Notes on the elementary geometry of real involutions of complex projective spaces

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Abstract

This approach to real projective geometry takes complex projective geometry as a starting point. [1]

1 Introduction

The containment of real projective spaces as subsets of complex projective spaces is considered from an elementary, algebraic point of view. By using complex linear algebra and "conjugate linear" maps, the goal is to construct a model of a real projective space, by taking the set of points to be the fixed point set of a certain type of involution of $\mathbb{C}P^m$. The lines of the model will be defined as intersections of complex projective lines with this subset of $\mathbb{C}P^m$. This linear algebraic approach generalizes the special case considered in [C], where the involution of $\mathbb{C}P^m$ was induced by complex conjugation of the coordinates of $\mathbb{C}^{m+1}$ with respect to a specific coordinate system.

The main results of the last two Sections find some conditions under which a collineation of $\mathbb{C}P^m$ must be a projective transformation or a projective transformation composed with a conjugation (Theorem 4.6 for $m \neq 1$, Theorem 5.12 for $m = 1$). Corollary 4.7 shows that for $m \neq 1$, these are the only collineations that restrict to collineations of a real projective subspace.

2 The complex foundation

Most of the constructions we'll need are standard in elementary projective geometry, and we briefly recall them here to fix notation.

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Let $\mathbb{C}$ be the field of complex numbers, and let $m \geq 0$ be an integer, so $\mathbb{C}^{m+1}$ is a complex vector space. The complex projective m-space, $\mathbb{C}P^m$, is the set of one-dimensional subspaces in $\mathbb{C}^{m+1}$. Denote the usual projection $\pi^m : \mathbb{C}^{m+1} \setminus \{\mathbf{0}\} \to \mathbb{C}P^m$, so that a non-zero column vector $\mathbf{z}$ spans the one-dimensional subspace $\pi^m(\mathbf{z})$. An element $z \in \mathbb{C}P^m$ with representative non-zero vector $\mathbf{z} = (z_0, z_1, \ldots, z_m)^T$ will have homogeneous coordinates $[z_0 : z_1 : \ldots : z_m]$.

For $d = 0, \ldots, m$, define a d-dimensional complex projective subspace of $\mathbb{C}P^m$ to be the image of some $(d+1)$-dimensional complex linear subspace $\mathbf{V} \subseteq \mathbb{C}^{m+1}$, $V = \pi^m(\mathbf{V} \setminus \{\mathbf{0}\})$. So, for example, a complex projective line in $\mathbb{C}P^m$ is, by definition, the image of a 2-dimensional complex linear subspace of $\mathbb{C}^m$.

Let $\mathbf{f} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ be any function. Given $\mathbf{z} \in \mathbb{C}^{m+1} \setminus \{\mathbf{0}\}$, suppose $\mathbf{f}$ has the following two properties: first,

$$\mathbf{f}(\mathbf{z}) \neq \mathbf{0}, \tag{1}$$

and second, for any $\lambda \in \mathbb{C} \setminus \{0\}$, there exists $\mu \in \mathbb{C} \setminus \{0\}$ so that

$$\mathbf{f}(\lambda \cdot \mathbf{z}) = \mu \cdot \mathbf{f}(\mathbf{z}). \tag{2}$$

Then $\mathbf{f}$ will also have these two properties at every non-zero scalar multiple of $\mathbf{z}$. If $\mathbf{U} \subseteq \mathbb{C}^{m+1} \setminus \{\mathbf{0}\}$ is the set of points where $\mathbf{f}$ has the two properties, then we will say "$\mathbf{f}$ induces a map from $\mathbb{C}P^m$ to $\mathbb{C}P^m$ which is well-defined on the set $\pi^m(\mathbf{U})$," and we will denote the induced map, which takes $\pi^m(\mathbf{z})$ to $\pi^m(\mathbf{f}(\mathbf{z}))$, by $f : z \mapsto f(z)$. It should also be mentioned that the map of projective spaces induced by a composition of maps is equal to the composition of the induced maps.

Example 2.1. If $\mathbf{f} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ is complex linear, then f is well-defined on the lines not contained in the kernel of $\mathbf{f}$. If $\mathbf{f} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ is complex linear and invertible, then f is well-defined on all of $\mathbb{C}P^m$, and also invertible. Let $GL(m+1, \mathbb{C}) \subseteq M(m+1, \mathbb{C})$ denote the subset of nonsingular matrices in the set of $(m+1) \times (m+1)$ matrices with entries in $\mathbb{C}$. Let $PGL(m+1, \mathbb{C})$ denote the set of one-dimensional subspaces of $M(m+1, \mathbb{C})$ which are subsets of $GL(m+1, \mathbb{C}) \cup \{\mathbf{0}\}$. The following construction defines a group action of $PGL(m+1, \mathbb{C})$ on $\mathbb{C}P^m$. For any nonsingular matrix $\mathbf{A}$, there is a corresponding invertible complex linear transformation, which in turn induces a well-defined map $\mathbb{C}P^m \to \mathbb{C}P^m$, denoted A, and called a projective transformation. Any non-zero scalar multiple of $\mathbf{A}$ induces the same map $A : \mathbb{C}P^m \to \mathbb{C}P^m$, so this notation is consistent with the above conventions: a nonsingular matrix $\mathbf{A}$ spans a line $A \in PGL(m+1, \mathbb{C})$, and the projective transformation of $\mathbb{C}P^m$ induced by $\mathbf{A}$ will be denoted $A : z \mapsto A(z)$.

Example 2.2. Suppose $\phi : \mathbb{C} \to \mathbb{C}$ is a field isomorphism, and define a map:

$$\tilde{\boldsymbol{\phi}} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1} : \tilde{\boldsymbol{\phi}}((z_0, \ldots, z_m)^T) = (\phi(z_0), \ldots, \phi(z_m))^T.$$

Since $\tilde{\boldsymbol{\phi}}$ satisfies properties (1, 2) at every point of $\mathbb{C}^{m+1} \setminus \{\mathbf{0}\}$, it induces a map

$$\tilde{\phi} : \mathbb{C}P^m \to \mathbb{C}P^m : \tilde{\phi}([z_0 : \ldots : z_m]) = [\phi(z_0) : \ldots : \phi(z_m)].$$

One example of a field isomorphism of $\mathbb{C}$ is complex conjugation, $\kappa : z_0 \mapsto \bar{z}_0$. The induced maps are $\tilde{\boldsymbol{\kappa}} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ and $\tilde{\kappa} : \mathbb{C}P^m \to \mathbb{C}P^m$, and both are involutions.

Definition 2.3. A map $g : \mathbb{C}P^m \to \mathbb{C}P^m$ is a "collineation" if it is well-defined on $\mathbb{C}P^m$, invertible, and it has the property that, for $V \subseteq \mathbb{C}P^m$, $g(V)$ is a complex projective subspace of $\mathbb{C}P^m$ if and only if V is a complex projective subspace of $\mathbb{C}P^m$.

The following Proposition is the Fundamental Theorem of Projective Geometry. See [Csikós] §6, [Samuel], [Seidenberg].

Proposition 2.4. *For $m \neq 1$, the group of projective transformations of $\mathbb{C}P^m$ is a normal subgroup of the group of collineations, and an invertible map $g : \mathbb{C}P^m \to \mathbb{C}P^m$ is a collineation if and only if it is of the form $g = \tilde{\phi} \circ A$ for some field isomorphism ϕ and projective transformation A.* ∎

Example 2.5. A map $\mathbf{B} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ will be called "conjugate linear" if it is additive, and satisfies the identity $\mathbf{B}(\lambda\mathbf{z}) = \bar{\lambda}\mathbf{B}(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{C}^{m+1}$ and all scalars $\lambda \in \mathbb{C}$. Since $\tilde{\boldsymbol{\kappa}} \circ \mathbf{B}$ is complex linear, it has some matrix representation $\mathbf{A} \in M(m+1, \mathbb{C})$, and we can conclude that every conjugate linear map $\mathbf{B}$ is of the form $\tilde{\boldsymbol{\kappa}} \circ \mathbf{A}$. An invertible conjugate linear map satisfies properties (1, 2) at every point of $\mathbb{C}^{m+1} \setminus \{\mathbf{0}\}$, so it induces a map B, and the induced map is a collineation of $\mathbb{C}P^m$, of the form $B = \tilde{\kappa} \circ A$ for some projective transformation A.

3 Hints at reality

Notation 3.1. Given a conjugate linear involution $\mathbf{B}$ (so that $\mathbf{B} \circ \mathbf{B}$ is the identity map), its fixed point set will be denoted $\mathbf{R_B} \subseteq \mathbb{C}^{m+1}$. The set of vectors $\mathbf{z}$ such that $\mathbf{B}(\mathbf{z}) = -\mathbf{z}$ will be denoted $\mathbf{I_B} \subseteq \mathbb{C}^{m+1}$.

Lemma 3.2. *Given a conjugate linear involution $\mathbf{B}$, these sets are equal:*

$$\mathbf{R_B} = \{\frac{1}{2}(\mathbf{z} + \mathbf{B}(\mathbf{z})) : \mathbf{z} \in \mathbb{C}^{m+1}\} = i\mathbf{I_B},$$

and every element $\mathbf{z} \in \mathbb{C}^{m+1}$ is uniquely expressible as a sum $\mathbf{x} + \mathbf{y}$, for some $\mathbf{x} \in \mathbf{R_B}$ and $\mathbf{y} \in \mathbf{I_B}$.

Proof. The claimed equalities of sets are easy to check, using the properties of $\mathbf{B}$. The identity $\mathbf{z} = \frac{1}{2}(\mathbf{z}+\mathbf{B}(\mathbf{z})) + \frac{1}{2}(\mathbf{z}-\mathbf{B}(\mathbf{z}))$ gives the claimed decomposition. The uniqueness easily follows from the fact that $\mathbf{R_B}$ and $\mathbf{I_B}$ intersect only at the origin. ∎

Theorem 3.3. *Given a complex linear map* $\mathbf{A} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ *and conjugate linear involutions* $\mathbf{B_1}, \mathbf{B_2}$*, if* $\mathbf{A}(\mathbf{R}_{\mathbf{B_1}}) \subseteq \mathbf{R}_{\mathbf{B_2}}$*, then* $\mathbf{A} = \mathbf{B_2} \circ \mathbf{A} \circ \mathbf{B_1}$*. If, further,* $\mathbf{A}$ *is invertible, then* $\mathbf{A}(\mathbf{R}_{\mathbf{B_1}}) = \mathbf{R}_{\mathbf{B_2}}$*.*

Proof. For any $\mathbf{z} \in \mathbf{R}_{\mathbf{B_1}}$, $\mathbf{z} = \mathbf{B_1}(\mathbf{z}) \implies \mathbf{A}(\mathbf{z}) = \mathbf{A}(\mathbf{B_1}(\mathbf{z})) = \mathbf{B_2}(\mathbf{A}(\mathbf{B_1}(\mathbf{z})))$. So, the complex linear maps $\mathbf{A}$ and $\mathbf{B_2} \circ \mathbf{A} \circ \mathbf{B_1}$ agree at every element of $\mathbf{R}_{\mathbf{B_1}}$. It follows from Lemma 3.2 that $\mathbf{R}_{\mathbf{B_1}}$ spans $\mathbb{C}^{m+1}$, so $\mathbf{A} = \mathbf{B_2} \circ \mathbf{A} \circ \mathbf{B_1}$.

To check the claimed equality, it will be enough to show that if $\mathbf{z} \in \mathbf{R}_{\mathbf{B_2}}$, then $\mathbf{A}^{-1}(\mathbf{z}) \in \mathbf{R}_{\mathbf{B_1}}$.

$$\mathbf{B_1}(\mathbf{A}^{-1}(\mathbf{z})) = \mathbf{B_1}((\mathbf{B_2} \circ \mathbf{A} \circ \mathbf{B_1})^{-1}(\mathbf{z})) = (\mathbf{B_1} \circ \mathbf{B_1} \circ \mathbf{A}^{-1} \circ \mathbf{B_2})(\mathbf{z}) = \mathbf{A}^{-1}(\mathbf{z}).$$

∎

Theorem 3.4. *Given a conjugate linear map* $\mathbf{B} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ *and conjugate linear involutions* $\mathbf{B_1}, \mathbf{B_2}$*, if* $\mathbf{B}(\mathbf{R}_{\mathbf{B_1}}) \subseteq \mathbf{R}_{\mathbf{B_2}}$*, then* $\mathbf{B} = \mathbf{B_2} \circ \mathbf{B} \circ \mathbf{B_1}$*. If, further,* $\mathbf{B}$ *is invertible, then* $\mathbf{B}(\mathbf{R}_{\mathbf{B_1}}) = \mathbf{R}_{\mathbf{B_2}}$*.*

Proof. The composition $\mathbf{B_2} \circ \mathbf{B}$ is complex linear, and satisfies $(\mathbf{B_2} \circ \mathbf{B})(\mathbf{R}_{\mathbf{B_1}}) \subseteq \mathbf{R}_{\mathbf{B_2}}$, so the previous Theorem applies, and $\mathbf{B_2} \circ \mathbf{B} = \mathbf{B_2} \circ (\mathbf{B_2} \circ \mathbf{B}) \circ \mathbf{B_1}$. It follows that $\mathbf{B_2} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{B_1}$, and $\mathbf{B} = \mathbf{B_2} \circ \mathbf{B} \circ \mathbf{B_1}$.

When $\mathbf{B}$ is invertible, the previous Theorem showed $(\mathbf{B_2} \circ \mathbf{B})(\mathbf{R}_{\mathbf{B_1}}) = \mathbf{R}_{\mathbf{B_2}}$, and the conclusion follows from applying $\mathbf{B_2}$ to both sides. ∎

Definition 3.5. A conjugate linear involution $\mathbf{B} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ induces a map $B : \mathbb{C}P^m \to \mathbb{C}P^m$, which will be called a "real involution." The fixed point set of a real involution B will be denoted $R_B \subseteq \mathbb{C}P^m$.

Theorem 3.6. *Given a conjugate linear involution* $\mathbf{B}$*, the following subsets of* $\mathbb{C}P^m$ *are non-empty and equal:*

$$R_B = \pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\}) = \pi^m(\mathbf{I_B} \setminus \{\mathbf{0}\}).$$

Proof. It follows from $m \geq 0$ and Lemma 3.2 that both $\mathbf{R_B}$ and $\mathbf{I_B}$ contain non-zero elements, and that they have the same non-empty image under π^m. To show $\pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\}) \subseteq R_B$, if $z = \pi^m(\mathbf{z})$ for some $\mathbf{z} \in \mathbf{R_B} \setminus \{\mathbf{0}\}$, then $\mathbf{z} = \mathbf{B}(\mathbf{z}) \implies z = \pi^m(\mathbf{z}) = \pi^m(\mathbf{B}(\mathbf{z})) = B(\pi^m(\mathbf{z})) = B(z)$. For the other direction, suppose $z \in R_B$, so $B(z) = z = \pi^m(\mathbf{z})$ for some representative $\mathbf{z} \neq \mathbf{0}$. It will be enough to show that some non-zero complex scalar multiple of $\mathbf{z}$ is an element of $\mathbf{R_B}$. $B(z) = B(\pi^m(\mathbf{z})) = \pi^m(\mathbf{B}(\mathbf{z})) = \pi^m(\mathbf{z}) \implies \mathbf{B}(\mathbf{z}) = \mu\mathbf{z}$ for some non-zero scalar μ. $\frac{1}{2}(\mathbf{z} + \mathbf{B}(\mathbf{z})) = \frac{1}{2}(\mathbf{z} + \mu\mathbf{z}) = \frac{1+\mu}{2}\mathbf{z} \in \mathbf{R_B}$, which is enough unless $\mu = -1$, in which case $\mathbf{z} \in \mathbf{I_B}$, and $i\mathbf{z} \in \mathbf{R_B}$. ∎

Theorem 3.7. *If* $\mathbf{B}$ *is a conjugate linear involution and* $\mathbf{V}$ *is a complex linear subspace of* $\mathbb{C}^{m+1}$*, then* $\pi^m((\mathbf{R_B} \cap \mathbf{V}) \setminus \{\mathbf{0}\}) = \pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\}) \cap \pi^m(\mathbf{V} \setminus \{\mathbf{0}\})$*.*

Proof. The inclusion $\pi^m((\mathbf{R_B} \cap \mathbf{V}) \setminus \{\mathbf{0}\}) \subseteq \pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\}) \cap \pi^m(\mathbf{V} \setminus \{\mathbf{0}\})$ is elementary set theory. It remains to show that if $z \in \pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\}) \cap \pi^m(\mathbf{V} \setminus \{\mathbf{0}\})$, and z has representative $\mathbf{z} \in \mathbb{C}^{m+1} \setminus \{\mathbf{0}\}$, then some non-zero complex scalar multiple of $\mathbf{z}$ is in $(\mathbf{R_B}\cap\mathbf{V})\setminus\{\mathbf{0}\}$. From $z\in\pi^m(\mathbf{V}\setminus\{\mathbf{0}\})$, we get a representative $\mathbf{z}\in\mathbf{V}\setminus\{\mathbf{0}\}$, and any non-zero complex scalar multiple of $\mathbf{z}$ is still in $\mathbf{V}\setminus\{\mathbf{0}\}$. By Theorem 3.6, any vector $\mathbf{z}$ such that $\pi^m(\mathbf{z})\in\pi^m(\mathbf{R_B}\setminus\{\mathbf{0}\})$ has some complex scalar multiple in $\mathbf{R_B}\setminus\{\mathbf{0}\}$. ∎

Theorem 3.8. *Given a real involution B, and a map $A:\mathbb{C}P^m\to\mathbb{C}P^m$ which is either a projective transformation or induced by an invertible conjugate linear map, the composite $A\circ B\circ A^{-1}$ is a real involution, with fixed point set $R_{A\circ B\circ A^{-1}}=A(R_B)$.*

Proof. If $\mathbf{A}$ and $\mathbf{B}$ induce A and B, then clearly $\mathbf{A}\circ\mathbf{B}\circ\mathbf{A}^{-1}$ is a conjugate linear involution inducing $A\circ B\circ A^{-1}$.

To show $R_{A\circ B\circ A^{-1}}\subseteq A(R_B)$, if $z=(A\circ B\circ A^{-1})(z)$, then $B(A^{-1}(z))=A^{-1}(z)\in R_B$, so z is the image under A of the point $A^{-1}(z)\in R_B$. To show $A(R_B)\subseteq R_{A\circ B\circ A^{-1}}$, if $z=A(x)$ for some $x\in R_B$, then $(A\circ B\circ A^{-1})(z)=A(B(A^{-1}(A(x))))=A(B(x))=A(x)=z$, so $z\in R_{A\circ B\circ A^{-1}}$. ∎

Proposition 3.9 ([Csikós] Lemma 6.1.3). *Given a basis $\{\mathbf{z}_0,\ldots,\mathbf{z}_m\}$ of $\mathbb{C}^{m+1}$, let $\mathbf{q}=\mathbf{z}_0+\ldots+\mathbf{z}_m$. If $\mathbf{A}\in M(m+1,\mathbb{C})$ is a matrix such that every vector $\mathbf{z}_0,\ldots,\mathbf{z}_m,\mathbf{q}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{A}$ is a multiple of the identity matrix.* ∎

Theorem 3.10. *Given a projective transformation A of $\mathbb{C}P^m$, and conjugate linear involutions $\mathbf{B_1}$, $\mathbf{B_2}$, if $A(R_{B_1})\subseteq R_{B_2}$, then there exists $\mathbf{A}\in GL(m+1,\mathbb{C})$ so that $\mathbf{A}$ induces A, and $\mathbf{A}=\mathbf{B_2}\circ\mathbf{A}\circ\mathbf{B_1}$.*

Proof. There is some invertible matrix $\mathbf{A_0}$ that induces A. Given a non-zero element $\mathbf{z}\in\mathbf{R_{B_1}}$, $z=\pi^m(\mathbf{z})\in R_{B_1}$ by Theorem 3.6, and $A(z)=A(B_1(z))=B_2(A(B_1(z)))$. This implies

$$A(\pi^m(\mathbf{z}))=\pi^m(\mathbf{A_0}(\mathbf{z}))=B_2(A(B_1(\pi^m(\mathbf{z}))))=\pi^m(\mathbf{B_2}(\mathbf{A_0}(\mathbf{B_1}(\mathbf{z})))).$$

So, there is some complex scalar μ (depending on $\mathbf{z}$) so that $\mathbf{B_2}(\mathbf{A_0}(\mathbf{B_1}(\mathbf{z})))=\mu\mathbf{A_0}(\mathbf{z})$, and it follows that every non-zero element $\mathbf{z}\in\mathbf{R_{B_1}}$ is an eigenvector of the complex linear map $\mathbf{A_0}^{-1}\circ\mathbf{B_2}\circ\mathbf{A_0}\circ\mathbf{B_1}$. Since $\mathbf{R_{B_1}}$ spans $\mathbb{C}^{m+1}$ by Lemma 3.2, and is closed under addition, it contains elements $\mathbf{z}_0,\ldots,\mathbf{z}_m,\mathbf{q}$ satisfying the hypothesis of Proposition 3.9, which gives a non-zero constant $\lambda\in\mathbb{C}$ such that $\lambda\cdot\mathbf{A_0}=\mathbf{B_2}\circ\mathbf{A_0}\circ\mathbf{B_1}$. Multiplying both sides by $\bar\lambda$ gives

$$\bar\lambda\cdot\lambda\cdot\mathbf{A_0}=\bar\lambda\cdot\mathbf{B_2}\circ\mathbf{A_0}\circ\mathbf{B_1}=\mathbf{B_2}\circ(\lambda\cdot\mathbf{A_0})\circ\mathbf{B_1}=\mathbf{B_2}\circ(\mathbf{B_2}\circ\mathbf{A_0}\circ\mathbf{B_1})\circ\mathbf{B_1}=\mathbf{A_0},$$

so $\bar\lambda\cdot\lambda=1$, and $\lambda=e^{i\theta}$ for some $\theta=\bar\theta$. Let $\mathbf{A}=e^{i\theta/2}\cdot\mathbf{A_0}$, so that

$$\mathbf{B_2}\circ\mathbf{A}\circ\mathbf{B_1}=\mathbf{B_2}\circ(e^{i\theta/2}\cdot\mathbf{A_0})\circ\mathbf{B_1}=e^{-i\theta/2}\cdot\mathbf{B_2}\circ\mathbf{A_0}\circ\mathbf{B_1}=e^{-i\theta/2}\cdot(e^{i\theta}\mathbf{A_0})=\mathbf{A}.$$

∎

Theorem 3.11. *Given a map $B:\mathbb{C}P^m\to\mathbb{C}P^m$ which is induced by some invertible conjugate linear map $\mathbb{C}^{m+1}\to\mathbb{C}^{m+1}$, and conjugate linear involutions $\mathbf{B_1}$, $\mathbf{B_2}$, if $B(R_{B_1})\subseteq R_{B_2}$, then there exists an invertible conjugate linear map $\mathbf{B}:\mathbb{C}^{m+1}\to\mathbb{C}^{m+1}$ so that $\mathbf{B}$ induces B, and $\mathbf{B}=\mathbf{B_2}\circ\mathbf{B}\circ\mathbf{B_1}$.*

Proof. B is induced by some $\mathbf{B_0}$, and $\mathbf{B_2} \circ \mathbf{B_0}$ is an invertible complex linear map which induces a projective transformation $B_2 \circ B$ such that $(B_2 \circ B)(R_{B_1}) \subseteq R_{B_2}$. The previous Theorem applies, so there is some $\mathbf{A}$ so that $\mathbf{A} = \mathbf{B_2} \circ \mathbf{A} \circ \mathbf{B_1}$ and $\mathbf{A}$ induces $B_2 \circ B$. Let $\mathbf{B} = \mathbf{B_2} \circ \mathbf{A}$, so it is an invertible conjugate linear map that induces $B_2 \circ (B_2 \circ B) = B$, and $\mathbf{B_2} \circ \mathbf{B} \circ \mathbf{B_1} = \mathbf{B_2} \circ (\mathbf{B_2} \circ \mathbf{A}) \circ \mathbf{B_1} = \mathbf{B_2} \circ \mathbf{A} = \mathbf{B}$. ∎

Theorem 3.12. *Given a projective transformation A of $\mathbb{C}P^m$, and conjugate linear involutions $\mathbf{B_1}$, $\mathbf{B_2}$, if $A(R_{B_1}) \subseteq R_{B_2}$, then $A(R_{B_1}) = R_{B_2}$.*

Proof. Let $\mathbf{A}$ be as in Theorem 3.10, so it induces A and $\mathbf{A} = \mathbf{B_2} \circ \mathbf{A} \circ \mathbf{B_1}$. It follows that $A = B_2 \circ A \circ B_1$. It will be enough to show that if $z \in R_{B_2}$, then $A^{-1}(z) \in R_{B_1}$.

$$B_1(A^{-1}(z)) = B_1((B_2 \circ A \circ B_1)^{-1}(z)) = (B_1 \circ B_1 \circ A^{-1} \circ B_2)(z) = A^{-1}(z).$$

∎

Theorem 3.13. *Given a map $B : \mathbb{C}P^m \to \mathbb{C}P^m$ induced by an invertible conjugate linear map, and conjugate linear involutions $\mathbf{B_1}$, $\mathbf{B_2}$, if $B(R_{B_1}) \subseteq R_{B_2}$, then $B(R_{B_1}) = R_{B_2}$.*

Proof. B is induced by some $\mathbf{B}$, and $\mathbf{B_2} \circ \mathbf{B}$ induces a projective transformation $B_2 \circ B$ so that $(B_2 \circ B)(R_{B_1}) \subseteq R_{B_2}$. The previous Theorem showed $(B_2 \circ B)(R_{B_1}) = R_{B_2}$, and the conclusion follows from applying B_2 to both sides. ∎

Theorems 3.12 and 3.13 could also have been proved using Theorems 3.3, 3.4, 3.10, and 3.11.

Theorem 3.14. *Given a real involution B and projective transformations A_1, A_2, if $A_1(x) = A_2(x)$ for all $x \in R_B$, then $A_1 = A_2$.*

Proof. B is induced by some $\mathbf{B}$, and let $\mathbf{A_1}, \mathbf{A_2} \in GL(m+1, \mathbb{C})$ induce A_1, A_2, so $\mathbf{A_2}^{-1} \circ \mathbf{A_1}$ induces $A_2^{-1} \circ A_1$, which satisfies $(A_2^{-1} \circ A_1)(x) = (A_2^{-1} \circ A_2)(x) = x$ for all $x \in R_B$. Since any non-zero $\mathbf{x} \in \mathbf{R_B}$ satisfies $\pi^m((\mathbf{A_2}^{-1} \circ \mathbf{A_1})(\mathbf{x})) = (A_2^{-1} \circ A_1)(\pi^m(\mathbf{x})) = \pi^m(\mathbf{x})$, every non-zero element of $\mathbf{R_B}$ is an eigenvector of $\mathbf{A_2}^{-1} \circ \mathbf{A_1}$, including elements $\mathbf{z}_0, \ldots, \mathbf{z}_m, \mathbf{q} \in \mathbf{R_B}$ as in the Proof of Theorem 3.10. It follows from Proposition 3.9 that there is some non-zero $\lambda \in \mathbb{C}$ such that $\mathbf{A_1} = \lambda \mathbf{A_2}$, and $A_1 = A_2$. ∎

Theorem 3.15. *Given two real involutions B_1, B_2, there exists a projective transformation A such that $A(R_{B_1}) = R_{B_2}$.*

Proof. Let $\mathbf{B_1}, \mathbf{B_2}$ induce B_1, B_2. By Lemma 3.2, $\mathbf{R_{B_1}}$ spans $\mathbb{C}^{m+1}$, and therefore contains a basis $\{\mathbf{z}_0, \ldots, \mathbf{z}_m\}$ of $\mathbb{C}^{m+1}$. Since $\mathbf{R_{B_2}}$ also spans $\mathbb{C}^{m+1}$, it contains a basis, and there exists some $\mathbf{A} \in GL(m+1, \mathbb{C})$ taking the first basis to the second. If $\mathbf{x} \in \mathbf{R_{B_1}}$, then it is of the form $\sum c_p \mathbf{z}_p = \mathbf{B_1}(\sum c_p \mathbf{z}_p) = \sum \bar{c}_p \mathbf{B_1}(\mathbf{z}) = \sum \bar{c}_p \mathbf{z}_p$, so $c_p = \bar{c}_p$ for $p = 0, \ldots, m$. It is easy to check $\mathbf{A}(\mathbf{x}) \in \mathbf{R_{B_2}}$, so $\mathbf{A}(\mathbf{R_{B_1}}) \subseteq \mathbf{R_{B_2}}$, and $A(R_{B_1}) \subseteq R_{B_2}$. The equality follows from Theorem 3.12. ∎

Theorem 3.16. *The fixed point set of a real involution of $\mathbb{C}P^m$ does not contain any complex projective lines.*

Proof. Suppose, toward a contradiction, that $\mathbf{B}$ is a conjugate linear involution, and $\mathbf{V} \subseteq \mathbb{C}^{m+1}$ is a two-dimensional complex linear subspace such that $\pi^m(\mathbf{V} \setminus \{\mathbf{0}\}) \subseteq R_B$. By Theorem 3.6, $\pi^m(\mathbf{V} \setminus \{\mathbf{0}\}) \subseteq \pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\})$, so every element of $\mathbf{V}$ is a complex scalar multiple of an element of $\mathbf{R_B}$. Let $\{\mathbf{z}_0, \mathbf{w}_0\} \subseteq \mathbf{V}$ be a linearly independent (over $\mathbb{C}$) set; there are scalars $\lambda, \mu \in \mathbb{C}$ so that $\{\mathbf{z}_1 = \lambda \mathbf{z}_0, \mathbf{w}_1 = \mu \mathbf{w}_0\} \subseteq \mathbf{R_B}$, and $\{\mathbf{z}_1, \mathbf{w}_1\}$ is a basis of $\mathbf{V}$. The vector $\mathbf{z}_1 + i\mathbf{w}_1 \in \mathbf{V}$ is non-zero, and is a non-zero multiple of an element of $\mathbf{R_B}$, so for some $\nu \neq 0$,

$$\nu(\mathbf{z}_1 + i\mathbf{w}_1) = \mathbf{B}(\nu(\mathbf{z}_1 + i\mathbf{w}_1)) = \bar{\nu}\mathbf{B}(\mathbf{z}_1 + i\mathbf{w}_1) = \bar{\nu}(\mathbf{z}_1 - i\mathbf{w}_1).$$

Since $\{\mathbf{z}_1, \mathbf{w}_1\}$ is a basis of $\mathbf{V}$, the coefficients must be equal: $\nu = \bar{\nu}$ and $i\nu = -i\bar{\nu}$. However, these equations imply $\nu = 0$, a contradiction. ∎

Theorem 3.17. *Given two distinct complex projective lines V_1, V_2 in $\mathbb{C}P^m$, and a real involution B, if V_1 intersects R_B in at least two points, and V_2 also intersects R_B in at least two points, then $V_1 \cap V_2 \subseteq R_B$.*

Proof. First, $V_1 \cap V_2$ could be the empty set, and the claim trivially follows.

Otherwise, denote two distinct points $u, v \in V_1 \cap R_B$, with $V_1 = \pi^m(\mathbf{V_1} \setminus \{\mathbf{0}\})$ for some 2-dimensional complex linear subspace $\mathbf{V_1}$ in $\mathbb{C}^{m+1}$, and let B be induced by some conjugate linear involution $\mathbf{B}$. Similarly, denote two distinct points $x, y \in V_2 \cap R_B$, with $V_2 = \pi^m(\mathbf{V_2} \setminus \{\mathbf{0}\})$. It follows from the hypothesis $V_1 \neq V_2$ that $\mathbf{V_1} \neq \mathbf{V_2}$, and from the current assumption $V_1 \cap V_2 \neq \emptyset$ that $\mathbf{V_1} \cap \mathbf{V_2}$ is a one-dimensional complex linear subspace of $\mathbb{C}^{m+1}$.

By Theorem 3.7, $u, v \in \pi^m(\mathbf{V_1} \setminus \{\mathbf{0}\}) \cap \pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\}) = \pi^m((\mathbf{V_1} \cap \mathbf{R_B}) \setminus \{\mathbf{0}\})$, so there are some representatives $\mathbf{u}, \mathbf{v} \in (\mathbf{V_1} \cap \mathbf{R_B}) \setminus \{\mathbf{0}\}$ which form a basis of $\mathbf{V_1}$ (since if they were not independent, then $u = v$). Similarly, there are $\mathbf{x}, \mathbf{y} \in (\mathbf{V_2} \cap \mathbf{R_B}) \setminus \{\mathbf{0}\}$ which form a basis of $\mathbf{V_2}$. Let $\mathbf{z}$ be any representative of $z \in V_1 \cap V_2$, so $\mathbf{z} \in (\mathbf{V_1} \cap \mathbf{V_2}) \setminus \{\mathbf{0}\}$. There are coefficients $\alpha, \beta \in \mathbb{C}$, not both 0, so that $\mathbf{z} = \alpha\mathbf{u} + \beta\mathbf{v}$, and similarly, $\mathbf{z} = \gamma\mathbf{x} + \delta\mathbf{y}$. One of the following elements of $\mathbf{R_B}$ must be non-zero:

$$\begin{aligned}
\frac{1}{2}(\mathbf{z} + \mathbf{B}(\mathbf{z})) &= \frac{1}{2}(\alpha\mathbf{u} + \beta\mathbf{v} + \mathbf{B}(\alpha\mathbf{u} + \beta\mathbf{v})) = \frac{1}{2}(\alpha + \bar{\alpha})\mathbf{u} + \frac{1}{2}(\beta + \bar{\beta})\mathbf{v} \\
&= \frac{1}{2}(\gamma\mathbf{x} + \delta\mathbf{y} + \mathbf{B}(\gamma\mathbf{x} + \delta\mathbf{y})) = \frac{1}{2}(\gamma + \bar{\gamma})\mathbf{x} + \frac{1}{2}(\delta + \bar{\delta})\mathbf{y}, \\
\frac{1}{2i}(\mathbf{z} - \mathbf{B}(\mathbf{z})) &= \frac{1}{2i}(\alpha\mathbf{u} + \beta\mathbf{v} - \mathbf{B}(\alpha\mathbf{u} + \beta\mathbf{v})) = \frac{1}{2i}(\alpha - \bar{\alpha})\mathbf{u} + \frac{1}{2i}(\beta - \bar{\beta})\mathbf{v} \\
&= \frac{1}{2i}(\gamma\mathbf{x} + \delta\mathbf{y} - \mathbf{B}(\gamma\mathbf{x} + \delta\mathbf{y})) = \frac{1}{2i}(\gamma - \bar{\gamma})\mathbf{x} + \frac{1}{2i}(\delta - \bar{\delta})\mathbf{y}.
\end{aligned}$$

These linear combinations show that there is a non-zero element of $\mathbf{R_B}$ which is on the one-dimensional complex linear subspace $\mathbf{V_1} \cap \mathbf{V_2}$, so it must be a complex scalar multiple of $\mathbf{z}$, and $z = \pi^m(\mathbf{z}) \in \pi^m((\mathbf{V_1} \cap \mathbf{V_2} \cap \mathbf{R_B}) \setminus \{\mathbf{0}\}) \subseteq \pi^m(\mathbf{R_B} \setminus \{\mathbf{0}\}) = R_B$. ∎

4 B-lines

At this point, we will take for granted some of the incidence axioms for complex projective geometry. They are straightforward consequences of the linear algebra definition of complex projective subspaces.

1. For any two distinct points in $\mathbb{C}P^m$, there exists a complex projective line containing them. The fact is true but trivial for $m = 0$ or 1.

2. If the intersection of a complex projective line L and a complex projective subspace V contains more than one point, then $L \subseteq V$. (In particular, the line from the previous axiom is unique.)

3. Any two complex projective lines contained in a complex projective plane have a non-empty intersection. Conversely, if two complex projective lines have a non-empty intersection, then they coincide or are contained in a complex projective plane.

Definition 4.1. Given a real involution B, and two distinct points in R_B, define the "B-line" through the two points to be the intersection of R_B with the unique complex line containing the two given points.

Theorem 4.2. *Given a real involution B, and two B-lines k and ℓ, either $k = \ell$ or k intersects ℓ in at most one point in R_B. If k and ℓ are both contained in some complex projective plane U, then k intersects ℓ in at least one point.*

Proof. Let $k = V_1 \cap R_B$ and $\ell = V_2 \cap R_B$, for complex projective lines V_1, V_2, so $k \cap \ell = (V_1 \cap V_2) \cap R_B$. As a consequence of the incidence axioms, $V_1 \cap V_2$ is equal to either V_1, or a set containing at most one point, and the first claim follows. If k is contained in a complex projective plane U, then $V_1 \cap U$ contains at least two points in k, so $V_1 \subseteq U$ (the incidence axioms again), and similarly if $\ell \subseteq U$, then $V_2 \subseteq U$. By the third incidence axiom, there is some element $z \in V_1 \cap V_2$. If $V_1 = V_2$, then $k = \ell$, so $k \cap \ell$ contains at least two points. If $V_1 \neq V_2$, z is unique, and Theorem 3.17 applies (since $k = V_1 \cap R_B$ and $\ell = V_2 \cap R_B$ each contain at least two points by the definition of B-line), to show that $z \in (V_1 \cap V_2) \cap R_B = k \cap \ell$. ∎

Any R_B, together with the set of B-lines contained in R_B, can be considered as an abstract projective space satisfying various incidence properties. Without going into the details, we list some of these axioms for a given set $R_B \subseteq \mathbb{C}P^m$.

1. For any two distinct points in R_B, there exists exactly one B-line containing them.

2. R_B contains at least one point, and there exist infinitely many points on each B-line.

3. For $m > 0$, R_B contains infinitely many points and at least one B-line.

4. For $m > 1$, R_B contains infinitely many B-lines, and not all points are on one B-line.

5. For $m > 2$, not every pair of B-lines meets.

6. (Pasch-Veblen-Young [Seidenberg]) Given points $u, v, x, y \in R_B$, no three of which lie on the same B-line, let k denote the B-line through u, v, and ℓ the B-line through x, y, and p the B-line through u, y, and q the B-line through v, x. If $k \cap \ell \neq \emptyset$, then $p \cap q \neq \emptyset$.

7. The theorems of Pappus and Desargues ([Csikós], [Seidenberg]) hold in R_B.

The first above item summarizes Theorem 4.2, and the next few properties are obvious and proofs are omitted. The PVY, Pappus, and Desargues properties follow as a consequence of these properties of $\mathbb{C}P^m$, together with Theorems 3.17 and 4.2.

Theorem 4.3 ([E] Proposition III.10). *Given real involutions B_1, B_2, a B_1-line ℓ, and a B_2-line k, if ℓ intersects k in three distinct points, then $\ell = k$.*

Proof. Let $\ell = V \cap R_{B_1}$, $k = V' \cap R_{B_2}$. Since V' meets V in distinct points, $V' = V$ and $k = V \cap R_{B_2}$. Let x, y, z be distinct points in $\ell \cap k$, with representatives $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{V} \cap \mathbf{R_{B_1}}$ as in Theorem 3.17. Any two of these representatives span $\mathbf{V}$, so, for example, there are non-zero complex constants α, β with $\mathbf{z} = \alpha\mathbf{x} + \beta\mathbf{y}$. It follows from $\mathbf{z} = \mathbf{B_1}(\mathbf{z})$ that $\alpha = \bar{\alpha}$ and $\beta = \bar{\beta}$.

Since $x, y, z \in k \subseteq R_{B_2}$, they are also represented by $\lambda\mathbf{x}, \mu\mathbf{y}, \nu\mathbf{z} \in \mathbf{R_{B_2}}$, for some non-zero complex numbers λ, μ, ν.

$$\begin{aligned}
\nu\mathbf{z} &= \mathbf{B_2}(\nu\mathbf{z}) \\
\implies \nu(\alpha\mathbf{x} + \beta\mathbf{y}) &= \mathbf{B_2}(\nu(\alpha\mathbf{x} + \beta\mathbf{y})) \\
\implies (\nu \cdot \alpha)\mathbf{x} + (\nu \cdot \beta)\mathbf{y} &= \mathbf{B_2}((\nu \cdot \alpha/\lambda)\lambda\mathbf{x} + (\nu \cdot \beta/\mu)\mu\mathbf{y}) \\
&= \overline{\left(\frac{\nu \cdot \alpha}{\lambda}\right)}\mathbf{B_2}(\lambda\mathbf{x}) + \overline{\left(\frac{\nu \cdot \beta}{\mu}\right)}\mathbf{B_2}(\mu\mathbf{y}) \\
&= \left(\frac{\bar{\nu} \cdot \bar{\alpha}}{\bar{\lambda}} \cdot \lambda\right)\mathbf{x} + \left(\frac{\bar{\nu} \cdot \bar{\beta}}{\bar{\mu}} \cdot \mu\right)\mathbf{y}.
\end{aligned}$$

From the equality of the coefficients, the properties $\alpha = \bar{\alpha}$, $\beta = \bar{\beta}$, and the fact that α and β are both non-zero (which is where we are using the assumption that x, y, z are distinct), a little calculation shows that $\frac{\lambda}{\bar{\lambda}} = \frac{\mu}{\bar{\mu}} = \frac{\nu}{\bar{\nu}}$.

To show $\ell \subseteq k$, let u be any point in ℓ, with representative $\mathbf{u} = \delta\mathbf{x} + \epsilon\mathbf{y} \in \mathbf{R_{B_1}}$, $\delta = \bar{\delta}$, $\epsilon = \bar{\epsilon}$. Then $B_2(u) = \pi^m(\mathbf{B_2}(\mathbf{u})) = \pi^m(\mathbf{B_2}(\delta\mathbf{x} + \epsilon\mathbf{y})) = \pi^m(\mathbf{B_2}((\delta/\lambda)\lambda\mathbf{x} + (\epsilon/\mu)\mu\mathbf{y})) = \pi^m((\bar{\delta} \cdot \frac{\lambda}{\bar{\lambda}})\mathbf{x} + (\bar{\epsilon} \cdot \frac{\mu}{\bar{\mu}})\mathbf{y})) = \pi^m(\frac{\lambda}{\bar{\lambda}}(\delta\mathbf{x} + \epsilon\mathbf{y})) = u$, so $u \in V \cap R_{B_2} = k$. An analogous argument shows $k \subseteq \ell$. ∎

Theorem 4.4. *Given a projective transformation A of $\mathbb{C}P^m$, real involutions B_1, B_2, and a B_1-line ℓ, if there are three distinct points in ℓ whose images under A are elements of R_{B_2}, then $A(\ell)$ is a B_2-line.*

Proof. $\ell = V \cap R_{B_1}$, for some complex projective line V, so $A(\ell) = A(V \cap R_{B_1}) \subseteq A(V)$, where $A(V)$ is also a complex projective line. Let k denote the B_2-line $A(V) \cap R_{B_2}$, so $A(\ell)$ meets k in three distinct points (since A is one-to-one).

The composite map $B_3 = A \circ B_1 \circ A^{-1}$ is a real involution, by Theorem 3.8, such that $R_{B_3} = A(R_{B_1})$. Since A is one-to-one, $A(\ell) = A(V \cap R_{B_1}) = A(V) \cap A(R_{B_1}) = A(V) \cap R_{B_3}$. Since $A(\ell)$ is a B_3-line that meets the B_2-line k in three distinct points, $A(\ell) = k$ by Theorem 4.3. ∎

Theorem 4.5. *Given a map $B : \mathbb{C}P^m \to \mathbb{C}P^m$ induced by an invertible conjugate linear map, real involutions B_1, B_2, and a B_1-line ℓ, if there are three distinct points in ℓ whose images under B are elements of R_{B_2}, then $B(\ell)$ is a B_2-line.*

Proof. The previous Theorem applies to the projective transformation $B_2 \circ B$, which takes three distinct points in $\ell = V \cap R_{B_1}$ into R_{B_2}. The conclusion is that $(B_2 \circ B)(\ell) \subseteq R_{B_2}$ is a B_2-line, equal to the set $B(\ell) = B(V) \cap R_{B_2}$. ∎

Theorem 4.6. *Given $m \neq 1$, a collineation g of $\mathbb{C}P^m$, real involutions B_1, B_2, and a B_1-line ℓ, if $g(\ell) \subseteq R_{B_2}$, then g is either a projective transformation, or it is induced by some invertible conjugate linear map $\mathbf{B} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$.*

Proof. Let x, y be distinct points on $\ell = V \cap R_{B_1} = \pi^m((\mathbf{V} \cap \mathbf{R_{B_1}}) \setminus \{\mathbf{0}\})$, with complex linearly independent representatives $\mathbf{x}, \mathbf{y} \in \mathbf{V} \cap \mathbf{R_{B_1}}$ as in the Proof of Theorem 3.17.

Let g be induced by some $\tilde{\boldsymbol{\phi}} \circ \mathbf{A} : \mathbb{C}^{m+1} \to \mathbb{C}^{m+1}$ as in Proposition 2.4. By the hypothesis that $g(x) \in R_{B_2}$, $g(x) = \pi^m((\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x})) \in \pi^m(\mathbf{R_{B_2}})$, so there is some non-zero complex number λ so that $\lambda(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x}) \in \mathbf{R_{B_2}}$. Similarly, there is some non-zero μ so that $\mu(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{y}) \in \mathbf{R_{B_2}}$. The vectors $(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x})$, $(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{y})$ are independent (since otherwise, $g(x) = g(y)$, but g is assumed to be one-to-one).

Let ζ be any complex number such that $\zeta = \bar{\zeta}$, and consider $\mathbf{z} = \mathbf{x} + \zeta\mathbf{y}$. It is easy to check that $\mathbf{z} \in (\mathbf{V} \cap \mathbf{R_{B_1}}) \setminus \{\mathbf{0}\}$, so $z = \pi^m(z) \in \ell$, $g(z) \in R_{B_2}$, and as with $\mathbf{x}$ and $\mathbf{y}$, there is some non-zero ν_ζ (depending on ζ) so that $\nu_\zeta(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{z}) \in \mathbf{R_{B_2}}$.

We also have the following expression:

$$\begin{aligned}
\nu_\zeta(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{z}) &= \nu_\zeta(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x} + \zeta\mathbf{y}) \\
&= \nu_\zeta((\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x}) + (\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\zeta\mathbf{y})) \\
&= \nu_\zeta((\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x}) + \phi(\zeta)(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{y})) \\
&= \nu_\zeta(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x}) + (\nu_\zeta \cdot \phi(\zeta))(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{y}) \\
&= (\nu_\zeta \cdot \frac{1}{\lambda})(\lambda(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{x})) + (\nu_\zeta \cdot \phi(\zeta) \cdot \frac{1}{\mu})(\mu(\tilde{\boldsymbol{\phi}} \circ \mathbf{A})(\mathbf{y})),
\end{aligned}$$

which should be invariant under $\mathbf{B_2}$, so:

$$\begin{aligned}
\nu_\zeta(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{z}) &= \mathbf{B_2}(\nu_\zeta(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{z}))\\
&= \mathbf{B_2}(\frac{\nu_\zeta}{\lambda}(\lambda(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{x}))+\frac{\nu_\zeta\cdot\phi(\zeta)}{\mu}(\mu(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{y})))\\
&= \overline{\left(\frac{\nu_\zeta}{\lambda}\right)}\mathbf{B_2}(\lambda(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{x}))+\overline{\left(\frac{\nu_\zeta\cdot\phi(\zeta)}{\mu}\right)}\mathbf{B_2}(\mu(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{y}))\\
&= \frac{\overline{\nu_\zeta}}{\bar{\lambda}}(\lambda(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{x}))+\frac{\overline{\nu_\zeta}\cdot\overline{\phi(\zeta)}}{\bar{\mu}}(\mu(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{y})).
\end{aligned}$$

From the uniqueness of the coefficients of the linearly independent vectors $(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{x})$ and $(\tilde{\boldsymbol{\phi}}\circ\mathbf{A})(\mathbf{y})$, we get $\nu_\zeta=\overline{\nu_\zeta}\cdot\lambda/\bar{\lambda}$ and $\nu_\zeta\cdot\phi(\zeta)=\overline{\nu_\zeta}\cdot\overline{\phi(\zeta)}\cdot\mu/\bar{\mu}$, and it follows from a short calculation that $\phi(\zeta)=\overline{\phi(\zeta)}\cdot\frac{\bar{\lambda}\cdot\mu}{\lambda\cdot\bar{\mu}}$. In particular, that equation holds for $\zeta=1$, when $\phi(1)=1=\overline{\phi(1)}$, so $\frac{\bar{\lambda}\cdot\mu}{\lambda\cdot\bar{\mu}}=1$, independent of ζ. The conclusion is that $\phi(\zeta)=\overline{\phi(\zeta)}$ for all ζ such that $\zeta=\bar{\zeta}$, but it is known ([Y]) that the only such field isomorphisms with that property are the identity map, so $\tilde{\boldsymbol{\phi}}\circ\mathbf{A}=\mathbf{A}$, and complex conjugation, $\phi=\kappa$, so $\tilde{\boldsymbol{\kappa}}\circ\mathbf{A}=\mathbf{B}$ is a conjugate linear map. ∎

It also follows from the hypothesis of Theorem 4.6 that $g(\ell)$ is a B_2-line, by Theorems 4.4 and 4.5.

Corollary 4.7. *Given $m\neq 1$, a collineation g of $\mathbb{C}P^m$, and conjugate linear involutions $\mathbf{B_1}$, $\mathbf{B_2}$ of $\mathbb{C}^{m+1}$, if $g(R_{B_1})\subseteq R_{B_2}$, then there exists an invertible map $\mathbf{A}:\mathbb{C}^{m+1}\to\mathbb{C}^{m+1}$ such that $\mathbf{A}$ induces g, $\mathbf{A}=\mathbf{B_2}\circ\mathbf{A}\circ\mathbf{B_1}$, and $\mathbf{A}$ is either complex linear or conjugate linear. It also follows that $g(R_{B_1})=R_{B_2}$.*

Proof. The $m=0$ case is trivial. For $m>1$, R_{B_1} contains at least two points, and some B_1-line ℓ, so the previous Theorem applies. In the first case, where $g=A$, Theorem 3.10 gives the required complex linear $\mathbf{A}$. In the second case, where g is induced by a conjugate linear map $\mathbf{B}$, Theorem 3.11 gives the required conjugate linear map, which turns out to be some complex scalar multiple of $\mathbf{B}$. The equality of sets follows from Theorems 3.12, 3.13. ∎

Corollary 4.8. *Given $m\neq 1$, collineations g and h of $\mathbb{C}P^m$, and a real involution B, if $h(x)=g(x)$ for all $x\in R_B$, then either $g=h$ or $g=h\circ B$.*

Proof. For $x\in R_B$, $g^{-1}(h(x))=g^{-1}(g(x))=x$, so $g^{-1}\circ h$ is a collineation such that $(g^{-1}\circ h)(R_B)\subseteq R_B$.

The previous Corollary applies. In the first case, where $g^{-1}\circ h$ is a projective transformation, it agrees with the identity transformation on R_B, and $g=h$ by Theorem 3.14. In the second case, $g^{-1}\circ h\circ B$ is a projective transformation that agrees with the identity on R_B, so $g=h\circ B$. ∎

5 The cross ratio

Definition 5.1. Define the cross ratio of four points $a = [a_0 : a_1]$, $b = [b_0 : b_1]$, $c = [c_0 : c_1]$, $d = [d_0 : d_1]$ on the complex projective line by the following rational map.

$$\begin{aligned} \chi \quad &: \quad \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^1 : \\ (a,b,c,d) \quad &\mapsto \quad [(a_1d_0 - a_0d_1)(b_1c_0 - b_0c_1) : (a_1c_0 - c_1a_0)(b_1d_0 - b_0d_1)]. \end{aligned}$$

The function χ is well-defined except for those quadruples of points where three coincide.

Notation 5.2. Any function $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ induces the obvious product map, denoted $f^4 : (a,b,c,d) \mapsto (f(a), f(b), f(c), f(d))$.

The next two facts are well-known and each is proved by a short calculation.

Proposition 5.3. *For any* $d \in \mathbb{C}P^1$, $\chi(([0:1],[1:0],[1:1],d)) = d$. ∎

Proposition 5.4. *For any* $A \in PGL(2,\mathbb{C})$, $\chi \circ A^4 = \chi$. ∎

Theorem 5.5. *If* $B : \mathbb{C}P^1 \to \mathbb{C}P^1$ *is induced by an invertible conjugate linear map* $\mathbf{B} : \mathbb{C}^2 \to \mathbb{C}^2$, *then* $\chi \circ B^4 = \tilde{\kappa} \circ \chi$.

Proof. This follows from $B = \tilde{\kappa} \circ A$ for some A, the easily checked identity $\chi \circ \tilde{\kappa}^4 = \tilde{\kappa} \circ \chi$, and the above Proposition, applied to $\chi \circ (\tilde{\kappa} \circ A)^4 = \chi \circ \tilde{\kappa}^4 \circ A^4$. ∎

Lemma 5.6. *Given three distinct points* $a, b, c \in \mathbb{C}P^1$, *there exists some* $A \in PGL(2,\mathbb{C})$ *such that*

$$A(a) = [0:1], \ A(b) = [1:0], \ A(c) = [1:1].$$

Proof. Consider representative vectors $\mathbf{a} = (a_0, a_1)^T$, $\mathbf{b} = (b_0, b_1)^T$, $\mathbf{c} = (c_0, c_1)^T \in \mathbb{C}^2 \setminus \{\mathbf{0}\}$ for a, b, and c. Any two of these three representatives are linearly independent because a, b, c are distinct, and there exists some complex linear map $\mathbf{A_1}$ such that $\mathbf{A_1}(\mathbf{a}) = (0,1)^T$, $\mathbf{A_1}(\mathbf{b}) = (1,0)^T$, and $\mathbf{A_1}(\mathbf{c}) = (c_0', c_1')^T$, with $c_0' \neq 0$ and $c_1' \neq 0$. Then let $\mathbf{A_2} = \begin{pmatrix} 1/c_0' & 0 \\ 0 & 1/c_1' \end{pmatrix}$, so that $A = A_2 \circ A_1$ is the required map. ∎

The following converse of Proposition 5.4, that maps preserving the cross ratio must be projective transformations, is also well known.

Theorem 5.7. *Given any map* $f : \mathbb{C}P^1 \to \mathbb{C}P^1$, *if* $\chi \circ f^4 = \chi$, *then* $f \in PGL(2,\mathbb{C})$.

Proof. First, f must be one-to-one; otherwise, let $z, w \in \mathbb{C}P^1$ be points such that $z \neq w$ and $f(z) = f(w)$, and pick any point $x \in \mathbb{C}P^1$ such that $x \neq z$. Then, $\chi((x,z,z,w))$ is defined, but $\chi((f(x), f(z), f(z), f(w)))$ is not.

Since f is one-to-one, the points $a = f([0:1])$, $b = f([1:0])$, $c = f([1:1])$ are distinct. By Lemma 5.6, there is some $A \in PGL(2,\mathbb{C})$ so that $A \circ f$ fixes each of the points $[0:1]$, $[1:0]$, $[1:1]$. By Proposition 5.4 and the hypothesis, $\chi \circ (A \circ f)^4 = \chi \circ A^4 \circ f^4 = \chi$, so for any point $d \in \mathbb{C}P^1$, expanding $(\chi \circ (A \circ f)^4)(([0:1],[1:0],[1:1],d))$ in two different ways gives:

$$\begin{aligned}\chi(([0:1],[1:0],[1:1],(A\circ f)(d))) &= \chi(([0:1],[1:0],[1:1],d)) \\ = (A\circ f)(d) &= d.\end{aligned}$$

This shows $f = A^{-1}$. ∎

Corollary 5.8. *Given any map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$, if $\chi \circ f^4 = \tilde{\kappa} \circ \chi$, then f is induced by some invertible conjugate linear map $\mathbb{C}^2 \to \mathbb{C}^2$.*

Proof. If $\chi \circ f^4 = \tilde{\kappa} \circ \chi$, then $\chi \circ (\tilde{\kappa} \circ f)^4 = \chi \circ \tilde{\kappa}^4 \circ f^4 = \tilde{\kappa} \circ \chi \circ f^4 = \tilde{\kappa} \circ \tilde{\kappa} \circ \chi = \chi$, so the previous Theorem applies to show $\tilde{\kappa} \circ f \in PGL(2,\mathbb{C})$. ∎

Theorem 5.9. *Given four points $a, b, c, d \in \mathbb{C}P^1$, at least three distinct, there exists a real involution B such that $\{a,b,c,d\} \subseteq R_B$ if and only if $\chi((a,b,c,d)) = (\tilde{\kappa} \circ \chi)((a,b,c,d))$.*

Proof. One direction is easy: if $B^4((a,b,c,d)) = (a,b,c,d)$, then by Theorem 5.5, $\chi((a,b,c,d)) = (\chi \circ B^4)((a,b,c,d)) = (\tilde{\kappa} \circ \chi)((a,b,c,d))$.

Conversely, suppose $\chi((a,b,c,d)) = (\tilde{\kappa} \circ \chi)((a,b,c,d))$, and since the other cases will be similar, suppose a, b, and c are distinct. Let $A \in PGL(2,\mathbb{C})$ be the map from Lemma 5.6, so that

$$A^4((a,b,c,d)) = ([0:1],[1:0],[1:1],[d_0:d_1]).$$

Finding the cross ratio of both sides, and then using the hypothesis, gives

$$\begin{aligned}(\chi \circ A^4)((a,b,c,d)) &= \chi(([0:1],[1:0],[1:1],[d_0:d_1])) \\ = \chi((a,b,c,d)) &= [d_0:d_1] \\ = (\tilde{\kappa} \circ \chi)((a,b,c,d)) &= \tilde{\kappa}([d_0:d_1]).\end{aligned}$$

So, $[0:1]$, $[1:0]$, $[1:1]$, and $[d_0:d_1]$ are all in the fixed point set $R_{\tilde{\kappa}}$. Using Theorem 3.8, a, b, c, d are in $A^{-1}(R_{\tilde{\kappa}}) = R_{A^{-1}\circ\tilde{\kappa}\circ A}$. ∎

To establish the last result, Theorem 5.12, we depart from our attempt to use only complex projective geometry. We will need some facts about real affine geometry, including the following real affine version of the Fundamental Theorem. See [H], [Samuel].

Proposition 5.10. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is invertible and takes real affine lines to real affine lines, then f is an affine transformation, of the form $f(\vec{x}) = \mathbf{L}(\vec{x}) + \vec{t}$, for some invertible real linear transformation $\mathbf{L}$ and vector $\vec{t}$.* ∎

The next Theorem will make a connection between complex projective geometry and real affine geometry, using the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$, so that the complex number $x+iy$ corresponds to the ordered pair (x,y). $\mathbb{C}$ will also be considered as an affine neighborhood of $\mathbb{C}P^1$, the complement of the singleton set $\{[0:1]\}$, so any complex number z corresponds to $[1:z] \in \mathbb{C}P^1 \setminus \{[0:1]\}$.

Theorem 5.11. *Given a real involution B of $\mathbb{C}P^1$, if $[0:1] \in R_B$, then the intersection of R_B with the affine neighborhood $\mathbb{C}P^1 \setminus \{[0:1]\} = \mathbb{C} = \mathbb{R}^2$ is a real affine line. Conversely, every real affine line is such an intersection.*

Proof. Recall the definition of a real affine line, to be the set of points (x,y) in $\mathbb{R}^2$ satisfying an equation $Ex+Fy+G=0$, for $E,F,G \in \mathbb{R}$, with E, F not both 0. Converting to the z coordinate system gives $E\frac{z+\bar{z}}{2} + F\frac{z-\bar{z}}{2i} + G = \frac{E-iF}{2}z + \frac{E+iF}{2}\bar{z} + G = 0$, a self-conjugate inhomogeneous linear equation. Similarly converting any equation of the form $\lambda z + \mu\bar{z} + G = 0$, with $\bar{\lambda} = \mu = \frac{E+iF}{2} \neq 0$ and $\bar{G} = G$, back to the (x,y) coordinate system gives the equation of a real affine line.

Supposing B is induced by a conjugate linear map, it is of the form $B([z_0 : z_1]) = [a\bar{z}_0 + b\bar{z}_1 : c\bar{z}_0 + d\bar{z}_1]$, for complex numbers a,b,c,d such that $ad-bc \neq 0$. Since B fixes $[0:1]$, $b=0$, $a \neq 0$, $d \neq 0$. $(B \circ B)([z_0 : z_1]) = [a\bar{a}z_0 : (c\bar{a} + d\bar{c})z_0 + d\bar{d}z_1]$, and since B is an involution, $a\bar{a} = d\bar{d}$ and $c\bar{a} + d\bar{c} = 0$. Since B is invertible, its restriction to the complement of $\{[0:1]\}$, $[1:z] \mapsto [a : c + d\bar{z}]$ is also invertible, and can be written $B : \mathbb{C} \to \mathbb{C} : B(z) = \frac{d}{a}\bar{z} + \frac{c}{a}$. $|\frac{d}{a}|^2 = 1$, so B can also be written $B(z) = -e^{i\theta}\bar{z} + \beta$, where $-e^{i\theta}\bar{\beta} + \beta = \frac{d}{a}\frac{\bar{c}}{\bar{a}} + \frac{c}{a} = 0$.

The intersection of R_B with the affine neighborhood $\mathbb{C}$ is the solution set of $z = -e^{i\theta}\bar{z} + \beta$, and multiplying this equation by $e^{-i\theta/2}$ gives

$$e^{-i\theta/2}z + e^{i\theta/2}\bar{z} - e^{-i\theta/2}\beta = 0. \tag{3}$$

Multiplying the equation $-e^{i\theta}\bar{\beta} + \beta = 0$ by the same quantity gives $-e^{i\theta/2}\bar{\beta} + e^{-i\theta/2}\beta = 0$, so $e^{-i\theta/2}\beta \in \mathbb{R}$, and (3) is the self-conjugate equation of a real affine line.

Conversely, given any real affine line, with equation $\lambda z + \bar{\lambda}\bar{z} + G = 0$, $\lambda \neq 0$, $G = \bar{G}$, define $B : \mathbb{C}P^1 \to \mathbb{C}P^1$ by $B([z_0 : z_1]) = [-\lambda\bar{z}_0 : G\bar{z}_0 + \bar{\lambda}\bar{z}_1]$. B is a real involution that fixes $[0:1]$, and restricts to $B(z) = -\frac{\bar{\lambda}}{\lambda}\bar{z} - \frac{G}{\lambda}$ on $\mathbb{C}$. The equation for the fixed point set in $\mathbb{C}$ is $z = -\frac{\bar{\lambda}}{\lambda}\bar{z} - \frac{G}{\lambda} \iff \lambda z + \bar{\lambda}\bar{z} + G = 0$. ∎

Theorem 5.12. *Given an invertible map $f : \mathbb{C}P^1 \to \mathbb{C}P^1$, the following are equivalent.*

1. *For every real involution B, there exists some real involution B' such that $f(R_B) \subseteq R_{B'}$.*

2. *For every quadruple of points (a,b,c,d), if $\chi((a,b,c,d)) \in R_{\tilde{\kappa}}$, then $(\chi \circ f^4)((a,b,c,d)) \in R_{\tilde{\kappa}}$.*

3. *Either f or $\tilde{\kappa} \circ f$ is a projective transformation.*

Proof. The equivalence of *1.* and *2.* follows from Theorem 5.9. *3.* $\implies$ *2.* by Proposition 5.4 and Theorem 5.5, and *3.* $\implies$ *1.* by Theorem 3.8. Showing *1.* and *2.* imply *3.* is enough for the claimed equivalence.

As in the Proof of Theorem 5.7, since f is one-to-one, the points $a = f([0 : 1])$, $b = f([1 : 0])$, $c = f([1 : 1])$ are distinct, and by Lemma 5.6, there is some $A \in PGL(2, \mathbb{C})$ so that $A \circ f$ fixes each of the points $[0 : 1]$, $[1 : 0]$, $[1 : 1]$. $A \circ f$ also satisfies *1.* and *2.* by Theorem 3.8 and Proposition 5.4. Let g denote the restriction of $A \circ f$ to the affine neighborhood $\mathbb{C}P^1 \setminus \{[0 : 1]\}$, so g is an invertible map $\mathbb{C} \to \mathbb{C}$.

Given any real affine line in $\mathbb{C}$, there is, by Theorem 5.11, some real involution B such that R_B is the union of the line and the point $[0 : 1]$. By *1.*, $(A \circ f)(R_B) \subseteq R_{B'}$ for some real involution B', and since $A \circ f$ is one-to-one and fixes $[0 : 1]$, the restriction g takes the given line to the complement of $[0 : 1]$ in $R_{B'}$, which by Theorem 5.11 again, is another real affine line. Proposition 5.10 applies, and since g fixes $0 = (0, 0)$ and $1 = (1, 0)$ in $\mathbb{C} = \mathbb{R}^2$, it is of the form $g((x, y)) = (x + py, qy)$, for some $p, q \in \mathbb{R}$, with $q \neq 0$.

For any $\alpha \in \mathbb{C}$ such that $\alpha\bar{\alpha} = 1$, $\alpha \neq \pm 1$, it is easy to check that the points $[1 : 1]$, $[1 : \alpha]$, $[1 : -1]$, $[1 : -\alpha]$ are distinct fixed points of the real involution $B([z_0 : z_1]) = [\bar{z}_1 : \bar{z}_0]$. It can also be checked directly that the four points in that order have cross ratio in $R_{\tilde{\kappa}}$. In terms of the complex affine neighborhood, the four points are 1, α, -1, and $-\alpha$, and g fixes 1 and -1. By *2.*, the following cross ratio should also be in $R_{\tilde{\kappa}}$:

$$\begin{aligned}
& \chi(([1 : 1], [1 : g(\alpha)], [1 : -1], [1 : g(-\alpha)])) \\
= \; & [(1 + g(\alpha))^2 : 4g(\alpha)] \\
= \; & [1 : \frac{4g(\alpha)(1 + \overline{g(\alpha)})^2}{(1 + g(\alpha))^2(1 + \overline{g(\alpha)})^2}],
\end{aligned}$$

which is equivalent to $g(\alpha) + |g(\alpha)|^2\overline{g(\alpha)} \in \mathbb{R}$. Let $\alpha = r + is$, with $r^2 + s^2 = 1$, $s \neq 0$, so $g(\alpha) = r + ps + iqs$. The imaginary part of $g(\alpha) + |g(\alpha)|^2\overline{g(\alpha)}$ is $qs(1 - ((r + ps)^2 + (qs)^2))$, which is zero if and only if

$$(r + ps)^2 + (qs)^2 = 1. \tag{4}$$

This equation holds for $\alpha = i$, so $r = 0$, $s = 1$, and $p^2 + q^2 = 1$. Expanding (4) gives $r^2 + 2rps + (p^2 + q^2)s^2 = 1$, which is equivalent to $2rps = 0$. Since this holds for some α with non-zero r and s, p must be 0, and it follows that $q = \pm 1$.

The conclusion is that $g(z)$ is either the identity function on $\mathbb{C}$, or complex conjugation, so $A \circ f$ is either the identity function on $\mathbb{C}P^1$, or $\tilde{\kappa}$, and *3.* follows. ∎

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