Notes on first semester calculus

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These notes supplement the textbooks that have been required for first semester calculus (MA 16500) at Purdue Fort Wayne — Briggs et al ([BCGS]), or, previously, Stewart ([S]) at IPFW. Page or Section numbers cited here refer to [BCGS].

1.1 Increasing Functions and Decreasing Functions

Theorem 1.1. *If $f(x)$ is increasing on its domain $D \subseteq \mathbb{R}$, then $f^{-1}(x)$ is also increasing on its domain.*

Proof. First, check that f is one-to-one, so that f^{-1} exists. Given two points in the domain of f, $x \neq t$, we want to show $f(x) \neq f(t)$. From $x \neq t$, there are two cases: $x < t$ or $x > t$. If $x < t$, then $f(x) < f(t)$ because f is increasing, and similarly if $t < x$, then $f(t) < f(x)$. In either case, $f(x) \neq f(t)$.

Second, suppose, toward a contradiction, that $f^{-1}(x)$ is not increasing. This means the following (the "negation" of the definition of increasing): there are two points (a, b) and (c, d) on the graph of f^{-1} with $a < c$ and $b \geq d$. However, then there are two points (b, a) and (d, c) on the graph of f with $b \geq d$ and $a < c$. The case $b = d$ and $a < c$ is impossible since then f would fail the vertical line test, but the case $b > d$ and $a < c$ is also excluded since f is increasing. This contradiction shows $f^{-1}(x)$ must be increasing. ∎

Similarly, if f is decreasing, then it is one-to-one, with a decreasing inverse.

Theorem 1.2. *Let D be any subset of $\mathbb{R}$ containing at least three different points, and suppose $f(x)$ is a function with domain D, that has the following property: for any three points x, y, z in D, if $x < y < z$, then either $f(x) < f(y) < f(z)$ or $f(x) > f(y) > f(z)$. Then we can conclude $f(x)$ is either increasing on D or decreasing on D.*

Proof. First, consider the following two cases: f is either decreasing, or it's not decreasing. In the first case, if f is decreasing, then the conclusion of the Theorem holds and we're done. In the second case, if f is not decreasing, we'll show that f is increasing, which will prove the Theorem.

So, the assumption that f is not decreasing means: there exist two points x, z in D so that $x < z$ and $f(x) \leq f(z)$. Pick any number y in D which is different from x and z (this is where we use the assumption that D has at least three points). There are three cases: if $y < x < z$, then either $f(y) < f(x) < f(z)$ or $f(y) > f(x) > f(z)$. However, the second possibility is excluded since we already know $f(x) \leq f(z)$. If $x < y < z$, then either $f(x) < f(y) < f(z)$ or

$f(x) > f(y) > f(z)$, and again the second possibility is excluded. Finally, if $x < z < y$, then $f(x) < f(z) < f(y)$ must hold since $f(x) > f(z) > f(y)$ cannot. So, we can conclude in each of these three cases that $f(x) < f(z)$.

The next step is to show that f is increasing on D, which means that for any two points a and b in D, if $a < b$, then $f(a) < f(b)$. We will be able to show $f(a) < f(b)$ using the property of f given in the hypothesis, together with the information $f(x) < f(z)$ proved in the previous paragraph, but there are several cases, depending on the relative position of the intervals $[a, b]$ and $[x, z]$.

Case 1. $x < z < a < b$. Then $x < z < a \implies f(x) < f(z) < f(a)$ or $f(x) > f(z) > f(a)$, but $f(x) > f(z)$ is excluded, so $f(z) < f(a)$. Also, $z < a < b \implies f(z) < f(a) < f(b)$ or $f(z) > f(a) > f(b)$, but we just showed $f(z) < f(a)$, so $f(a) < f(b)$.

Case 2. $x < z = a < b$. Then $f(x) < f(z) = f(a) < f(b)$ or $f(x) > f(z) = f(a) > f(b)$, but the second possibility is excluded since $f(x) < f(z)$, so $f(a) < f(b)$.

Case 3. $x < a < z < b$. Then $x < a < z \implies f(x) < f(a) < f(z)$ or $f(x) > f(a) > f(z)$, but $f(x) > f(z)$ is excluded, so $f(a) < f(z)$. Also, $a < z < b \implies f(a) < f(z) < f(b)$ or $f(a) > f(z) > f(b)$, but we just showed $f(a) < f(z)$, so $f(a) < f(b)$.

Case 4. $x < a < z = b$. Then $f(x) < f(a) < f(z) = f(b)$ or $f(x) > f(a) > f(z) = f(b)$, but the second possibility is excluded since $f(x) < f(z)$, so $f(a) < f(b)$.

Case 5. $x < a < b < z$. Then $x < a < z \implies f(x) < f(a) < f(z)$ or $f(x) > f(a) > f(z)$, but $f(x) > f(z)$ is excluded, so $f(a) < f(z)$. Also, $a < b < z \implies f(a) < f(b) < f(z)$ or $f(a) > f(b) > f(z)$, but we just showed $f(a) < f(z)$, so $f(a) < f(b)$.

Case 6. $x = a < z < b$. Then $f(x) = f(a) < f(z) < f(b)$ or $f(x) = f(a) > f(z) > f(b)$, but the second possibility is excluded since $f(x) < f(z)$, so $f(a) < f(b)$.

Case 7. $x = a < z = b$. Then $f(x) = f(a) < f(z) = f(b)$.

Case 8. $x = a < b < z$. Then $f(x) = f(a) < f(b) < f(z)$ or $f(x) = f(a) > f(b) > f(z)$, but the second possibility is excluded since $f(x) < f(z)$, so $f(a) < f(b)$.

Case 9. $a < x < z < b$. Then $a < x < z \implies f(a) < f(x) < f(z)$ or $f(a) > f(x) > f(z)$, but $f(x) > f(z)$ is excluded, so $f(a) < f(z)$. Also, $a < z < b \implies f(a) < f(z) < f(b)$ or $f(a) > f(z) > f(b)$, but we just showed $f(a) < f(z)$, so $f(a) < f(b)$.

Case 10. $a < x < z = b$. Then $f(a) < f(x) < f(z) = f(b)$ or $f(a) > f(x) > f(z) = f(b)$, but the second possibility is excluded since $f(x) < f(z)$, so $f(a) < f(b)$.

Case 11. $a < x < b < z$. Then $a < x < z \implies f(a) < f(x) < f(z)$ or $f(a) > f(x) > f(z)$, but $f(x) > f(z)$ is excluded, so $f(a) < f(z)$. Also, $a < b < z \implies f(a) < f(b) < f(z)$ or $f(a) > f(b) > f(z)$, but we just showed $f(a) < f(z)$, so $f(a) < f(b)$.

Case 12. $a < x = b < z$. Then $f(a) < f(x) = f(b) < f(z)$ or $f(a) > f(x) = f(b) > f(z)$, but the second possibility is excluded since $f(x) < f(z)$, so $f(a) < f(b)$.

Case 13. $a < b < x < z$. Then $a < x < z \implies f(a) < f(x) < f(z)$ or $f(a) > f(x) > f(z)$, but $f(x) > f(z)$ is excluded, so $f(a) < f(z)$. Also, $a < b < z \implies f(a) < f(b) < f(z)$ or $f(a) > f(b) > f(z)$, but we just showed $f(a) < f(z)$, so $f(a) < f(b)$.

In each case, $f(a) < f(b)$, which establishes f is increasing on D. ∎

1.2 Limits

Theorem 1.3. *If* $\lim_{x\to c} g(x) = L$*, and there is some* $\delta > 0$ *so that* $g(x) > 0$ *for all* x *such that* $0 < |x - c| < \delta$*, then* $L \geq 0$*.*

Proof. Suppose, toward a contradiction, that $L \geq 0$ is false, so $L < 0$. Let $\epsilon = -L > 0$, so from the definition of limit, there is some $\delta' > 0$ so that if $0 < |x - c| < \delta'$, then $|g(x) - L| < \epsilon = -L$. Solving this inequality gives $-(-L) < g(x) - L < -L \implies g(x) < 0$ for all x within δ' of c, but there is some x within $\min\{\delta, \delta'\}$ of c where $g(x) > 0$ by hypothesis, and $g(x) < 0$ from the previous inequality, a contradiction. Since assuming $L < 0$ leads to a contradiction, we must conclude $L \geq 0$. ■

Theorem 1.4. *If* $\lim_{x\to c} g(x) = L > 0$*, then there is some* $\delta_1 > 0$ *so that if* $0 < |x - c| < \delta_1$*, then* $g(x) > 0$*.*

Proof. Let $\epsilon = L > 0$, so from the definition of limit, there is some $\delta_1 > 0$ so that if $0 < |x-c| < \delta_1$, then $|g(x) - L| < \epsilon = L$. Solving this inequality gives $-L < g(x) - L < L \implies 0 < g(x)$. ■

Here's a proof of the Product Rule for limits (a similar proof appears in Appendix A, starting on page AP-1). The idea is: I assume I can get $f(x)$ close to L, and $g(x)$ close to M, and I want to get $f(x) \cdot g(x)$ within a given output tolerance ϵ of $L \cdot M$. So how close do f and g have to be to L and M, respectively? Recall that in the proof of the Sum Rule for limits (Example 2.7.6, p. 122), we were trying to get the sum $f + g$ within ϵ of $L + M$, and it was enough to get f and g within $\epsilon/2$ of L and M. However, that $\epsilon/2$ is not necessarily close enough to get $f \cdot g$ within ϵ of $L \cdot M$ so the calculation in the following proof follows the same general idea but chooses different tolerances.

Theorem 1.5. *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$*, then* $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot M$*.*

Proof. Given any $\epsilon > 0$, we want to find $\delta > 0$ so that if $0 < |x-c| < \delta$, then $|f(x) \cdot g(x) - L \cdot M| < \epsilon$. We'll use the positive number $\frac{\epsilon}{2(1+|M|)}$ as an output tolerance for f, so there is some $\delta_1 > 0$ so that if $0 < |x - c| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2(1+|M|)}$. Since 1 and $\frac{\epsilon}{2(1+|L|)}$ are both positive, use the smaller of these two numbers, $\min\{1, \frac{\epsilon}{2(1+|L|)}\} > 0$ as an output tolerance for g, so there is some $\delta_2 > 0$ so that if $0 < |x - c| < \delta_2$, then $|g(x) - M| < \min\{1, \frac{\epsilon}{2(1+|L|)}\}$. Let $\delta = \min\{\delta_1, \delta_2\} > 0$, so that if $0 < |x - c| < \delta$, then all these inequalities hold:

$$|f(x) - L| < \frac{\epsilon}{2(1 + |M|)}, \quad |g(x) - M| < 1, \quad |g(x) - M| < \frac{\epsilon}{2(1 + |L|)}.$$

These inequalities, together with the triangle inequality, will be enough to get $f(x) \cdot g(x)$ within

ϵ of $L \cdot M$:

$$\begin{aligned}
|f(x) \cdot g(x) - L \cdot M| &= |f(x) \cdot g(x) - L \cdot g(x) + L \cdot g(x) - L \cdot M| \\
&= |(f(x) - L) \cdot g(x) + L \cdot (g(x) - M)| \\
&\leq |(f(x) - L) \cdot g(x)| + |L \cdot (g(x) - M)| \\
&= |f(x) - L| \cdot |g(x)| + |L| \cdot |g(x) - M| \\
&= |f(x) - L| \cdot |(g(x) - M) + M| + |L| \cdot |g(x) - M| \\
&\leq |f(x) - L| \cdot (|g(x) - M| + |M|) + |L| \cdot |g(x) - M| \\
&< \frac{\epsilon}{2(1 + |M|)} \cdot (1 + |M|) + |L| \cdot \frac{\epsilon}{2(1 + |L|)} \\
&= \frac{\epsilon}{2} + \frac{\epsilon \cdot |L|}{2(1 + |L|)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

∎

As one of the steps toward proving the Quotient Rule for limits, the following argument gives an answer to this question: for $M \neq 0$, how close does $g(x)$ have to be to M, in order to get $\frac{1}{g(x)}$ within ϵ of $\frac{1}{M}$? This is the "Reciprocal Rule" for limits and a similar proof appears in Appendix A, page AP-2.

Theorem 1.6. *If* $\lim_{x \to a} g(x) = M$ *and* $M \neq 0$*, then* $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$.

Proof. Given $\epsilon > 0$, and $M \neq 0$, the following number is positive: $\min\{\frac{|M|}{2}, \frac{\epsilon |M|^2}{2}\}$, so we can use it as an output tolerance for $g(x)$, and there exists some $\delta > 0$ so that if $0 < |x - a| < \delta$, then $|g(x) - M| < \min\{\frac{|M|}{2}, \frac{\epsilon |M|^2}{2}\}$. From the inequality $|g(x) - M| < \frac{|M|}{2}$, and the triangle inequality, we get:

$$\begin{aligned}
|M| &= |M - g(x) + g(x)| \leq |M - g(x)| + |g(x)| < \frac{|M|}{2} + |g(x)| \\
\implies |g(x)| &> |M| - \frac{|M|}{2} = \frac{|M|}{2} \\
\implies \frac{1}{|g(x)|} &< \frac{2}{|M|}.
\end{aligned}$$

This, and the inequality $|g(x) - M| < \frac{\epsilon |M|^2}{2}$, are used as steps to show that for x within δ of a, $\frac{1}{g(x)}$ is within ϵ of $\frac{1}{M}$:

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| = \left| \frac{M - g(x)}{g(x) \cdot M} \right| = \frac{1}{|g(x)|} \cdot \frac{1}{|M|} \cdot |g(x) - M| < \frac{2}{|M|} \cdot \frac{1}{|M|} \cdot \frac{\epsilon |M|^2}{2} = \epsilon.$$

∎

The Quotient Rule for limits (Limit Rule #5 from Theorem 2.3, Section 2.3, page 72) follows immediately from the previous Theorem and the Product Rule for limits (#4 on page 72), so we call it a "Corollary."

Corollary 1.7 (The Quotient Rule for Limits). *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to c} g(x) = M$ *and* $M \neq 0$, *then* $\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$.

Proof. We just showed, under the condition $M \neq 0$, $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$, so we can use the product rule for limits to get

$$\lim_{x\to c} \left(\frac{f(x)}{g(x)}\right) = \lim_{x\to c} \left(f(x) \cdot \frac{1}{g(x)}\right) = L \cdot \frac{1}{M} = \frac{L}{M}.$$

∎

The next result is the Composite Limit Theorem. (This is Theorem 2.12.2 in Section 2.6, on page 106, with a proof on page AP-3 in Appendix A.) Again, we are going to try to find a δ that will get the function $f(g(x))$ close to its limit. Since the output of g is the input of f in the composite function, it is not too surprising that the input tolerance for f will also be used as an output tolerance for g.

Theorem 1.8. *If* $f(x)$ *is continuous at* b, *and* $\lim_{x\to a} g(x) = b$, *then* $\lim_{x\to a} f(g(x)) = f(b)$.

Proof. Given any $\epsilon > 0$, we want to find $\delta > 0$ so that if $0 < |x-c| < \delta$, then $|f(g(x))-f(b)| < \epsilon$. From the definition of continuous ($\lim_{u\to b} f(u) = f(b)$), we know there is some $\delta_1 > 0$ so that if $0 < |u-b| < \delta_1$, then $|f(u)-f(b)| < \epsilon$. Obviously if $u = b$, then $|f(u)-f(b)| = 0 < \epsilon$, so whether u is close to b or equal to b, we get the implication $(*)$: if $|u - b| < \delta_1$, then $|f(u) - f(b)| < \epsilon$. Using this $\delta_1 > 0$ as an output tolerance for g, there is some input tolerance $\delta > 0$ so that if $0 < |x - a| < \delta$, then $|g(x) - b| < \delta_1$. Since $u = g(x)$ then satisfies the hypothesis of $(*)$, for x within δ of a, we get the conclusion of $(*)$ with $u = g(x)$: $|f(g(x)) - f(b)| < \epsilon$, which is all we needed. ∎

1.3 More on Increasing and Decreasing

Theorem 1.9. *If* $f(x)$ *is weakly increasing on* (a, b) *and* $\lim_{x\to b^-} f(x) = f(b)$, *then* $f(x)$ *is weakly increasing on* $(a, b]$.

Proof. We only need to show that if $a < x < b$, then $f(x) \leq f(b)$. Suppose, toward a contradiction, that there is some t in (a, b) so that $f(t) > f(b)$. We can use the positive number $f(t) - f(b)$ as an output tolerance from the definition of limit: there is some $\delta_1 > 0$ so that if $b - \delta_1 < x < b$, then $|f(x) - f(b)| < f(t) - f(b)$. Let $\delta_2 = \min\{\delta_1, b - t\} > 0$, so there is some point x in $(b - \delta_2, b)$ so that $f(x) - f(b) < f(t) - f(b)$ because $b - \delta_1 \leq b - \delta_2 < x < b$, so $f(x) < f(t)$, but $b - (b - t) \leq b - \delta_2 < x < b \implies t < x \implies f(t) \leq f(x)$ because f is weakly increasing. This is a contradiction. ∎

Theorem 1.10. *If $f(x)$ is increasing on (a, b) and $\lim\limits_{x \to b^-} f(x) = f(b)$, then $f(x)$ is increasing on $(a, b]$.*

Proof. Since f is increasing, it is also weakly increasing, and so the previous Theorem applies to show that if $a < x < b$, then $f(x) \leq f(b)$. So, given any x in (a, b) we only need to check that $f(x) < f(b)$. Let t be any number between x and b, so $a < x < t < b$. Then, $f(x) < f(t)$ because f is increasing on (a, b) and $f(t) \leq f(b)$ because f is weakly increasing on $(a, b]$. We can conclude $f(x) < f(t) \leq f(b)$, which is enough. ∎

Similarly, the "increasing" and "weakly increasing" properties extend to a left endpoint of an interval as long as the function is continuous from the right at that endpoint. Without rewriting the proof, we state this fact as a "Proposition."

Proposition 1.11. *If $f(x)$ is weakly increasing on (c, b) and $\lim\limits_{x \to c^+} f(x) = f(c)$, then f is weakly increasing on $[c, b)$.* ∎

The "decreasing" and "weakly decreasing" properties also extend to endpoints. The following Theorem considers a function increasing on interval and decreasing on another interval.

Theorem 1.12. *If $f(x)$ is weakly increasing on (a, c) and $f(x)$ is weakly decreasing on (c, b), and f is continuous at c, then $f(x) \leq f(c)$ for all x in (a, b).*

Proof. By Theorem 1.9, f is weakly increasing on $(a, c]$, so $f(x) \leq f(c)$ for all x in $(a, c]$. An analogous result for weakly decreasing functions on (c, b) shows $f(x) \leq f(c)$ for all x in $[c, b)$. The conclusion is that $f(x) \leq f(c)$ for all x in $(a, c] \cup [c, b) = (a, b)$. ∎

The conclusion of the above Theorem could be stated as "The global maximum value of $f(x)$ on the interval (a, b) is $f(c)$." The following Proposition, on a minimum value, is similar.

Proposition 1.13. *If $f(x)$ is weakly decreasing on (a, c) and $f(x)$ is weakly increasing on (c, b), and f is continuous at c, then $f(x) \geq f(c)$ for all x in (a, b).* ∎

1.4 Consequences of the Intermediate Value Theorem

The proof of the following Theorem is outside the scope of this course (maybe you'll see it in Math 441).

Proposition 1.14 (The Intermediate Value Theorem)**.** *Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.* ∎

Recall that continuity on a closed interval includes one-side limits at the endpoints.

Theorem 1.15. *Let $I \subseteq \mathbb{R}$ be an interval or a ray or the whole line $\mathbb{R}$. If $f(x)$ is continuous and one-to-one on I, then $f(x)$ is either increasing on I or decreasing on I.*

Proof. Consider three points in I, $x < y < z$. Since $f(x)$ is one-to-one, no two of the values $f(x)$, $f(y)$, $f(z)$ can be equal. There are six cases for the relative positions of these three different values:

Case 1. $f(x) < f(y) < f(z)$.

Case 2. $f(x) > f(y) > f(z)$.

Case 3. $f(x) < f(z) < f(y)$. However, in this case, f is continuous on $[x, y]$, so by IVT there is some $c \in (x, y)$ such that $f(c) = f(z)$. Since $x < c < y < z$, $c \neq z$, but $f(c) = f(z)$, contradicting the assumption that f is one-to-one.

Case 4. $f(y) < f(x) < f(z)$. However, in this case, f is continuous on $[y, z]$, so by IVT there is some $c \in (y, z)$ such that $f(c) = f(x)$. Since $x < y < c < z$, $c \neq x$, but $f(c) = f(x)$, contradicting the assumption that f is one-to-one.

Case 5. $f(z) < f(x) < f(y)$. However, in this case, f is continuous on $[y, z]$, so by IVT there is some $c \in (y, z)$ such that $f(c) = f(x)$. Since $x < y < c < z$, $c \neq x$, but $f(c) = f(x)$, contradicting the assumption that f is one-to-one.

Case 6. $f(y) < f(z) < f(x)$. However, in this case, f is continuous on $[x, y]$, so by IVT there is some $c \in (x, y)$ such that $f(c) = f(z)$. Since $x < c < y < z$, $c \neq z$, but $f(c) = f(z)$, contradicting the assumption that f is one-to-one.

So, only Cases 1 and 2 do not lead to a contradiction. This means that for any three points $x < y < z$ in I, either $f(x) < f(y) < f(z)$, or $f(x) > f(y) > f(z)$, and this is exactly the hypothesis of Theorem 1.2. The conclusion from that Theorem is that f is either increasing or decreasing on I. ∎

Theorem 1.16. *If $f(x)$ is continuous and one-to-one on an open interval (a, b), then f^{-1} is continuous at every point in its domain.*

Proof. By Theorem 1.15, $f(x)$ is either increasing or decreasing. We will consider just the increasing case. The decreasing case can be given a similar proof.

Given any $\epsilon > 0$ and c in the domain of f^{-1}, we want to find a $\delta > 0$ so that if $|x - c| < \delta$, then $|f^{-1}(x) - f^{-1}(c)| < \epsilon$.

Suppose (d, c) is a point on the graph of f, so $c = f(d)$, and (c, d) is on the graph of f^{-1}, so $d = f^{-1}(c)$. Let $\epsilon_1 > 0$ be less than or equal to ϵ, and also small enough so that $[d-\epsilon_1, d+\epsilon_1]$ is contained in (a, b) (for example, let $\epsilon_1 = \min\{\epsilon, \frac{1}{2}(d-a), \frac{1}{2}(b-d)\} > 0$). Since f is increasing, $f(d - \epsilon_1) < f(d) = c < f(d + \epsilon_1)$. Then, we can pick some $\delta > 0$ so that $f(d - \epsilon_1) < c - \delta < c < c + \delta < f(d + \epsilon_1)$ (for example, let $\delta = \frac{1}{2}\min\{c - f(d - \epsilon_1), f(d + \epsilon_1) - c\} > 0$). If x is any point such that $c - \delta < x < c + \delta$, then $f(d - \epsilon_1) < x < f(d + \epsilon_1)$, and since f is continuous on $[d - \epsilon_1, d + \epsilon_1]$, the Intermediate Value Theorem applies to show there exists some point k (depending on x) such that $f(k) = x$. This means x is in the domain of f^{-1}, with $f^{-1}(x) = k$. Since f^{-1} is also increasing (by Theorem 1.1), if x is any point such that $c - \delta < x < c + \delta$, then we can apply f^{-1} to get the inequalities:

$$f^{-1}(f(d - \epsilon_1)) < f^{-1}(x) < f^{-1}(f(d + \epsilon_1)).$$

This simplifies to $d-\epsilon_1 < f^{-1}(x) < d+\epsilon_1$, so we can conclude $|f^{-1}(x)-d| = |f^{-1}(x)-f^{-1}(c)| < \epsilon_1 \leq \epsilon$. ∎

1.5 Derivatives

Definition 1.17. A function f is <u>differentiable</u> at c means that $f'(c)$ exists — this means the limit $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists, and we denote the value of the limit by $f'(c)$. Using the definition of limit gives the following: For any $\epsilon > 0$, there exists some $\delta > 0$ so that if $0 < |h-0| < \delta$, then $\left|\frac{f(c+h)-f(c)}{h} - f'(c)\right| < \epsilon$.

Since $f(c)$ and $f(c+h)$ must both be defined for this Definition to work, we can conclude that δ must be small enough so that if $-\delta < h < \delta$, then $c+h$ is in the domain of $f(x)$, and so the interval $(c-\delta, c+\delta)$ is contained in the domain of $f(x)$.

Given f and c, we can also conclude that if the domain of $f(x)$ does not contain any interval of the form $(c-\delta, c+\delta)$, then f is not differentiable at c. For example, if the domain of $f(x)$ is a closed interval $[c,d]$ (so c is an endpoint), then no matter how small δ is, there isn't any neighborhood $(c-\delta, c+\delta)$ contained in $[c,d]$. This is why we include endpoints of closed intervals as "critical points" for the purposes of Fermat's theorem on local extreme values.

Here's a proof of the Composite Rule for derivatives (the "chain rule"). This rule is stated as Theorem 3.13 in Section 3.7 on page 192, with a proof of a special case appearing on page 196 (it makes an extra assumption to avoid a division by zero problem). The idea for the proof without using any extra assumptions is sketched as Exercise 115 on page 200. The following proof uses the same idea, but different notation, and fills in all the details.

Theorem 1.18 (The Chain Rule)**.** *If g is differentiable at a and f is differentiable at $g(a)$, then $f\circ g$ is differentiable at a, and $(f\circ g)'(a) = f'(g(a))\cdot g'(a)$.*

Proof. Since f is differentiable at $g(a)$, it must be defined near $g(a)$ as previously described, so there is some $\delta_0 > 0$ so that the neighborhood $(g(a)-\delta_0, g(a)+\delta_0)$ is contained in the domain of f. Since g is differentiable at a, it is also continuous at a, meaning $\lim_{x\to a} g(x) = g(a)$. By the definition of limit (with output tolerance $\delta_0 > 0$) there is some $\delta_1 > 0$ so that if $0 < |x-a| < \delta_1$, then $|g(x)-g(a)| < \delta_0$. We can conclude that the neighborhood $(a-\delta_1, a+\delta_1)$ is in the domain of g, and that the values $g(x)$, for x in this neighborhood, satisfy $g(a)-\delta_0 < g(x) < g(a)+\delta_0$, so they are in the domain of f. It follows that $(a-\delta_1, a+\delta_1)$ is contained in the domain of $f(g(x))$.

Define the following function on the domain $\Delta x \in (-\delta_1, \delta_1)$:

$$\varepsilon_2(\Delta x) = \begin{cases} \dfrac{f(g(a+\Delta x)) - f(g(a))}{g(a+\Delta x) - g(a)} - f'(g(a)) & \text{if } g(a+\Delta x) \neq g(a) \\ \\ 0 & \text{if } g(a+\Delta x) = g(a). \end{cases}$$

The two cases are set up to avoid a division by 0 problem, and the domain $\Delta x \in (-\delta_1, \delta_1)$ uses the δ_1 from the previous paragraph to ensure that $a+\Delta x$ is in the domain of g and $g(a+\Delta x)$ is in the domain of f.

At $\Delta x = 0$, $g(a+0) = g(a)$, so the second case applies, and $\varepsilon_2(0) = 0$. To show that ε_2 is continuous at 0, we need to find, for any given $\epsilon > 0$, some $\delta > 0$ so that if $0 < |\Delta x - 0| < \delta$, then $|\varepsilon_2(\Delta x) - 0| < \epsilon$. Again using the hypothesis that f is differentiable at $g(a)$, there is some $\delta_2 > 0$

so that if $0 < |\Delta u| < \delta_2$, then $\left|\frac{f(g(a)+\Delta u)-f(g(a))}{\Delta u} - f'(g(a))\right| < \epsilon$. Again using the continuity of g at a, there is some $\delta_3 > 0$ so that if $0 < |x-a| < \delta_3$, then $|g(x)-g(a)| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_3\}$, so that if $0 < |\Delta x| < \delta$, then Δx is in the domain of ε_2 and $0 < |(a + \Delta x) - a| < \delta_3$, so $|g(a + \Delta x) - g(a)| < \delta_2$.

This leads to two cases, the first where $g(a + \Delta x) = g(a)$, so $|\varepsilon_2(\Delta x) - 0| = |0 - 0| = 0 < \epsilon$, and the second where $g(a+\Delta x) \neq g(a)$, so we define $\Delta u = g(a+\Delta x)-g(a)$. Since $\Delta u \neq 0$ and $|\Delta u| < \delta_2$, $\left|\frac{f(g(a)+\Delta u)-f(g(a))}{\Delta u} - f'(g(a))\right| < \epsilon$, and we get: if $0 < |\Delta x| < \delta$ and $g(a+\Delta x) \neq g(a)$, then

$$\begin{aligned}
|\varepsilon_2(\Delta x) - 0| &= \left|\frac{f(g(a + \Delta x)) - f(g(a))}{g(a + \Delta x) - g(a)} - f'(g(a))\right| \\
&= \left|\frac{f(g(a) + \Delta u) - f(g(a))}{\Delta u} - f'(g(a))\right| \\
&< \epsilon.
\end{aligned}$$

In either of the above two cases, if $0 < |\Delta x| < \delta$, then $|\varepsilon_2(\Delta x)| < \epsilon$, which is what we wanted to show; the conclusion here is that $\lim\limits_{\Delta x\to 0} \varepsilon_2(\Delta x) = 0$.

Now for $0 < |\Delta x| < \delta_1$, both of the following quantities are defined, and I want to show the left hand side (LHS) is equal to the right hand side (RHS):

$$\frac{f(g(a + \Delta x)) - f(g(a))}{\Delta x} = (f'(g(a)) + \varepsilon_2(\Delta x)) \cdot \frac{g(a + \Delta x) - g(a)}{\Delta x}.$$

If $g(a+\Delta x) = g(a)$, then LHS simplifies to $\frac{0}{\Delta x} = 0$, and RHS simplifies to $(f'(g(a))+0)\cdot \frac{0}{\Delta x} = 0$. If $g(a + \Delta x) \neq g(a)$, RHS equals

$$\left(f'(g(a)) + \left(\frac{f(g(a + \Delta x)) - f(g(a))}{g(a + \Delta x) - g(a)} - f'(g(a))\right)\right) \cdot \frac{g(a + \Delta x) - g(a)}{\Delta x},$$

which simplifies to be the same as the LHS. So, the equality holds for all Δx such that $0 < |\Delta x| < \delta_1$.

The definition of the derivative of the composite is:

$$\begin{aligned}
(f \circ g)'(a) &= \lim_{\Delta x\to 0} \frac{f(g(a + \Delta x)) - f(g(a))}{\Delta x} \\
&= \lim_{\Delta x\to 0} \left((f'(g(a)) + \varepsilon_2(\Delta x)) \cdot \frac{g(a + \Delta x) - g(a)}{\Delta x}\right) \\
&= \left(\lim_{\Delta x\to 0} (f'(g(a)) + \varepsilon_2(\Delta x))\right) \cdot \left(\lim_{\Delta x\to 0} \frac{g(a + \Delta x) - g(a)}{\Delta x}\right) \\
&= (f'(g(a)) + 0) \cdot g'(a).
\end{aligned}$$

The above steps used the sum and product rules for limits, the hypothesis that g is differentiable at a, and the previously established limit $\lim\limits_{\Delta x\to 0} \varepsilon_2(\Delta x) = 0$. $\blacksquare$

The following Theorem is part of the Inverse Function Theorem:

Theorem 1.19. *Given a function $f(x)$ which is continuous and one-to-one on an open interval (a,b), let (d,c) be a point on the graph of f with $a < d < b$. If f is differentiable at d and $f'(d) \neq 0$, then f^{-1} is differentiable at c, and $(f^{-1})'(c) = \frac{1}{f'(d)}$.*

Proof. Starting with any $\epsilon > 0$, the number $\epsilon_2 = \min\{\frac{1}{2}|f'(d)|, \frac{\epsilon}{2}|f'(d)|^2\}$ is positive, using the assumption that $f'(d) \neq 0$. The differentiability assumption means that there is some $\delta_2 > 0$ so that if $0 < |x-d| < \delta_2$, then $\left|\frac{f(x)-f(d)}{x-d} - f'(d)\right| < \epsilon_2$. By Theorem 1.16 (which used the Intermediate Value Theorem), f^{-1} is continuous at c, so using $\delta_2 > 0$ as an output tolerance for f^{-1}, there is some $\delta > 0$ so that if $|y-c| < \delta$, then $|f^{-1}(y) - f^{-1}(c)| < \delta_2$. For any y with $0 < |y-c| < \delta$, let $x = f^{-1}(y)$, so $|x-d| < \delta_2$ by continuity, and $x \neq d$ because $y \neq c$ and f^{-1} is one-to-one. So, $0 < |x-d| < \delta_2$, which implies $\left|\frac{f(x)-f(d)}{x-d} - f'(d)\right| < \epsilon_2$, and $\left|\frac{y-c}{f^{-1}(y)-f^{-1}(c)} - f'(d)\right| < \min\{\frac{1}{2}|f'(d)|, \frac{\epsilon}{2}|f'(d)|^2\}$. By the calculation from the Proof of Theorem 1.6, this shows

$$\left|\frac{1}{\left(\frac{y-c}{f^{-1}(y)-f^{-1}(c)}\right)} - \frac{1}{f'(d)}\right| < \epsilon,$$

and the conclusion is that

$$\lim_{y\to c} \frac{f^{-1}(y) - f^{-1}(c)}{y-c} = \frac{1}{f'(d)},$$

as claimed. ∎

Using $c = t$ and $d = f^{-1}(t)$, the above formula can also be written as the following derivative rule for inverse functions:

$$(f^{-1})'(t) = \frac{1}{f'(f^{-1}(t))}.$$

1.6 The Mean Value Theorem and some applications

Theorem 1.20 (Rolle)**.** *If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b), and $f(a) = f(b) = 0$, then there is some point c in (a,b) so that $f'(c) = 0$.*

Proof. From the Extreme Value Existence Theorem (Theorem 4.1 in Section 4.1, page 243), $f(x)$ has a maximum value $f(c_1)$ and a minimum value $f(c_2)$ on $[a,b]$. If one of c_1 or c_2 is not an endpoint (a or b), then c_1 or c_2 is a critical point (Fermat's Theorem, Theorem 4.2 in Section 4.1, page 244), and since f' exists at every point of (a,b), either $f'(c_1) = 0$ or $f'(c_2) = 0$, which establishes the claim. The other possibility is that the maximum value occurs at an endpoint, and so does the minimum value, so $f(c_1) = f(c_2) = f(a) = f(b) = 0$. By definition of max and min, $0 = f(c_2) \leq f(x) \leq f(c_1) = 0$ for all x in $[a,b]$, so $f(x)$ is the constant function 0, with $f'(x) = 0$ for all x in (a,b). ∎

Theorem 1.21. *If $\lim\limits_{x\to c^+} f(x) = 0$ and f is differentiable on (c,b), with $f'(x) \neq 0$ for all x in (c,b), then $f(x) \neq 0$ for all x in (c,b).*

Proof. Suppose, toward a contradiction, that there is some point p in (c, b) where $f(p) = 0$. Then, define a new function $h(x)$ on the interval $[c, p]$, by $h(c) = 0$ and $h(x) = f(x)$ for $c < x \leq p$. $h(x)$ is continuous on $[c, p]$, because f is continuous at every point of $(c, p]$, and $\lim_{x \to c^+} h(x) = \lim_{x \to c^+} f(x) = 0 = h(c)$. Also, $h(x)$ is differentiable on (c, p), with $h'(x) = f'(x)$, so Rolle's Theorem applies, and there is some d in (c, p) with $h'(d) = f'(d) = 0$. However, this contradicts the hypothesis that $f'(x) \neq 0$ for all x in (c, b); the conclusion is that there is no such point p with $f(p) = 0$. ∎

Theorem 1.22 (Cauchy's Mean Value Theorem)**.** *If $f(x)$ and $g(x)$ are both continuous on $[a, b]$ and differentiable on (a, b), then there is some point c in (a, b) so that $f'(c) \cdot (g(b) - g(a)) = g'(c) \cdot (f(b) - f(a))$.*

Proof. Define a new function $h(x)$ on $[a, b]$ by the formula

$$h(x) = f(x) \cdot (g(b) - g(a)) - g(x) \cdot (f(b) - f(a)) - f(a) \cdot g(b) + g(a) \cdot f(b).$$

$h(x)$ is continuous on $[a, b]$, and differentiable on (a, b), with $h'(x) = f'(x) \cdot (g(b) - g(a)) - g'(x) \cdot (f(b) - f(a))$. It is straightforward to check that $h(a) = h(b) = 0$, so Rolle's Theorem applies, and there is some c in (a, b) so that $h'(c) = f'(c) \cdot (g(b) - g(a)) - g'(c) \cdot (f(b) - f(a)) = 0$, which proves the claim. ∎

When $g(x)$ is just the function $g(x) = x$, this Theorem says $f'(c) \cdot (b - a) = 1 \cdot (f(b) - f(a))$, which implies the usual form of the Mean Value Theorem:

Theorem 1.23 (Mean Value Theorem)**.** *If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b), then there is some point c in (a, b) so that $f'(c) = \frac{f(b)-f(a)}{b-a}$.* ∎

Theorem 1.24. *If $f'(x) \geq 0$ for all x in (c, b), then f is weakly increasing on (c, b).*

Proof. First, since f is differentiable on (c, b), f is continuous at every point in (c, b). Given any two points t and x in (c, b) with $t < x$, f is continuous on the closed interval $[t, x]$. The Mean Value Theorem applies to f on $[t, x]$, giving some point d in (t, x) with $f'(d) = \frac{f(x)-f(t)}{x-t}$, so $f(x) - f(t) = f'(d) \cdot (x - t)$. Since $x - t > 0$ and $f'(d) \geq 0$, we get $f(x) - f(t) \geq 0$ for any t and x with $c < t < x < b$, which implies $f(t) \leq f(x)$, the definition of weakly increasing on (c, b). ∎

If we assume the derivative is positive (never 0), a similar argument would show that f is increasing ($f(t) < f(x)$ for $c < t < x < b$), not just weakly increasing.

Proposition 1.25. *If $f'(x) > 0$ for all x in (c, b), then f is increasing on (c, b).* ∎

If we include just one endpoint in the domain and assume that f is continuous from one side at that endpoint, we can draw a similar conclusion:

Theorem 1.26. *If $\lim_{x \to c^+} f(x) = f(c)$ and $f'(x) \geq 0$ for all x in (c, b), then f is weakly increasing on $[c, b)$ and $f(c)$ is the minimum value of f on $[c, b)$.*

Proof. By Theorem 1.24, f is weakly increasing on (c, b). Then Proposition 1.11 applies, to show f is weakly increasing on $[c, b)$. By definition of weakly increasing, this means $f(c) \leq f(x)$ for all x in $[c, b)$. ∎

Again, if we assume the derivative is positive, a similar argument would show that f is increasing, not just weakly increasing.

Proposition 1.27. *If* $\lim_{x\to c^+} f(x) = f(c)$ *and* $f'(x) > 0$ *for all* x *in* (c,b)*, then* f *is increasing on* $[c,b)$ *and* $f(c)$ *is the minimum value of* f *on* $[c,b)$. ∎

The interval in Theorem 1.26 could be changed to include b, or to be a ray. The proofs would not require any major changes, and the following Propositions could also be modified to conclude that f is increasing when the derivative is positive.

Proposition 1.28. *If* $\lim_{x\to c^+} f(x) = f(c)$ *and* $f'(x) \geq 0$ *for all* x *in* (c,∞)*, then* f *is weakly increasing on* $[c,\infty)$ *and* $f(c)$ *is the minimum value of* f *on* $[c,\infty)$. ∎

Proposition 1.29. *If* $\lim_{x\to c^+} f(x) = f(c)$ *and* $\lim_{x\to b^-} f(x) = f(b)$ *and* $f'(x) \geq 0$ *for all* x *in* (c,b)*, then* f *is weakly increasing on* $[c,b]$ *and* $f(c)$ *is the minimum value of* f *on* $[c,b]$. ∎

We could also change Theorem 1.26 to refer to a maximum value, or to let c be the right endpoint instead of the left. Each of the following Propositions has a proof similar to the proof of Theorem 1.26, and each could be modified to apply to closed intervals or rays, and to conclude increasing or decreasing when the derivative is strictly positive or negative.

Proposition 1.30. *If* $\lim_{x\to c^+} f(x) = f(c)$ *and* $f'(x) \leq 0$ *for all* x *in* (c,b)*, then* f *is weakly decreasing on* $[c,b)$ *and* $f(c)$ *is the maximum value of* f *on* $[c,b)$. ∎

Proposition 1.31. *If* $\lim_{x\to c^-} f(x) = f(c)$ *and* $f'(x) \geq 0$ *for all* x *in* (a,c)*, then* f *is weakly increasing on* $(a,c]$ *and* $f(c)$ *is the maximum value of* f *on* $(a,c]$. ∎

Proposition 1.32. *If* $\lim_{x\to c^-} f(x) = f(c)$ *and* $f'(x) \leq 0$ *for all* x *in* (a,c)*, then* f *is weakly decreasing on* $(a,c]$ *and* $f(c)$ *is the minimum value of* f *on* $(a,c]$. ∎

Definition 1.33. A function $f(x)$ has a local minimum value at c means: there is some $\delta > 0$ so that if x is in the domain of f and $|x - c| < \delta$, then $f(x) \geq f(c)$.

Definition 1.34. A function $f(x)$ has a local maximum value at c means: there is some $\delta > 0$ so that if x is in the domain of f and $|x - c| < \delta$ then $f(x) \leq f(c)$.

Definition 1.35. A function $f(x)$ has an interior local minimum value at c means: there is some $\delta > 0$ so that if $|x - c| < \delta$, then x is in the domain of f and $f(x) \geq f(c)$.

Definition 1.36. A function $f(x)$ has an interior local maximum value at c means: there is some $\delta > 0$ so that if $|x - c| < \delta$, then x is in the domain of f and $f(x) \leq f(c)$.

Note that any $f(c)$ satisfying Definition 1.35 must also satisfy Definition 1.33, so any interior local minimum value is also a local minimum value (and similarly for local maximum). These definitions are a little different from the definitions appearing on page 244 in Section 4.1, but the only practical difference in the calculus problems that we'll see is that Definition 1.33 allows for a local minimum to occur at an endpoint when the domain of f is a closed interval. This way, every absolute (or "global") minimum value (as on page 241 in Section 4.1) is also a local minimum value, even if it occurs at an endpoint.

Theorem 1.37. *If $f(x)$ is continuous at c and there is some $\delta_2 > 0$ so that $f'(x) \geq 0$ for $c < x < c + \delta_2$ and $f'(x) \leq 0$ for $c - \delta_2 < x < c$, then f has an interior local minimum value at c.*

Proof. The continuity of f at c implies $\lim_{x \to c^+} f(x) = f(c)$, so Theorem 1.26 applies, to show that $f(c)$ is the minimum value of f on $[c, c + \delta_2)$. Similarly using Proposition 1.32, $f(c)$ is the minimum value of f on $(c - \delta_2, c]$, and this is enough to show $f(c)$ is a local minimum value. ∎

Proposition 1.38. *If $f(x)$ is continuous at c and there is some $\delta_2 > 0$ so that $f'(x) \leq 0$ for $c < x < c + \delta_2$ and $f'(x) \geq 0$ for $c - \delta_2 < x < c$, then f has an interior local maximum value at c.* ∎

Note that the above Theorem and Proposition (together, called the "First Derivative Test") do not require that f is differentiable at c, only near c on both sides.

Theorem 1.39. *If $f'(c) = 0$ and $f''(c) > 0$, then f has an interior local minimum value at c.*

Proof. From the definition of derivative,

$$\lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} = f''(c) > 0.$$

We can conclude from Theorem 1.4 that there is some $\delta_1 > 0$ so that $\frac{f'(x) - f'(c)}{x - c} > 0$ for all x such that $0 < |x - c| < \delta_1$. If $c < x < c + \delta_1$, then $x - c > 0$, and so $f'(x) - f'(c) = f'(x) - 0 > 0$. If $c - \delta_1 < x < c$, then $x - c < 0$, and so $f'(x) < 0$. Since $f(x)$ is continuous at c (because it is differentiable at c), Theorem 1.37 applies to show that $f(c)$ is a local minimum value. ∎

Proposition 1.40. *If $f'(c) = 0$ and $f''(c) < 0$, then f has an interior local maximum value at c.* ∎

The above Theorem and Proposition are together called the "Second Derivative Test" as on page 265 in Section 4.3, although it is not required that f'' exists for x near c, only at c (so the textbook is making an unnecessary extra assumption).

Theorem 1.41. *If $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to a^+} f'(x) = L$, then $\lim_{x \to a^+} \dfrac{f(x) - f(a)}{x - a} = L$.*

Proof. The hypothesis $\lim_{x \to a^+} f'(x) = L$ means that $f'(x)$ exists for x in some interval $(a, a + \delta_1)$, and that for any $\epsilon > 0$ there is some δ, $0 < \delta < \delta_1$, so that if $0 < x - a < \delta$, then $|f'(x) - L| < \epsilon$. So, for any x such that $0 < x - a < \delta$, $f(x)$ satisfies the hypothesis of the Mean Value Theorem on the interval $[a, x]$ (it is differentiable as just mentioned, and continuous on the closed interval by the first hypothesis). There exists some c in (a, x) so that $f'(c) = \frac{f(x) - f(a)}{x - a}$. Since c is also in the interval $(a, a + \delta)$, $|f'(c) - L| < \epsilon$. The conclusion is that for any x such that $0 < x - a < \delta$, $\left| \frac{f(x) - f(a)}{x - a} - L \right| = |f'(c) - L| < \epsilon$, which is what we wanted to show. ∎

There is an analogous result for left-side limits. However, the easy example $f(x) = |x|$ shows that f can be continuous, and f' can have a left-side and a right-side limit at $a = 0$, without f being differentiable at a. If we make the additional assumption that f is differentiable at a, then the following result shows that $f'(a)$ is equal to the limit of f'.

Theorem 1.42. *If $f(x)$ is differentiable at a and $\lim_{x\to a^+} f'(x) = L$, then $L = f'(a)$.*

Proof. The function f satisfies the conditions of Theorem 1.41, so $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = L$. Since f is differentiable at a, its (two-side) derivative is equal to this right-side limit. ∎

There are analogous versions of the Theorem for the $x \to a^-$ and $x \to a$ limits.

Theorem 1.43. *If $\lim_{x\to a^+} f'(x) = +\infty$, then f is not differentiable at a.*

Proof. Suppose, toward a contradiction, that f is differentiable at a, with $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$. By definition of limit, there is some $\delta_1 > 0$ so that for all x in $(a, a+\delta_1)$, $\left|\frac{f(x)-f(a)}{x-a} - f'(a)\right| < 1$. By the definition of the $\lim = +\infty$ hypothesis, there is some $\delta_2 > 0$ so that for all x in $(a, a+\delta_2)$, $f'(x) > f'(a) + 1$. Let x_0 be a particular point in the interval $(a, a + \min\{\delta_1, \delta_2\})$, so f is differentiable on (a, x_0), and continuous on $[a, x_0]$. By the Mean Value Theorem, there is some point c in (a, x_0) so that $f'(c) = \frac{f(x_0)-f(a)}{x-a}$. Since $a < c < x_0 < a + \delta_2$, $f'(c) > f'(a) + 1$, which implies $f'(c) - f'(a) > 1 \implies |f'(c) - f'(a)| > 1$. However, since $a < x_0 < a + \delta_1$,

$$|f'(c) - f'(a)| = \left|\frac{f(x_0)-f(a)}{x-a} - f'(a)\right| < 1.$$

This contradiction ($|f'(c) - f'(a)|$ can't be < 1 and > 1) shows f isn't differentiable at a. ∎

Example 1.44. A function can be differentiable at a while f' does not have a $x \to a$ limit, either L or $+\infty$, as in the previous two Theorems. For example, let

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

This function is differentiable on $(-\infty, \infty)$, with

$$f'(x) = \begin{cases} 2x \sin(\frac{1}{x^2}) + \frac{-2}{x}\cos(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

This derivative is not bounded in any neighborhood of 0, and $\lim_{x\to 0} f'(x)$ DNE.

Theorem 1.45. *If the left-side limit $\lim_{t\to b^-} f(t) = -\infty$, then there is no interval $(b-\delta, b)$ on which $f'(t)$ is bounded below.*

Proof. Suppose toward a contradiction that there is some $K < 0$ and some interval $(b-\delta, b)$ so that $f'(t) \geq K$ for $b - \delta < t < b$. By the definition of $\lim = -\infty$, there is some δ_2 with $0 < \delta_2 < \delta/2$ so that for all $t \in (b-\delta_2, b)$, $f(t) < f(b-\delta/2) + K\delta/2$. For any $t_1 \in (b-\delta_2, b)$, applying the MVT to f on $[b-\delta/2, t_1]$ gives some t_2 with $b - \delta < t_2 < b$ and, using $K < 0$,

$$f'(t_2) = \frac{f(t_1) - f(b-\delta/2)}{t_1 - (b-\delta/2)} < \frac{f(b-\delta/2) + K\delta/2 - f(b-\delta/2)}{t_1 - (b-\delta/2)} < \frac{K\delta/2}{\delta/2} = K,$$

contradicting $f'(t_2) \geq K$. ∎

Theorem 1.46 (The Inverse Function Theorem)**.** *Given a function $f(x)$ and a point (d, c) on the graph of f, suppose f is differentiable at d and* *__either__:*

1. *There is some interval (a, b) with $a < d < b$ where $f(x)$ is continuous and one-to-one on (a, b);*

   __or__,

2. *The limit $\lim_{x \to d} f'(x)$ exists.*

If $f'(d) \neq 0$, then f^{-1} is defined on some interval $(c - \delta, c + \delta)$, and f^{-1} is differentiable at c with $(f^{-1})'(c) = \frac{1}{f'(d)}$.

Proof. In Case 1., the statement is copied from Theorem 1.19. In Case 2., the differentiability of f and the existence of the limit imply that $\lim_{x \to d} f'(x) = f'(d)$ by the two-sided version of Theorem 1.42, which used the Mean Value Theorem. So, f' is continuous at d, and corresponding to the positive number $\epsilon = \frac{1}{2}|f'(d)|$, there is some $\delta_1 > 0$ so that if $|x - d| < \delta_1$, then $|f'(x) - f'(d)| < \frac{1}{2}|f'(d)|$. This expands to two inequalities:

$$-\frac{1}{2}|f'(d)| < f'(x) - f'(d) < \frac{1}{2}|f'(d)|.$$

If $f'(d) > 0$, the left inequality leads to $\frac{1}{2}f'(d) < f'(x)$, so $f'(x) > 0$ for $d - \delta_1 < x < d + \delta_1$, and by Proposition 1.25, which also used the Mean Value Theorem, f is increasing on $(d - \delta_1, d + \delta_1)$. Similarly, if $f'(d) < 0$, the right inequality leads to $f'(x) < \frac{1}{2}f'(d)$, so $f'(x)$ is negative and f is decreasing on $(d - \delta_1, d + \delta_1)$. Either way, f is one-to-one on $(d - \delta_1, d + \delta_1)$. The existence of f' on $(d - \delta_1, d + \delta_1)$ also implies f is continuous at every point on that interval, and this is enough to satisfy the assumptions of Theorem 1.19. ∎

Example 1.47. Let $g(x) = x + f(x)$, where $f(x)$ is from Example 1.44, so

$$g(x) = \begin{cases} x + x^2 \sin(\frac{1}{x^2}) & \text{if } x \neq 0 \\ 0 + 0 = 0 & \text{if } x = 0 \end{cases}.$$

This function g is differentiable on $(-\infty, \infty)$, with $g'(x) = 1 + f'(x)$ and $g'(0) = 1 + f'(0) = 1$. However, g is neither increasing nor one-to-one on any interval $(0 - \delta, 0 + \delta)$. Like Example 1.44, $\lim_{x \to 0} g'(x)$ DNE, and Theorem 1.46 does not apply; g does not have an inverse function near $(0, 0)$.

Theorem 1.48. *If f is differentiable at a and $f'(x)$ is weakly increasing on (a, b), then $f'(x) \geq f'(a)$ for all x in (a, b).*

Proof. Suppose, toward a contradiction, that there is some point x_0 in (a, b) so that $f'(x_0) < f'(a)$. Corresponding to the positive number $f'(a) - f'(x_0)$ used as an output tolerance for the definition of $\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = f'(a)$, there is some $\delta_1 > 0$ so that for all x in $(a, a + \delta_1)$,

$$\left| \frac{f(x) - f(a)}{x - a} - f'(a) \right| < f'(a) - f'(x_0).$$

Let $\delta_2 = \min\{\delta_1, x_0 - a\} > 0$, and pick a specific x_1 in $(a, a+\delta_2)$. Then, since $a < x_1 < a + \delta_2 \leq a + \delta_1$, $\left|\frac{f(x_1) - f(a)}{x_1 - a} - f'(a)\right| < f'(a) - f'(x_0)$. The Mean Value Theorem applies to $f(x)$ on $[a, x_1]$ (since f is differentiable on (a, x_1) and continuous at a and x_1), so there is some x_2 in (a, x_1) so that $f'(x_2) = \frac{f(x_1) - f(a)}{x_1 - a}$. It follows that $|f'(x_2) - f'(a)| < f'(a) - f'(x_0)$. Since $a < x_2 < a + \delta_2 \leq a + (x_0 - a) = x_0 < b$, and f' is weakly increasing on (a, b), $f'(x_2) \leq f'(x_0)$, and since $f'(x_0) < f'(a)$, $f'(x_2) < f'(a)$. It follows that $|f'(x_2) - f'(a)| = f'(a) - f'(x_2) \geq f'(a) - f'(x_0)$. However, this is a contradiction, so we can conclude there is no such point x_0. ∎

The conclusion is that f' is weakly increasing on $[a, b)$. It can similarly be shown that if $f'(a)$ exists and f' is weakly decreasing on (a, b), then f' is weakly decreasing on $[a, b)$, or that if $f'(b)$ exists and f' is weakly decreasing on (a, b), then f' is weakly decreasing on $(a, b]$. There are analogous results for increasing and decreasing functions, for example:

Corollary 1.49. *If f is differentiable at a and $f'(x)$ is increasing on (a, b), then $f'(x) > f'(a)$ for all x in (a, b).*

Proof. f' is also weakly increasing, so by the previous Theorem, $f'(x) \geq f'(a)$ for all x in (a, b). It remains only to show that $f'(x) > f'(a)$. Suppose, toward a contradiction, that there is some x_0 in (a, b) where $f'(x_0) = f'(a)$. Then, there is some x_1 in (a, x_0) where $f'(x_1) < f'(x_0)$, because f' is increasing. However, $f'(x_1) < f'(x_0) = f'(a)$ contradicts $f'(x_1) \geq f'(a)$. ∎

Theorem 1.50. *If $p(x)$ satisfies $p''(x) \geq 0$ on (a, b) then for any $c \in (a, b)$, p satisfies $p(x) \geq p(c) + p'(c)(x - c)$ for all $x \in (a, b)$.*

Proof. Let $q(x) = p(x) - (p(c) + p'(c)(x - c))$; we want to show $q(x) \geq 0$. By the rules for derivatives, $q'(x) = p'(x) - p'(c)$ and $q''(x) = p''(x) \geq 0$ on (a, b). By Theorem 1.24, q' is weakly increasing on (a, b), so $q'(x) \geq q'(c) = 0$ for $x \in (c, b)$ and by Theorem 1.24 again, q is weakly increasing on (c, b). Similarly, for $x \in (a, c)$, $q'(x) \leq q'(c) = 0$, so q is weakly decreasing on (a, c). Because q is continuous at c, Proposition 1.13 applies: $q(x) \geq q(c) = 0$ for all $x \in (a, b)$. ∎

Definition 1.51. Given a function $f(x)$ with domain D and any subset $S \subseteq D$, $f(x)$ is <u>concave up</u> on S means: $f'(x)$ is increasing on S. Similarly, $f(x)$ is <u>concave down</u> on S means: $f'(x)$ is decreasing on S.

The above Definition appears on page 262 in Section 4.3, although it may be different from definitions appearing in some other textbooks.

Definition 1.52. A function $f(x)$ has an <u>inflection point</u> at c means: f is continuous at c, and there is some $\delta > 0$ so that one of the following two cases holds: (1.) $f(x)$ is concave up on $(c - \delta, c)$ and $f(x)$ is concave down on $(c, c + \delta)$, or (2.) $f(x)$ is concave down on $(c - \delta, c)$ and $f(x)$ is concave up on $(c, c + \delta)$.

The Definition does not require that f is differentiable at c, only near c on both sides. This excludes endpoints of closed intervals.

Theorem 1.53. *If c is an inflection point of f and f is differentiable at c then $f'(x)$ has either an interior local max at c or an interior local min at c.*

Proof. From case (2.) of the definition of inflection point, suppose $f'(x)$ is increasing on some interval $(c, c+\delta)$. Then, since f is differentiable at c, Corollary 1.49 applies, and $f'(x) > f'(c)$ for all x in $(c, c+\delta)$, so $f'(c)$ is the minimum value on $[c, c+\delta)$. By an analogous result for functions with a decreasing derivative on $(c-\delta, c)$, $f'(x) > f'(c)$ for x in $(c-\delta, c)$, so $f'(c)$ is the minimum value on $(c-\delta, c+\delta)$ Case (1.) is similar, where f' has a local max value. ∎

Example 1.54. The function

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x - x^3 & \text{if } x \geq 0 \end{cases}$$

is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. It has an inflection point at the origin, but $f'(x)$ is defined only for $x \neq 0$ and f' does not have any local max or min value.

Theorem 1.55. *If c is an inflection point of f, and $f''(c)$ exists, then $f''(c) = 0$.*

Proof. Since $f''(c)$ exists, f' is differentiable at c, and in particular exists at c, so the previous Theorem applies, and f' has a local max or local min at c. By Fermat's Theorem on local max/min, c is a critical point of f' at a point where f' is differentiable, so $f''(c) = 0$. ∎

This means solutions of $f'' = 0$ are candidates for inflection points, but not every such point is an inflection point. Points in the domain where f'' does not exist are also candidates for inflection points.

Example 1.56. The function $f(x) = x^4$ is concave up on $(-\infty, \infty)$ because its derivative, $f'(x) = 4x^3$, is increasing on $(-\infty, \infty)$. So, x^4 has no inflection points. However, $f''(0) = 0$.

Example 1.57. The function

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ -x^4 & \text{if } x \geq 0 \end{cases}$$

is differentiable on $(-\infty, \infty)$, concave up on $(-\infty, 0]$, and concave down on $[0, \infty)$, where

$$f'(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ -4x^3 & \text{if } x \geq 0 \end{cases} .$$

f has an inflection point at 0, and $f'(x)$ has a local max value at 0, but $f''(0)$ DNE.

Theorem 1.58. *Given $a < b$, suppose:*

- $\lim_{x \to a^+} f(x) = f(a)$*;*
- *there is some $x_0 \in (a, b)$ so that $f(x_0) > f(a)$;*
- *the limit exists: $\lim_{x \to a^+} f'(x) = L$;*
- *$f''(x) \leq 0$ for all $x \in (a, b)$.*

Then $L > 0$.

Proof. f' is differentiable, and therefore continuous on (a, b), and we can extend f' to a continuous function on $[a, b)$ by defining $g(x) = f'(x)$ for $a < x < b$ and $g(a) = L$. The Mean Value Theorem applies to g on $[a, x]$ for x in (a, b): $g(x) - g(a) = g'(c)(x - a)$ for some $a < c < x$, so $f'(x) - L = f''(c)(x - a) \leq 0$. Suppose, toward a contradiction, that $L \leq 0$. Then $f'(x) \leq L \leq 0$, for any x. By Proposition 1.30, f is weakly decreasing on $[a, b)$, which contradicts the assumption that $f(x_0) > f(a)$. ∎

Theorem 1.59. *Given $a < b$, suppose:*

- $\lim_{x \to a^+} f(x) = f(a)$*;*
- *there are constants $C > 0$ and $k < 2$ such that $f(x) \geq f(a) + C(x - a)^k$ for all x in (a, b);*
- *the limit $\lim_{x \to a^+} f'(x) = L$ exists;*
- *f'' is bounded above on (a, b).*

Then $L > 0$.

Proof. For any c in (a, b), f is continuous on $[a, c]$ and differentiable on (a, c), so by the Mean Value Theorem, there is some y in (a, c) so that

$$f'(y) = \frac{f(c) - f(a)}{c - a} \geq \frac{(f(a) + C(c - a)^k) - f(a)}{c - a} = C(c - a)^{k-1}.$$

We can extend f' to a continuous function on $[a, b)$ by defining $g(x) = f'(x)$ for $a < x < b$ and $g(a) = L$. The Mean Value Theorem applies to g on $[a, y]$: there is some ℓ in (a, y) so that $g'(\ell) = \dfrac{g(y) - g(a)}{y - a} \implies f''(\ell) = \dfrac{f'(y) - L}{y - a}$. Suppose, toward a contradiction, that $L \leq 0$. Then,

$$f''(\ell) = \frac{f'(y) - L}{y - a} > \frac{C(c - a)^{k-1} - 0}{c - a} = C(c - a)^{k-2}.$$

However, since $C > 0$ and $k < 2$, $\lim_{x \to a^+} C(x - a)^{k-2} = +\infty$, so for any M, there are some c, ℓ with $a < \ell < c < b$ and $M < C(c - a)^{k-2} < f''(\ell)$, contradicting the hypothesis that f'' has an upper bound on (a, b). ∎

In the previous two Theorems, it follows from Theorem 1.41 that L is equal to the one-side derivative at a, and from Theorem 1.42 that if f is differentiable at a then $L = \lim_{x \to a^+} f'(x) = f'(a)$.

Theorem 1.60 (L'Hôpital's Rule)**.** *If* $\lim_{x \to c^+} f(x) = 0$ *and* $\lim_{x \to c^+} g(x) = 0$ *and* $\lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L$*, then* $\lim_{x \to c^+} \frac{f(x)}{g(x)} = L$*.*

Proof. We want to show that given $\epsilon > 0$, there is some $\delta > 0$ so that if $0 < x - c < \delta$, then $|\frac{f(x)}{g(x)} - L| < \epsilon$.

From the hypothesis $\lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L$, we can use the positive number $\epsilon/3$ to get $\delta_1 > 0$ so that if $0 < x - c < \delta_1$, then $|\frac{f'(x)}{g'(x)} - L| < \epsilon/3$.

Since the function $|x|$ is continuous, the Composite Limit Theorem implies $\lim_{x \to c^+} \left|\frac{f'(x)}{g'(x)}\right| = |L|$, and we can use 1 as an output tolerance to get $\delta_2 > 0$ so that if $0 < x - c < \delta_2$, then $\left|\left|\frac{f'(x)}{g'(x)}\right| - |L|\right| < 1$, which implies $\left|\frac{f'(x)}{g'(x)}\right| < |L| + 1$.

So, let $\delta = \min\{\delta_1, \delta_2\} > 0$, and then both of the above inequalities hold for any x in $(c, c + \delta)$. In particular, from $|\frac{f'(x)}{g'(x)} - L| < \epsilon/3$ we can conclude that $f(x)$, $f'(x)$, and $g(x)$ are defined for every x in $(c, c + \delta)$, and that $g'(x)$ is defined and non-zero for every x in $(c, c + \delta)$. Since $\lim_{x \to c^+} g(x) = 0$, Theorem 1.21 applies, to show that $g(x) \neq 0$ for all x in $(c, c + \delta)$. This means $\frac{f(x)}{g(x)}$ is defined for every x in $(c, c+\delta)$, and the claim is that this δ is the one we're trying to get; it remains only to show that $|\frac{f(x)}{g(x)} - L| < \epsilon$ for every x in $(c, c + \delta)$.

Given any x in $(c, c + \delta)$, and any number t between c and x, the functions f and g are continuous at t and x, and at every point in between, because they are differentiable on $(c, c+\delta)$. So, Cauchy's Mean Value Theorem applies to $f(x)$ and $g(x)$ on the interval $[t, x]$: there is some point, depending on t and x, so we call it $d_{t,x}$, so that $f'(d_{t,x}) \cdot (g(x) - g(t)) = g'(d_{t,x}) \cdot (f(x) - f(t))$. Since g' is non-zero on $(c, c + \delta)$, we can divide to get

$$f(x) = f(t) + \frac{f'(d_{t,x})}{g'(d_{t,x})} \cdot (g(x) - g(t)).$$

Since this formula works for any x and t with $c < t < d_{t,x} < x < c + \delta$, we take x to be fixed, so that $g(x) \neq 0$ (as previously mentioned), and we can choose t to be close to c.

From the hypothesis $\lim_{t \to c^+} f(t) = 0$, we can use the positive number $\epsilon \cdot |g(x)|/3$ as an output tolerance to get $\delta_3 > 0$ so that if $0 < t - c < \delta_3$, then $|f(t) - 0| < \epsilon \cdot |g(x)|/3$.

From the hypothesis $\lim_{t \to c^+} g(t) = 0$, we can use the positive number $\epsilon \cdot |g(x)|/(3 \cdot (|L| + 1))$ as an output tolerance to get $\delta_4 > 0$ so that if $0 < t - c < \delta_4$, then $|g(t) - 0| < \epsilon \cdot |g(x)|/(3 \cdot (|L| + 1))$.

For t such that $0 < t - c < \min\{\delta_3, \delta_4, x - c\}$, the inequalities for both $f(t)$ and $g(t)$ hold, as well as the estimates on f'/g', at $d_{t,x}$, which is in $(c, c + \delta)$ for any t and x. Combining these with the Triangle Inequality gives the result:

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &= \left|\frac{f(t) + \frac{f'(d_{t,x})}{g'(d_{t,x})}\cdot(g(x)-g(t))}{g(x)} - L\right| \\
&= \left|\frac{f(t)}{g(x)} - \frac{f'(d_{t,x})}{g'(d_{t,x})}\cdot\frac{g(t)}{g(x)} + \frac{f'(d_{t,x})}{g'(d_{t,x})} - L\right| \\
&\leq \left|\frac{f(t)}{g(x)}\right| + \left|-\frac{f'(d_{t,x})}{g'(d_{t,x})}\cdot\frac{g(t)}{g(x)}\right| + \left|\frac{f'(d_{t,x})}{g'(d_{t,x})} - L\right| \\
&= \frac{|f(t)|}{|g(x)|} + \left|\frac{f'(d_{t,x})}{g'(d_{t,x})}\right|\cdot\frac{|g(t)|}{|g(x)|} + \left|\frac{f'(d_{t,x})}{g'(d_{t,x})} - L\right| \\
&< \frac{\epsilon\cdot|g(x)|/3}{|g(x)|} + (|L|+1)\cdot\frac{\epsilon\cdot|g(x)|/(3\cdot(|L|+1))}{|g(x)|} + \epsilon/3 \\
&= \epsilon.
\end{aligned}$$

∎

The L'Hôpital rule for $\frac{0}{0}$ limits also holds when the $x \to c^+$ limit is replaced by $x \to c$, $x \to c^-$, $x \to \infty$, or $x \to -\infty$. There are also versions for limits of the form $\frac{\infty}{\infty}$.

The following Theorem applies to any limit of the form $\lim\limits_{x\to+\infty} = \frac{?}{\infty}$, where we are not assuming anything about the limit of the numerator:

Theorem 1.61. *Given $g(x)$ and $h(x)$ differentiable on (a, ∞), if $\lim\limits_{x\to\infty} g(x) = +\infty$ and $\lim\limits_{x\to\infty} \frac{h'(x)}{g'(x)} = L$, then $\lim\limits_{x\to\infty} \frac{h(x)}{g(x)} = L$.*

Proof. Let $\epsilon > 0$. From the hypothesis that $\lim\limits_{x\to\infty}\frac{h'(x)}{g'(x)} = L$, there is some $N_1 > a$ so that $\left|\frac{h'(x)}{g'(x)} - L\right| < \frac{2\epsilon}{9}$ for $x > N_1$ (in particular, $g'(x) \neq 0$ for $x > N_1$).

Corresponding to any $y > N_1$, from the hypothesis that $\lim\limits_{x\to\infty} g(x) = +\infty$, there is some $N \geq y$ so that if $x > N$, then

$$g(x) > \max\left\{2|g(y)|, \frac{3|h(y)|}{\epsilon}, \frac{3|L|\cdot|g(y)|}{\epsilon}\right\} \geq 0.$$

For such an x with $x > N$, $\left|\frac{g(y)}{g(x)}\right| < \frac{1}{2}$, $\left|\frac{h(y)}{g(x)}\right| < \frac{\epsilon}{3}$, and $\left|L\cdot\frac{g(y)}{g(x)}\right| < \frac{\epsilon}{3}$.

By Cauchy's Mean Value Theorem, there is some c, $y < c < x$, so that

$$g'(c)(h(x) - h(y)) = h'(c)(g(x) - g(y)),$$

and dividing by $g'(c)g(x) \neq 0$,

$$\frac{h(x)}{g(x)} - \frac{h(y)}{g(x)} = \frac{h'(c)}{g'(c)}\left(1 - \frac{g(y)}{g(x)}\right).$$

Using the above bounds and the triangle inequality completes the proof:

$$\begin{aligned}
\left|\frac{h(x)}{g(x)} - L\right| &= \left|\frac{h'(c)}{g'(c)}\left(1 - \frac{g(y)}{g(x)}\right) - L + \frac{h(y)}{g(x)}\right| \\
&= \left|\left(\frac{h'(c)}{g'(c)} - L\right)\left(1 - \frac{g(y)}{g(x)}\right) - L \cdot \frac{g(y)}{g(x)} + \frac{h(y)}{g(x)}\right| \\
&\leq \left|\frac{h'(c)}{g'(c)} - L\right|\left(1 + \left|\frac{g(y)}{g(x)}\right|\right) + \left|L \cdot \frac{g(y)}{g(x)}\right| + \left|\frac{h(y)}{g(x)}\right| \\
&< \frac{2\epsilon}{9} \cdot \frac{3}{2} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

∎

The remark in the above proof that g' must be non-zero at every point in some interval (N_1, ∞) is important, as shown by the following Example.

Example 1.62. Consider the following functions.

$$\begin{aligned}
f(x) &= \frac{1}{2}(x + \sin(x)\cos(x)) \\
f'(x) &= \cos^2(x) \\
g(x) &= \frac{1}{2}e^{\sin(x)}(x + \sin(x)\cos(x)) \\
g'(x) &= e^{\sin(x)}\cos^2(x) + \frac{1}{2}e^{\sin(x)}\cos(x)(x + \sin(x)\cos(x)) \\
&= \frac{1}{2}e^{\sin(x)}\cos(x)(2\cos(x) + x + \sin(x)\cos(x)).
\end{aligned}$$

By construction, f and g are non-zero (strictly positive) for $x > 0$, and $\lim_{x\to+\infty} g(x) = +\infty$.

The fraction:

$$\frac{f'(x)}{g'(x)} = \frac{\cos^2(x)}{\frac{1}{2}e^{\sin(x)}\cos(x)\left(2\cos(x) + x + \sin(x)\cos(x)\right)}$$

is undefined $(\frac{0}{0})$ at every point where $\cos(x) = 0$. So, the $x \to +\infty$ limit is undefined and the above version of L'Hôpital's Rule (Theorem 1.61) does not apply. However, if we try to cancel a $\cos(x)$ from the numerator and denominator, there is a function that equals f'/g' for all x with $\cos(x) \neq 0$:

$$q(x) = \frac{\cos(x)}{\frac{1}{2}e^{\sin(x)}\left(2\cos(x) + x + \sin(x)\cos(x)\right)},$$

and $\lim_{x\to+\infty} q(x) = 0$, since the numerator is bounded and the denominator $\to +\infty$.

We cannot conclude that $\lim_{x\to+\infty} f(x)/g(x)$ has the same limit; in fact, $\frac{f(x)}{g(x)} = \frac{1}{e^{\sin(x)}}$ for all $x > 0$, but $\lim_{x\to+\infty} f(x)/g(x)$ DNE because $e^{-\sin(x)}$ has periodic oscillation.

1.7 Definite integrals

Definition 1.63. Given $a < b$, a function $f(x)$ defined on the closed interval $[a, b]$, a positive integer n, any points $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, and a point x_k^* in each interval $[x_{k-1}, x_k]$ for $k = 1, \ldots, n$, the number

$$\sum_{k=1}^{n} \left(f(x_k^*) \cdot (x_k - x_{k-1})\right)$$

is a <u>Riemann Sum</u> of $f(x)$ on $[a, b]$.

It is convenient to denote $x_k - x_{k-1} = \Delta x_k$, and to call the points x_k^* the "sample points." A Riemann Sum is a way to estimate the "signed area" between a curve $f(x)$ and the interval $[a, b]$ on the x-axis, where there are many possible choices for both the partition points x_k and the sample points x_k^*.

Definition 1.64. A function $f(x)$ is <u>integrable</u> on $[a, b]$ means: there is some number A so that for any $\epsilon > 0$, there is some $\delta > 0$, where any Riemann Sum with $\Delta x_k < \delta$ for $k = 1, \ldots, n$ satisfies:

$$\left| \left(\sum_{k=1}^{n} f(x_k^*) \Delta x_k \right) - A \right| < \epsilon.$$

The number A from the definition is called the "definite integral" of f on $[a, b]$, and is denoted $\int_a^b f(x)dx$. The idea is that A can be approximated by Riemann Sums, and that the limit exists when we get a good approximation ($< \epsilon$) no matter what choices are made for the sample points x_k^* or the partition x_k, as long as the subintervals are narrow enough ($< \delta$).

It is convenient to allow $a = b$ in the definite integral symbol, in which case $\int_a^a f(x)dx = 0$ for any f.

Theorem 1.65. *If $f(x)$ is integrable on $[a, b]$, then $f(x)$ is bounded on $[a, b]$.*

Proof. Recall that a function f is bounded on a domain D means that there is some number M so that $|f(x)| \leq M$ for all x in D.

Assuming f is integrable, use $\epsilon = 1$ from Definition 1.64, to get some $\delta > 0$, so that any Riemann Sum with $\Delta x_k < \delta$ for all $k = 1, \ldots, n$ satisfies

$$\left| \left(\sum_{k=1}^{n} f(x_k^*) \Delta x_k \right) - A \right| < 1.$$

At this point, we pick a particular Riemann Sum satisfying this condition, so the number n and the points x_k and x_k^* are fixed.

The claim that f is bounded by M on $[a, b]$ is clearly equivalent to the statement that f is bounded by some number M_k on each subinterval $[x_{k-1}, x_k]$: if f is bounded by M on $[a, b]$, then it's bounded by M on each subinterval, and conversely, if it's bounded by M_k on subinterval $\#k$, then f is bounded by $\max\{M_1, \ldots, M_n\}$ on $[a, b]$.

So, we suppose, toward a contradiction, that there is (at least) one subinterval where f is not bounded, specifically, there is some number K so that f is not bounded on $[x_{K-1}, x_K]$.

Denote by $\sum_{k\neq K} f(x_k)\Delta x_k$ the sum of just $n-1$ out of n terms from the previously chosen Riemann Sum: all except for the $k=K$ term.

$$\begin{aligned}
\left|\left(\sum_{k\neq K} f(x_k^*)\Delta x_k\right) - A\right| &= \left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - f(x_K^*)\Delta x_K - A\right| \\
&\leq \left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - A\right| + |f(x_K^*)\Delta x_K| \\
&< 1 + |f(x_K^*)|\Delta x_K.
\end{aligned}$$

Since f is not bounded on $[x_{K-1}, x_K]$, we can pick a new sample point, x_K^{**} in $[x_{K-1}, x_K]$, so that $|f(x_K^{**})| > \frac{2}{\Delta x_K} + |f(x_K^*)|$. From Definition 1.64, changing one of the sample points from x_K^* to x_K^{**}, without changing any of the Δx_k, still gives a Riemann sum close to A:

$$\left|\left(\sum_{k\neq K} f(x_k^*)\Delta x_k\right) + f(x_K^{**})\Delta x_K - A\right| < 1.$$

However, this gives:

$$\begin{aligned}
&|f(x_K^{**})|\Delta x_K \\
= &\left|\left(\sum_{k\neq K} f(x_k^*)\Delta x_k\right) + f(x_K^{**})\Delta x_K - A - \left(\sum_{k\neq K} f(x_k^*)\Delta x_k\right) + A\right| \\
\leq &\left|\left(\sum_{k\neq K} f(x_k^*)\Delta x_k\right) + f(x_K^{**})\Delta x_K - A\right| + \left|\left(\sum_{k\neq K} f(x_k^*)\Delta x_k\right) - A\right| \\
< &\ 1 + 1 + |f(x_K^*)|\Delta x_K,
\end{aligned}$$

and dividing by $\Delta x_K > 0$ gives $|f(x_K^{**})| < \frac{2}{\Delta x_K} + |f(x_K^*)|$, which contradicts $|f(x_K^{**})| > \frac{2}{\Delta x_K} + |f(x_K^*)|$. This contradiction shows that the assumption that f is not bounded on $[x_{K-1}, x_K]$ must be false, and therefore f must be bounded on $[a, b]$. ∎

The contrapositive follows immediately: an unbounded function is not integrable. This does not necessarily mean that the "area under the curve" of every unbounded function is $+\infty$ — it only means that the Riemann Sum procedure to calculate area should not be applied to shapes defined by unbounded functions, because the limit constructed in Definition 1.64 fails to exist. Later in the textbook, there will be different limits we can calculate to extend the "area under the curve" idea to include some unbounded shapes (infinitely tall or infinitely wide).

Theorem 1.66. *Given $a < b < c$, and $f(x)$ defined on $[a, c]$, if f is integrable on $[a, b]$, and on $[b, c]$, then f is integrable on $[a, c]$, and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

Proof. Given $\epsilon > 0$, we need to find $\delta > 0$ so that any Riemann Sum of f on $[a,b]$ with $\Delta x_k < \delta$ satisfies

$$\left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \left(\int_a^b f(x)dx + \int_b^c f(x)dx\right)\right| < \epsilon.$$

Using Theorem 1.65, f is bounded by $M_1 > 0$ on $[a,b]$ and bounded by $M_2 > 0$ on $[b,c]$. Let $M = \max\{M_1, M_2\} > 0$. Corresponding to $\epsilon/3 > 0$, there is some $\delta_1 > 0$ so that any Riemann Sum of f on $[a,b]$ with $\Delta x_k < \delta_1$ satisfies

$$\left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \int_a^b f(x)dx\right| < \epsilon/3. \tag{1.1}$$

There is also some $\delta_2 > 0$ so that any Riemann Sum of f on $[b,c]$ with $\Delta x_k < \delta_2$ satisfies

$$\left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \int_b^c f(x)dx\right| < \epsilon/3. \tag{1.2}$$

Choose $\delta = \min\{\delta_1, \delta_2, \frac{\epsilon}{6M}, \frac{b-a}{2}, \frac{c-b}{2}\} > 0$, and consider any partition $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = c$, with $\Delta x_k < \delta$ for $k = 1, \ldots, n$, and any sample points x_k^*.

Note that $\Delta x_1 = x_1 - a < \frac{1}{2}(b-a) \implies x_1 < \frac{1}{2}(b+a) < \frac{1}{2}(b+b) = b$, and similarly, $\Delta x_n = c - x_{n-1} < \frac{1}{2}(c-b) \implies x_{n-1} > \frac{1}{2}(c+b) > b$, so $a < x_1 < b < x_{n-1} < c$. So, there are at least 3 subintervals of $[a,c]$, and at least one that contains b.

Case 1. b is one of the points x_K, for some K between 1 and $n-1$. Then, $a < x_1 < \ldots < x_K = b$ is a partition of $[a,b]$ into intervals with $\Delta x_k < \delta \le \delta_1$, and $b = x_K < \ldots < x_{n-1} < c$ is a partition of $[b,c]$ into intervals with $\Delta x_k < \delta \le \delta_2$, so we can use estimates (1.1) and (1.2):

$$\begin{aligned}
& \left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \left(\int_a^b f(x)dx + \int_b^c f(x)dx\right)\right| \\
= & \left|\left(\sum_{k=1}^{K} f(x_k^*)\Delta x_k + \sum_{k=K+1}^{n} f(x_k^*)\Delta x_k\right) - \left(\int_a^b f(x)dx + \int_b^c f(x)dx\right)\right| \\
\le & \left|\left(\sum_{k=1}^{K} f(x_k^*)\Delta x_k\right) - \int_a^b f(x)dx\right| + \left|\left(\sum_{k=K+1}^{n} f(x_k^*)\Delta x_k\right) - \int_b^c f(x)dx\right| \\
< & \ \epsilon/3 + \epsilon/3 < \epsilon.
\end{aligned}$$

Case 2. b is not one of the points x_K, so it is in exactly one of the subintervals: $x_{K-1} < b < x_K$, for K between 1 and n.

Use the points $a = x_0 < x_1 < \ldots < x_{K-1} < b$ as a partition of $[a,b]$, and $x_1^*, \ldots, x_{K-1}^*, b$ as sample points; then the subintervals have length $\Delta x_k < \delta \le \delta_1$ for $k < K$ and the last subinterval has length $b - x_{K-1} < x_K - x_{K-1} < \delta \le \delta_1$, so by (1.1),

$$\left|\left(\sum_{k=1}^{K-1} f(x_k^*)\Delta x_k\right) + f(b)\cdot(b - x_{K-1}) - \int_a^b f(x)dx\right| < \epsilon/3,$$

and

$$\begin{aligned}
& \left| \left(\sum_{k=1}^{K-1} f(x_k^*) \Delta x_k \right) - \int_a^b f(x)dx \right| \\
= & \left| \left(\sum_{k=1}^{K-1} f(x_k^*) \Delta x_k \right) + f(b) \cdot (b - x_{K-1}) - f(b) \cdot (b - x_{K-1}) - \int_a^b f(x)dx \right| \\
\leq & \left| \left(\sum_{k=1}^{K-1} f(x_k^*) \Delta x_k \right) + f(b) \cdot (b - x_{K-1}) - \int_a^b f(x)dx \right| + |f(b) \cdot (b - x_{K-1})| \\
< & \frac{\epsilon}{3} + |f(b)|(b - x_{K-1}).
\end{aligned}$$

Similarly, use the points $b < x_K < \ldots < x_{n-1} < x_n = b$ as a partition of $[b, c]$, and $b, x_{K+1}^*, \ldots, x_{n-1}^*, x_n^*$ as sample points; then the subintervals have length $\Delta x_k < \delta \leq \delta_2$ for $k > K$ and the first subinterval has length $x_K - b < x_K - x_{K-1} < \delta \leq \delta_2$, so by (1.2),

$$\left| \left(\sum_{k=K+1}^{n} f(x_k^*) \Delta x_k \right) + f(b) \cdot (x_K - b) - \int_b^c f(x)dx \right| < \epsilon/3,$$

and

$$\begin{aligned}
& \left| \left(\sum_{k=K+1}^{n} f(x_k^*) \Delta x_k \right) - \int_b^c f(x)dx \right| \\
= & \left| \left(\sum_{k=K+1}^{n} f(x_k^*) \Delta x_k \right) + f(b) \cdot (x_K - b) - f(b) \cdot (x_K - b) - \int_b^c f(x)dx \right| \\
\leq & \left| \left(\sum_{k=K+1}^{n} f(x_k^*) \Delta x_k \right) + f(b) \cdot (x_K - b) - \int_b^c f(x)dx \right| + |f(b) \cdot (x_K - b)| \\
< & \frac{\epsilon}{3} + |f(b)|(x_K - b).
\end{aligned}$$

Then, using the bound $|f(x)| \leq M$, we get the estimate we needed for the Riemann Sum of f

on $[a, c]$:

$$\begin{aligned}
& \left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \left(\int_a^b f(x)dx + \int_b^c f(x)dx\right)\right| \\
= & \left|\left(\sum_{k=1}^{K-1} f(x_k^*)\Delta x_k\right) + f(x_K^*)\Delta x_K + \left(\sum_{k=K+1}^{n} f(x_k^*)\Delta x_k\right)\right. \\
& \left. - \int_a^b f(x)dx - \int_b^c f(x)dx\right| \\
\leq & \left|\left(\sum_{k=1}^{K-1} f(x_k^*)\Delta x_k\right) - \int_a^b f(x)dx\right| + |f(x_K^*)\Delta x_K| \\
& + \left|\left(\sum_{k=K+1}^{n} f(x_k^*)\Delta x_k\right) - \int_b^c f(x)dx\right| \\
< & \frac{\epsilon}{3} + |f(b)|(b - x_{K-1}) + |f(x_K^*)|\Delta x_K + \frac{\epsilon}{3} + |f(b)|(x_K - b) \\
= & \frac{\epsilon}{3} + |f(b)|(x_K - x_{K-1}) + |f(x_K^*)|\Delta x_K + \frac{\epsilon}{3} \\
\leq & \frac{\epsilon}{3} + M\Delta x_K + M\Delta x_K + \frac{\epsilon}{3} \\
< & \frac{\epsilon}{3} + 2M\frac{\epsilon}{6M} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

■

Proposition 1.67. *Given $a < b < c$, and $f(x)$ defined on $[a, c]$, if $f(x)$ is integrable on $[a, c]$, then $f(x)$ is integrable on $[a, b]$.* ■

The proof is omitted. There are analogous versions of Proposition 1.67, so that if f is integrable on $[a, c]$, then f is integrable on any closed interval contained in $[a, c]$.

Theorem 1.68. *If $f(x)$ is integrable on $[a, b]$ and $f(x)$ has lower bound m and upper bound M on $[a, b]$, then*

$$m \cdot (b - a) \leq \int_a^b f(x)dx \leq M \cdot (b - a).$$

Proof. Because $m \leq f(x) \leq M$ for any x in $[a, b]$, any Riemann sum satisfies

$$m(b - a) = \sum_{k=1}^{n} m\Delta x_k \leq \sum_{k=1}^{n} f(x_k^*)\Delta x_k \leq \sum_{k=1}^{n} M\Delta x_k = M(b - a). \tag{1.3}$$

Suppose, toward a contradiction, that $\int_a^b f(x)dx > M(b-a)$. Then, we can use $\epsilon = \int_a^b f(x)dx - M(b-a) > 0$ in Definition 1.64 to get $\delta > 0$ so that any Riemann sum with $\Delta x_k < \delta$ for all k satisfies

$$\left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \int_a^b f(x)dx\right| < \int_a^b f(x)dx - M(b - a).$$

However, this implies

$$\begin{aligned}\int_a^b f(x)dx - \sum_{k=1}^{n} f(x_k^*)\Delta x_k &< \int_a^b f(x)dx - M(b-a)\\ \implies M(b-a) &< \sum_{k=1}^{n} f(x_k^*)\Delta x_k,\end{aligned}$$

which contradicts (1.3). Supposing $\int_a^b f(x)dx < m(b-a)$ leads to a similar contradiction with (1.3). ∎

1.8 The Fundamental Theorem of Calculus

Theorem 1.69 (The Evaluation Theorem)**.** *If $f(x)$ is integrable on $[a,b]$, and $F(t)$ is a continuous function on $[a,b]$ such that $F'(t) = f(t)$ for all t in (a,b), then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof. The idea is to show that for any $\epsilon > 0$, $|F(b) - F(a) - \int_a^b f(x)dx| < \epsilon$, which proves the claimed equality. Using the assumption that f is integrable, there is some $\delta > 0$ so that any Riemann Sum with $\Delta x_k < \delta$ satisfies

$$\left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \int_a^b f(x)dx\right| < \epsilon.$$

Choose any partition of $[a,b]$ with $\Delta x_k < \delta$; then apply the Mean Value Theorem to $F(x)$ on each interval $[x_{k-1}, x_k]$ to get a sample point x_k^* in (x_{k-1}, x_k) such that $F'(x_k^*) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}$ (this is where the hypothesis that F is continuous on $[a,b]$ and differentiable on (a,b) is used). Multiplying by $\Delta x_k = x_k - x_{k-1}$ and using $F' = f$ on (a,b) gives $f(x_k^*)\Delta x_k = F(x_k) - F(x_{k-1})$, the terms in a Riemann Sum:

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta x_k &= \sum_{k=1}^{n}(F(x_k) - F(x_{k-1}))\\ &= (F(x_1) - F(x_0)) + (F(x_2) - F(x_1)) + \ldots + (F(x_n) - F(x_{n-1}))\\ &= F(x_n) - F(x_0) = F(b) - F(a),\end{aligned}$$

and this gives $\left|\left(\sum_{k=1}^{n} f(x_k^*)\Delta x_k\right) - \int_a^b f(x)dx\right| = |F(b) - F(a) - \int_a^b f(x)dx| < \epsilon.$ ∎

There are examples of functions f which are integrable but which do not have an antiderivative as in the hypothesis of the Evaluation Theorem. For example, an integrable function could have a point where its one-side limit as $x \to c^+$ is not equal to its value $f(c)$, so it is not the derivative of any differentiable function F, by Theorem 1.42.

Theorem 1.70. *If $f(x)$ is integrable on $[a,b]$, then the function*

$$F(t) = \int_a^t f(x)dx$$

is continuous on $[a,b]$.

Proof. First, $F(t)$ is defined for all t in $(a, b]$ because f is integrable on $[a, t]$ by Proposition 1.67. Also, $F(a) = \int_a^a f(x)dx = 0$ as remarked after Definition 1.64. By Theorem 1.65, f is bounded on $[a, b]$, by some $M > 0$ so that $|f(x)| \le M$. We'll show first that for $a \le s < b$, $\lim_{t \to s^+} F(t) = F(s)$. For any $\epsilon > 0$, let $\delta = \min\{\epsilon/M, b - s\} > 0$: we need to show that if $0 < t - s < \delta$, then $|F(t) - F(s)| < \epsilon$. Since $a \le s < t < b$, f is integrable on $[s, t]$ by a version of Proposition 1.67, and $\int_a^s f(x)dx + \int_s^t f(x)dx = \int_a^t f(x)dx$ by Theorem 1.66. Subtracting gives $F(t) - F(s) = \int_s^t f(x)dx$, and because f satisfies $-M \le f(x) \le M$ on $[s, t]$, Theorem 1.68 applies:

$$-M(t-s) \le \int_s^t f(x)dx \le M(t-s)$$

$$\implies |F(t) - F(s)| = |\int_s^t f(x)dx| \le M(t-s) < M\delta = M\frac{\epsilon}{M} = \epsilon.$$

A similar argument for $a < s \le b$, using $\delta = \min\{\epsilon/M, s - a\}$ shows that $\lim_{t \to s^-} F(t) = F(s)$, which is enough to show F is continuous on $[a, b]$. ∎

Theorem 1.71. *Given $a < s < b$ and a function $f(x)$ defined on $[a, b)$, if $f(x)$ is integrable on $[a, t]$ for every t in (a, b), and f is continuous at s, then the function*

$$F(t) = \int_a^t f(x)dx$$

is differentiable at s, and $F'(s) = f(s)$.

Proof. $F(t)$ is defined for $a < t < b$ by hypothesis. For $a < s < b$, to show $\lim_{t \to s^+} \frac{F(t) - F(s)}{t - s} = f(s)$, let $\epsilon > 0$: we need to find $\delta > 0$ so that if $0 < t - s < \delta$, then $\left|\frac{F(t)-F(s)}{t-s} - f(s)\right| < \epsilon$. Since f is continuous at s, there is some $\delta > 0$ so that if $0 < x - s < \delta$, then $|f(x) - f(s)| < \frac{\epsilon}{2}$. This means $f(s) - \epsilon/2 < f(x) < f(s) + \epsilon/2$ for x in $[s, s + \delta)$, and Theorem 1.68 applies to f on $[s, t]$, for $s < t < s + \delta$:

$$(f(s) - \epsilon/2) \cdot (t - s) \le F(t) - F(s) = \int_s^t f(x)dx \le (f(s) + \epsilon/2) \cdot (t - s),$$

so dividing by $t - s > 0$ and subtracting $f(s)$ gives

$$-\epsilon < -\epsilon/2 \le \frac{F(t) - F(s)}{t - s} - f(s) \le \epsilon/2 < \epsilon,$$

which is what we needed to show. The $t \to s^-$ limit is similar. ∎

Proposition 1.72. *If $f(x)$ is continuous on $[a, b]$, then $f(x)$ is integrable on $[a, b]$.* ∎

The proof is omitted.

Theorem 1.73 (The Fundamental Theorem of Calculus)**.** *If $f(x)$ is continuous on $[a,b]$, then:*

- *There exists at least one function $F(t)$ which is continuous on $[a,b]$, and differentiable on (a,b), with $F'(t) = f(t)$ for t in (a,b). One such function is $F(t) = \int_a^t f(x)dx$.*
- *Given any function $F(t)$ which is continuous on $[a,b]$, and which satisfies $F'(t) = f(t)$ on (a,b), $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof. The statement of the Theorem is just a summary of earlier results. Using Proposition 1.72, the continuity of f on $[a,t]$ for any t in $(a,b]$ implies that f is integrable on $[a,t]$. Theorem 1.70 gives the continuity of F, and since f is continuous at every point s in (a,b), Theorem 1.71 gives the differentiability of F on (a,b) and the formula $F'(s) = f(s)$. The second item is Theorem 1.69, with the integrability again following from continuity by Proposition 1.72. ∎

1.9 More on L'Hôpital's Rule

The remark in the Proof of L'Hôpital's Rule that g' must be non-zero at every point in some interval $(c, c+\delta)$ is important, as shown by the following Example.

Example 1.74. Let $h(t)$ be defined by the formula $h(t) = t\sin^2(1/t^2)$ for $t \neq 0$, and $h(0) = 0$. Since $\lim_{t\to 0} h(t) = 0 = h(0)$, h is continuous on $\mathbb{R}$, and the Fundamental Theorem of Calculus applies. Consider the following function: $f(x) = \int_0^x h(t)dt$; it is continuous on $\mathbb{R}$ and positive on $(0,\infty)$, it satisfies $\lim_{x\to 0^+} f(x) = 0$, and $f'(x) = x\sin^2(1/x^2) \geq 0$ on $(0,\infty)$.

Let $g(x) = e^{\cos(1/x^2)} \cdot f(x)$. Since $e^{\cos(1/x^2)}$ is differentiable, positive, and bounded on $(0,\infty)$, $g(x)$ is also differentiable and positive on $(0,\infty)$, with $\lim_{x\to 0^+} g(x) = 0$ and for $x > 0$,

$$\begin{aligned} g'(x) &= e^{\cos(1/x^2)} \cdot f'(x) + e^{\cos(1/x^2)}(-\sin(1/x^2))(-2/x^3)f(x) \\ &= e^{\cos(1/x^2)}\sin(1/x^2)\left(2x^{-3}f(x) + x\sin(1/x^2)\right). \end{aligned}$$

The fraction:

$$\frac{f'(x)}{g'(x)} = \frac{x\sin^2(1/x^2)}{e^{\cos(1/x^2)}\sin(1/x^2)\left(2x^{-3}f(x) + x\sin(1/x^2)\right)}$$

is undefined $(\frac{0}{0})$ at every point where $\sin(1/x^2) = 0$, which is every point in the sequence $\{1/\sqrt{n\pi}\} \to 0$. So, the $x \to 0^+$ limit does not exist and L'Hôpital's Rule does not apply. However, if we try to cancel a $\sin(1/x^2)$ from the numerator and denominator, there is a function that equals f'/g' for all x with $\sin(1/x^2) \neq 0$:

$$q(x) = x \cdot \frac{\sin(1/x^2)}{e^{\cos(1/x^2)}} \cdot \frac{1}{2x^{-3}f(x) + x\sin(1/x^2)},$$

and $\lim_{x\to 0^+} q(x) = 0$, since $x \to 0$, the second fractional factor is bounded, and the last factor is of the form $\frac{1}{\infty}$, since $\lim_{x\to 0^+} x^{-3}f(x) = +\infty$ (checking this is non-trivial but it can be graphed).

We cannot conclude that $\lim_{x\to 0+} f(x)/g(x)$ has the same limit; in fact, $\frac{f(x)}{g(x)} = \frac{1}{e^{\cos(1/x^2)}}$ for all $x > 0$, but $\lim_{x\to 0^+} f(x)/g(x)$ DNE because $e^{-\cos(1/x^2)}$ oscillates infinitely many times as $x \to 0^+$.

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