/n)} = \frac{1}{2} \cdot (2^{(-1)^{n+1}})^{(1/n)}$, which has limit $\frac{1}{2}$. This particular sequence is an example where the Ratio test doesn't work:

$$\lim_{n\to\infty} |a_{n+1}/a_n| = \lim_{n\to\infty} \frac{1}{2^{n+1+(-1)^{n+1}}} \cdot 2^{n+(-1)^n} = \lim_{n\to\infty} \frac{1}{2} \cdot 2^{(-1)^n-(-1)^{n+1}}.$$

This limit doesn't exist, since the exponent $(-1)^n - (-1)^{n+1} = (-1)^n + (-1)^n = 2 \cdot (-1)^n$ oscillates between -2 and $+2$.

1.8 Power Series

Definition 1.40. An infinite series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$ is called a "power series." The c_n are called the coefficients, and a is called the "center" of the power series.

The coefficients c_n, the center a, and the variable x can all be complex numbers. The index n usually starts at 0 (the constant term is $c_0x^0 = c_0$), or, if the first few coefficients are 0, n may start at any positive integer. (The definition of power series excludes negative or non-integer exponents n.)

Definition 1.41. The "domain of convergence" of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the set of all (complex) numbers x so that the series is convergent.

Note that the domain of convergence cannot be the empty set, since any power series always converges at its center, $x = a$: $\sum_{n=0}^{\infty} c_n(a-a)^n = c_0 + c_1 0^1 + c_2 0^2 + \cdots = c_0$.

The following Lemma gives a "comparison" criterion for telling whether x is in the domain of convergence of a power series with coefficients c_n and center 0.

Lemma 1.42. *Suppose r is a positive constant, and the sequence c_nr^n is bounded. Then, $\sum_{n=0}^{\infty} c_nx^n$ is absolutely convergent for all x in the disc $\{x \in \mathbb{C} : |x| < r\}$.*

Proof. Recall "bounded" means there exists some number M so that $|c_nr^n| \le M$ for all n. Since r is positive, $|c_nr^n| = |c_n|r^n \le M \implies |c_n| \le M/r^n \implies |c_n| \cdot |x^n| \le |x^n|M/r^n \implies |c_nx^n| \le M|x/r|^n$. Since $|x| < r$, $|x/r| < 1$ and $\sum_{n=0}^{\infty} M|x/r|^n$ is a convergent geometric series, so $\sum_{n=0}^{\infty} |c_nx^n|$ converges by comparison, and $\sum_{n=0}^{\infty} c_nx^n$ is absolutely convergent. ∎

Observe that Lemma 1.42 doesn't say anything about the *series* $\sum c_nr^n$, only the *sequence* c_nr^n. However, if the series $\sum c_nr^n$ happens to be convergent, then the sequence c_nr^n must have limit 0 (Theorem 1.26), and must also be bounded (Corollary 1.24), so Lemma 1.42 applies.

This proves that if $r > 0$ is in the domain of convergence of a power series with center 0, and if $|x| < r$, then x is also in the domain. This is almost, but not quite, enough to prove a theorem called Abel's Lemma, that the domain of convergence of any power series with center 0 must be a disc centered at 0 (the domain may contain all, part, or none of its boundary circle). The Proof uses Lemma 1.42 twice.

Theorem 1.43 ("Abel's Lemma"). *For a power series centered at a, $\sum_{n=0}^{\infty} c_n(x-a)^n$, exactly one of the following holds:*

- $\sum_{n=0}^{\infty} c_n(x-a)^n$ *converges to c_0 at $x = a$, and diverges for all other x.*
- $\sum_{n=0}^{\infty} c_n(x-a)^n$ *is absolutely convergent for all $x \in \mathbb{C}$.*
- *There is a real number $R > 0$ so that $\sum_{n=0}^{\infty} c_n(x-a)^n$ is absolutely convergent for $|x-a| < R$, and the series is divergent for $|x-a| > R$.*

Definition 1.44. The number R is the "Radius of convergence" of the power series, and it must be nonnegative. The first two cases are referred to as $R = 0$ and $R = \infty$.

Note that the Theorem is inconclusive when both $0 < R < \infty$ and $|x-a| = R$. (Geometrically, this is the case where the domain of convergence is a disc in $\mathbb{C}$ with positive radius, and x is on the circular boundary of the disc.) The power series could be divergent, absolutely convergent, or conditionally convergent for x on the boundary. In the case where the center a is on the real number line, then the real values of x for which the series is convergent form an interval centered at a (the intersection of the disc and the real axis), and the points on the boundary are the two endpoints, $a - R$ and $a + R$.

Proof of Theorem 1.43. Since we can make the substitution $x - a$ instead of x, it will be enough to prove the Theorem when $a = 0$, so that the power series is $\sum_{n=0}^{\infty} c_n x^n$, and the three cases are $x = 0$, $x \in \mathbb{C}$, or $|x| < R$.

Given the coefficient sequence c_n, consider the set of real numbers $S = \{r \geq 0 : c_n r^n$ is a bounded sequence$\} \subseteq \mathbb{R}$. For example, 0 is an element of S because $c_n 0^n = 0$ is bounded, so S is not the empty set.

Suppose S does not have an upper bound; that is, for any $x \in \mathbb{R}$, there's some $r \in S$ so that $|x| < r$. Then $c_n r^n$ is bounded (by definition of $r \in S$), and $\sum_{n=0}^{\infty} c_n x^n$ is absolutely convergent by Lemma 1.42. This is the $R = \infty$ case.

If S does have an upper bound, then by the Completeness property of the real number system $\mathbb{R}$, S has a least upper bound: $R = \mathrm{lub}(S)$, and $R \geq 0$.

If $|x| > R$, then $|x| \notin S$, meaning $c_n|x|^n$ is not bounded. Since $c_n|x|^n$ and c_nx^n have the same absolute value, Corollary 1.25 applies, proving c_nx^n is a divergent sequence, so $\sum_{n=0}^{\infty} c_nx^n$ diverges by the "Divergence test."

If $|x| < R$, then there's some $r \in S$ such that $|x| < r \leq R$ (by Theorem 1.9, with $\epsilon = R - |x| > 0$). Again, $r \in S$ means c_nr^n is bounded, which is exactly what is needed for Lemma 1.42, so $\sum_{n=0}^{\infty} c_nx^n$ is absolutely convergent. ∎

Theorem 1.45. *If $\sum_{n=0}^{\infty} c_nx^n$ has radius of convergence R, then $\sum_{n=1}^{\infty} c_nnx^{n-1}$ also has radius of convergence R.*

Proof. Let S' be the set $\{r \geq 0 : c_nnr^{n-1}$ is a bounded sequence$\}$, similar to the set S from the previous Proof, so that if S' has a least upper bound R', then R' is the radius of convergence of $\sum_{n=1}^{\infty} c_nnx^{n-1}$. Also, if S' has no upper bound, $R' = \infty$. The Theorem claims that $R' = R$, or that they're both ∞. The strategy will be to show both $R' \leq R$ (Step 1.), and $R' \geq R$ (Step 2.), so that equality must hold.

Step 1. Suppose $x \in S'$ (there's at least one such x, for example $x = 0$), so that by definition of S', $|c_nnx^{n-1}| \leq M$. Then for $n \geq 1$, $|c_nx^n| \leq |c_nnx^n| = |c_nnx^{n-1}| \cdot |x| \leq M|x|$, so c_nx^n is a bounded sequence, and $x \in S$. This shows $S' \subseteq S$, so if S has a least upper bound R, then S' has least upper bound $R' \leq R$ by Theorem 1.11, which uses the Completeness property of $\mathbb{R}$. Since $x = 0$ is in the set, $R' \geq 0$ by definition of upper bound. If S' has no upper bound ($R' = \infty$), then S also has no upper bound (by Corollary 1.6), so $R = \infty = R'$. If S has no least upper bound, this calculation is inconclusive and R' could be finite or infinite. If $R = 0$, then $0 \leq R' \leq R$ implies R' is also 0.

Step 2. Assume $R > 0$ or $R = \infty$ (since in the $R = 0$ case, Step 1. just showed $R' = R$). Also assume S' has a least upper bound $R' \geq 0$ (since $R' = \infty$ was also covered in Step 1.). By Theorem 1.9, there is some $\rho \in S$ so that $0 < R - \epsilon < \rho \leq R$. ($\rho$=letter "rho", and Theorem 1.9 applies with any $\epsilon \in (0, R)$ if R is finite; if $R = \infty$, pick any positive $\rho \in S$.) Pick any r so that $0 < R - \epsilon < r < \rho$ (or, if $R = \infty$, pick any r so $0 < r < \rho$). Then since $\rho \in S$, $|c_n\rho^n| \leq M$, and

$$n|c_n|r^{n-1} = \frac{1}{r}|c_n|\rho^n n(\frac{r}{\rho})^n \leq \frac{1}{r}Mn(\frac{r}{\rho})^n,$$

and since $|\frac{r}{\rho}| < 1$, Lemma 1.37 shows the sequence $\frac{M}{r} \cdot n(\frac{r}{\rho})^n$ converges to 0, so it is bounded (Corollary 1.24), and the previous inequality implies nc_nr^{n-1} is also a bounded sequence, so $r \in S'$ and $r \leq R'$. The numbers r, R, and R' are this close: $R - \epsilon < r \leq R'$. Since $R < R' + \epsilon$ even when ϵ is very small, R must be less than or equal to R'. This, together with Step 1., proves $R = R'$. If $R = \infty$ (meaning S does not have an upper bound), then ρ, and r, could both be arbitrarily large, and $r \in S'$ means S' also has no upper bound, and $R' = \infty = R$. ∎