COMMENTS ON: THE ALGEBRA AND GEOMETRY OF STEINER AND OTHER QUADRATICALLY PARAMETRIZABLE SURFACES

ADAM COFFMAN

Last update: Sept. 29, 2024.

1. Errata

The following typos appear in the published paper, [CSS].

- (p. 270) The derivative, equation (34), should read

$$\frac{\partial f_3}{\partial x} = [x_2(x_3^2 + x_4^2) - 2x_1x_4^2, x_1(x_3^2 + x_4^2),$$
$$2x_1x_2x_3 - 4x_3(x_3^2 + x_4^2), 2x_1x_2x_4 - 2x_1^2x_4 - 4(x_3^2 + x_4^2)x_4].$$

  This corrects the lower right 2×2 block of the Hessian, equation (35), to

$$\begin{matrix} 2x_1x_2 - 12x_3^2 - 4x_4^2 & -8x_3x_4 \\ -8x_3x_4 & 2x_1x_2 - 2x_1^2 - 12x_4^2 - 4x_3^2 \end{matrix} \Bigg] .$$

- (p. 273) Below equation (58) should read "As in the case of Σ_7 and Σ_8"
- (p. 279) **Theorem 7.** should read "*The order of* Σ *is* $4 - \nu_\Sigma$."
- (p. 281) Above equation (105) should read "Since $\mathcal{P}(\Sigma) \nsubseteq \mathbb{M}_2$, $e \neq 0$."
- (p. 284) Step (2) in Section **6.** should read "$\det\{\lambda\mathbf{M} + \mu\mathbf{N}\}$."

2. Updates

The references mention unpublished notes of A. Schwartz and C. Stanton — a version from 1988 is available from me, on request. Another pre-1996 source on this topic, with some details on the matrix calculations leading to the classification theorem, is [C_1], which I made available online in 2018.

It should have been mentioned that Coffman's research was supported in part by a National Science Foundation Research Experience for Undergraduates program in the summer of 1990.

Since publication, the following related article has come to our attention: [D]. The topic of projections of the real Veronese variety has more recently been considered in [C_3]. Also see my web page on Steiner surfaces, currently at this address: [C_2].

3. Citations

Our article is cited in these academic papers: [A_1], [A_2], [A_3], [ABB], [AMT], [AS_1], [AS_2], [BJKL], [BOR], [BCF], [BEG], [CFRV], [EGL_1], [EGL_2], [GS], [G], [HJS], [H], [HW_1], [HK], [HW_2], [HW_3], [KO], [LG], [L], [M], [PA], [PL], [PO], [POS], [PR], [PT], [P], [RJ], [S], [SPS], [VMD], [WG], [WC], [WCD], [Y], [Zanella], [Z_1], [Z_2], [Z_3], as well as these books: [F], [KI], [OSG], and this computer technical manual: [T].

References

[A_1] G. Albrecht, *Determination and classification of triangular quadric patches*, Computer Aided Geometric Design (7) **15** (1998), 675–697. MR 1632733 (99e:65190).

[A_2] G. Albrecht, *The Veronese surface revisited*, J. Geom. (1–2) **73** (2002), 22–38. MR 1916463 (2003e:65019).

[A_3] G. Albrecht, *Geometric invariants of parametric triangular quadric patches*, Int. Electron. J. Geom. (2) **4** (2011), 63–84. MR 2854276.

[ABB] F. Aries, E. Briand, and C. Bruchou, *Some covariants related to Steiner surfaces*, in Geometric Modeling and Algebraic Geometry, 31–46, Springer, Berlin, 2008. MR 2381602 (2009b:14111), Zbl 1132.14336.

[AMT] F. Aries, B. Mourrain, and J.-P. Técourt, *Quadratically parametrized surfaces: algorithms and applications*, in Geometric Modeling and Computing: Seattle 2003, M. L. Lucian and M. Neamtu (eds.), Nashboro Press, Brentwood, TN, 2004, 21–40. MR 2122113, Zbl 1060.65528.

[AS_1] F. Aries and R. Senoussi, *Approximation de surfaces paramétriques par des carreaux rationnels du second degré en lancer de rayons*, Revue de CFAO et d'informatique graphique (6) **12** (1997), 627–645.

[AS_2] F. Aries and R. Senoussi, *An implicitization algorithm for rational surfaces with no base points*, J. Symbolic Comput. (4) **31** (2001), 357–365. MR 1823071 (2002a:14063), Zbl 0981.14030.

[BJKL] B. Bastl, B. Jüttler, J. Kosinka, M. Lávička, *Computing exact rational offsets of quadratic triangular Bezier surface patches*, Computer Aided Design (2) **40** (2008), 197–209.

[BOR] P. Benedini Riul, R. Oset Sinha, and M. A. S. Ruas, *Curvature loci of 3-manifolds*, Math. Nachr. (10) **296** (2023), 4656–4672. MR 4663196.

[BCF] R. Binotto, S. Costa, and M. C. Romero Fuster, *The curvature Veronese of a 3-manifold immersed in Euclidean space*, in Real and Complex Singularities, XIII International Workshop, São Carlos, 2014, Contemporary Mathematics **675**, 2016. MR 3578716.

[BEG] L. Busé, M. Elkadi, and A. Galligo, *Intersection and self-intersection of surfaces by means of Bezoutian matrices*, Computer Aided Geometric Design (2) **25** (2008), 53–68. MR 2388405 (2009b:65034).

[CFRV] A. Cantón, L. Fernández-Jambrina, E. Rosado María, and M. J. Vázquez-Gallo, *Implicit equations of non-degenerate rational Bezier quadric triangles*, in Curves and Surfaces, 8^{th} International Conference, Paris, 2014, Springer LNCS **9213**, 2015. MR 3440735.

[$CFRV_2$] A. Cantón, L. Fernández-Jambrina, E. Rosado María, and M. J. Vázquez-Gallo, *Geometric elements and classification of quadrics in rational Bézier form*, Journal of Computational and Applied Mathematics **300** (2016), 400–419. MR 3460308.

[C_1] A. Coffman, *A Classification of Quadratically Parametrized Maps of the Real Projective Plane*, B.S. Honors Thesis, University of Michigan, Ann Arbor, 1991.
`http://users.pfw.edu/CoffmanA/`

[C_2] A. Coffman, *Steiner Surfaces* web site.
`http://users.pfw.edu/CoffmanA/steinersurface.html`

[C_3] A. Coffman, *Real congruence of complex matrix pencils and complex projections of real Veronese varieties*, Linear Algebra and its Applications **370** (2003), 41–83. MR 1994320 (2004f:14026), Zbl 1049.14042.

[CSS] A. Coffman, A. J. Schwartz, and C. M. Stanton, *The algebra and geometry of Steiner and other quadratically parametrizable surfaces*, Computer Aided Geometric Design (3) **13** (1996), 257–286. MR 1390390 (97g:53012), Zbl 0875.68860.

[D] W. L. F. Degen, *The types of triangular Bézier surfaces*, in The Mathematics of Surfaces, VI (Brunel Univ., 1994), 153–170, Inst. Math. Appl. Conf. Ser. New Ser., **58**, G. Mullineux, ed., Oxford, 1996. MR 1430584 (98h:51033).

[EGL_1] M. Elkadi, A. Galligo, and T. H. Lê, *Parametrized surfaces in* $\mathbb{P}^3$ *of bidegree* $(1, 2)$, Proceedings of ISSAC (Santander, Spain), 141–148, ACM, New York, 2004. MR 2126936 (2005j:68135).

[EGL$_2$] M. Elkadi, A. Galligo, and T. H. Lê, *On parametric surfaces of low degree in* $\mathbb{P}^3(\mathbb{C})$, in Algebraic Geometry and Geometric Modeling, 151–168, Springer series on Mathematics and Visualization, Berlin, 2006. MR 2279849 (2007k:14125).

[F] R. Farouki, *Pythagorean-Hodograph Curves: Algebra and Geometry Inseparable.* Geometry and Computing **1**. Springer, Berlin, 2008. MR 2365013 (2008k:65027).

[GS] A. Galligo and M. Stillman, *On the geometry of parametrized bicubic surfaces*, J. Symbolic Computation (1–2) **42** (2007), 136–158. MR 2284289 (2008e:14054).

[G] G. H. Georgiev, *Rational generalized offsets of rational surfaces*, Mathematical Problems in Engineering **2012** Article ID 618148, (2012), 15 pages. MR 2878968.

[HJS] M. Hauer, B. Jüttler, and J. Schicho, *Projective and affine symmetries and equivalences of rational and polynomial surfaces*, Journal of Computational and Applied Mathematics **349** (2019), 424–437. MR 3886725.

[H] D. Henrion, *Semidefinite representation of convex hulls of rational varieties*, Acta Appl. Math. (3) **115** (2011), 319–327. MR 2823121 (2012f:14112).

[HW$_1$] J. W. Hoffman and H. Wang, *A note on quadratically parametrized surfaces*, Algebra Colloq. (3) **21** (2014), 461–476. MR 3224670.

[HK] I. Hoveijn and O. N. Kirillov, *Singularities on the boundary of the stability domain near* $1:1$*-resonance*, J. Differential Equations (10) **248** (2010), 2585–2607. MR 2600969 (2011h:34108).

[HW$_2$] Y. Huang and D. Wang, *Computing self-intersection loci of parametrized surfaces using regular systems and Gröbner bases*, SYNASC 2009 — 11^{th} International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, art. no. 5460872, 28–36.

[HW$_3$] Y. Huang and D. Wang, *Computing intersection and self-intersection loci of parametrized surfaces using regular systems and Gröbner bases*, Computer Aided Geometric Design (9) **28** (2011), 566–581. MR 2861556 (2012j:65046).

[KO] M. Kontsevich and A. Odesskii, *When the Fourier transform is one loop exact?* Selecta Math. (N.S.) (3) **30** (2024), Paper #35, 55 pp. MR 4715444.

[KI] S. Krivoshapko and V. N. Ivanov, *Encyclopedia of Analytical Surfaces*, Springer, 2015. MR 3309742.

[LG] T.-H. Lê and A. Galligo, *General classification of* $(1,2)$ *parametric surfaces in* $\mathbb{P}^3$, in Geometric Modeling and Algebraic Geometry, 93–113, Springer, Berlin, 2008. MR 2381606 (2009a:14075).

[L] N. Lubbes, *Self-intersections of surfaces that contain two circles through each point*, J. Symbolic Computation, 2024.

[M] J. Monterde, *Curvature locus of a 3-manifold having a normal bundle with some flat direction*, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas (2) **117** (2023), Article #60, 19 pp. MR 4540868.

[OSG] B. Odehnal, H. Stachel, and G. Glaeser, *The Universe of Quadrics*, Springer, 2020. MR 4215627.

[PA] R. Paul and S. Anand, *A new Steiner patch based file format for Additive Manufacturing processes*, Computer-Aided Design **63** (2015), 86–100.

[PL] M. Penz and R. van Leeuwen, *Geometry of degeneracy in potential and density space*, Quantum **7** article #918 (2023), 14 pp.

[PO] M. Peternell and B. Odehnal, *Convolution surfaces of quadratic triangular Bézier surfaces*, Computer Aided Geometric Design (2) **25** (2008), 116–129. MR 2388409 (2008m:65052).

[POS] M. Peternell, B. Odehnal, and M. Sampoli, *On quadratic two-parameter families of spheres and their envelopes*, Computer Aided Geometric Design (4–5) **25** (2008), 342–355. MR 2408097 (2009b:65046).

[PR] J. Peters and U. Reif, *The 42 equivalence classes of quadratic surfaces in affine* n*-space*, Computer Aided Geometric Design (5) **15** (1998), 459–473. MR 1624047 (99e:14063).

[PT] V. Petrushchenko and V. Timorin, *On maps taking lines to plane curves*, Arnold Mathematical Journal (1) **2** (2016), 1–20. MR 3460040.

[P] R. Piene, *Singularities of some projective rational surfaces*, in Computational Methods for Algebraic Spline Surfaces, 171–182, Springer, Berlin, 2005. MR 2127188 (2005m:14060).

[RJ] K. Rittenschober and B. Jüttler, *Rational mappings associated with space filling line congruences*, in Geometric Modeling and Computing: Seattle 2003, M. L. Lucian

and M. Neamtu (eds.), Nashboro Press, Brentwood, TN, 2004, 481–490. MR 2122140, Zbl 1060.65547.

[S] M. Saniga, *Cremona transformations and the conundrum of dimensionality and signature of macro-spacetime*, Chaos, Solitons and Fractals (12) **12** (2001), 2127–2142. MR 1859514 (2003a:51019).

[SPS] J. R. Sendra, M. Peternell, and J. Sendra, *Cissoid constructions of augmented rational ruled surfaces*, Computer Aided Geometric Design **60** (2018), 1–9. MR 3765807.

[T] M. Trott, *The Mathematica GuideBook for Graphics*, Springer, 2004.

[VMD] O. Vasilyev, A. Maciołek, and S. Dietrich, *Sensitivity of the thermodynamics of two-dimensional systems towards the topological classes of their surfaces*, Physica A: Statistical Mechanics and its Applications (2023), article #128960.

[WG] H. Wang and R. Goldman, *Syzygies for translational surfaces*, J. Symbolic Computation **89** (2018), 73–93. MR 3804807.

[WC] X. Wang and F. Chen, *Implicitization, parameterization and singularity computation of Steiner surfaces using moving surfaces*, J. Symbolic Computation (6) **47** (2012), 733–750. MR 2908591.

[WCD] X. Wang, F. Chen, and J. Deng, *Implicitization and parametrization of quadratic surfaces with one simple base point*, Proceedings of ISSAC (Linz/Hagenberg, Austria), 31–38, 2008. MR 2500370 (2010h:68232).

[Y] A. Yampolsky, *On projective classification of points of a submanifold in the Euclidean space*, Zh. Mat. Fiz. Anal. Geom. (3) **16** (2020), 364–371. MR 4252713.

[Zanella] C. Zanella, *On finite Steiner surfaces*, Discrete Mathematics (3) **312** (2012), 652–656. MR 2854812.

[Z_1] S. Zubė, *Bidegree* $(2,1)$ *parametrizable surfaces in projective* 3*-space*, Liet. Mat. Rink. (3) **38** (1998), 379–402; translation in Lithuanian Math. J. (3) **38** (1999), 291–308. MR 1657885 (99m:51030).

[Z_2] S. Zubė, *Real ruled degree four toric surfaces in projective* 3*-space*, Proceedings of 39^{th} Conference of Lithuanian Mathematical Society, 191–195, 1998.

[Z_3] S. Zubė, *Correspondence and* $(2,1)$*-Bézier surfaces*, Liet. Mat. Rink. (1) **43** (2003), 99–122; translation in Lithuanian Math. J. (1) **43** (2003), 83–102. MR 1996756 (2004k:14061).

Department of Mathematical Sciences, Purdue University Fort Wayne, Fort Wayne, IN 46805-1499

Email address: CoffmanA@pfw.edu