Some Proofs on an inequality related to a theorem by Malliavin

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Synopsis

This work is devoted to an inequality that's an estimated inversion formula related to the Cauchy integral of a distribution function on $\mathbb{R}_+ = (0, +\infty)$. It yields a theorem proven by Malliavin. Alternatives to the inequality are presented. The inequality (3) has been applied in spectral theory of differential and pseudo-differential operators and by extension to mathematical physics, notably quantum mechanics, e.g. atomic spectra, and more generally physics of vibrations.

Estimated Riesz means

Given a power asymptotics of the Cauchy integral of $N(\lambda)$ along a certain parabola-like curve in $\mathbb{C}$ that avoids $\mathbb{R}_+$, the asymptotics of the Riesz means as $\lambda \to +\infty$ can be recovered utilizing:

Theorem 2 (Theorem 1) *Let the function $N(\lambda)$ be constant in a neigbourhood of λ_0. Then for any $\alpha > 0$*

$$\left| N^{(\alpha)}(\lambda_0) | - \int_\Gamma \frac{S(\zeta)\left(1 - \frac{\zeta}{\lambda_0}\right)^\alpha d\zeta}{2\pi i} \right| \le \left(\frac{\eta_0}{\lambda_0}\right)^\alpha \frac{\eta_0\,|S(\zeta_0)|}{\alpha\pi}. \tag{5}$$

For $\alpha < 1$ the factor $(\alpha\pi)^{-1}$ in the RHS may be replaced by $\sqrt{\pi^{-2} + 1/4}$.[a]

The LHS vanishes if a closed contour of integration consisting of Γ and the segment $[\zeta, \bar{\zeta_0}]$ is used. Indeed, since $N(\lambda)$ is assumed constant in the vicinity of λ_0, one may change the order of integration:

$$\int_0^\infty dN(\lambda) \frac{1}{2\pi i} \oint \left(\frac{\lambda_0 - \zeta}{\lambda_0}\right)^\alpha \frac{d\zeta}{\lambda - \zeta}.$$

Henceforth the branch $z^\alpha = \exp(\alpha \ln z)$ with $-\pi < \operatorname{Im} \ln z \le \pi$ is assumed.

[a] A continuous dependence of the constant on α can be realized by substituting the term $1/4$ with $(1 - \alpha^{1+\varepsilon})/4$, with $\varepsilon > \varepsilon_0 \approx 1/16$, calculated numerically.

Remark

In Theorems 1 and 2, $N(\lambda)$ need not be continuous at λ_0. If λ_0 is a discontinuity point of $N(\lambda)$, Theorem 1 remains valid without change, while in Theorem 2 the value $N(\lambda_0)$ in the LHS can be replaced by any value between $N(\lambda_0 - 0)$ and $N(\lambda_0 + 0)$.

$N(\lambda)$ & its Cauchy Integral

Let $N(\lambda)$ be a nondecreasing function defined on $\mathbb{R}_+ = (0, +\infty)$ such that $N(\lambda) = 0$ for small λ and

$$\int_0^\infty \lambda^{-1} dN(\lambda) < \infty. \tag{1}$$

The *Cauchy Integral* of $N(\lambda)$ is defined as

$$S(\zeta) = \int_0^\infty (\lambda - \zeta)^{-1}\, dN(\lambda), \qquad \zeta \notin \mathbb{R}_+. \tag{2}$$

$\nexists$ Ordinary Cauchy Integral

If in lieu of (1) a weaker [a] condition with some integer $q > 1$

$$\int_0^\infty \lambda^{-q} dN(\lambda) < \infty \tag{6}$$

holds true, then the leading term of the asymptotics of $N(\lambda)$ can be recovered by means of the next theorem from the behaviour of its *generalized Cauchy integral*

$$S_q(\zeta) = \int_0^\infty (\lambda - \zeta)^{-q}\, dN(\lambda), \qquad \zeta \notin \mathbb{R}_+. \tag{7}$$

Theorem 3 (Theorem 2) *Let the function $N(\lambda)$ satisfy (6) and be constant in a neigbourhood of λ_0. There exist constants $C_0, C_1, \ldots, C_{q-2}$ (which depend only on q) such that*

$$\begin{aligned} &\left| N(\lambda_0) - \frac{1}{2\pi i} \int_\Gamma S_q(\zeta)(\zeta - \lambda_0)^{q-1}\, d\zeta \right| \\ &\qquad \le \sum_{m=0}^{q-2} C_m \eta_0^{q-1-m} \left| \int_\Gamma S_q(\zeta)(\lambda_0 - \zeta)^m\, d\zeta \right|. \end{aligned} \tag{8}$$

[a] Inequality (6) is weaker than (1) because it is taken for granted that $dN(\lambda) = 0$ near $\lambda = 0$.

Inequality: Estimated Inversion $\to$ Cauchy integral

Fix a point $\zeta_0 = \lambda_0 + i\eta_0$ in the first quadrant of the complex plane $\mathbb{C}$. Denote by Γ a contour that connects the point ζ_0 to $\bar{\zeta}_0 = \lambda_0 - i\eta_0$ and does not cross the integration path $\mathbb{R}_+$ of (2).[a]

The inequality related to a theorem by Malliavin is

$$\left| N(\lambda_0) | - \frac{1}{2\pi i} \int_\Gamma S(\zeta)\, d\zeta \right| \le \eta_0 \sqrt{1 + \pi^{-2}}\, |S(\zeta_0)| \tag{3}$$

and employed it to provide a brief proof of Malliavin's Tauberian theorem. For $\alpha > 0$, the *Riesz mean of order α of $N(\lambda)$* is

$$N^{(\alpha)}(\lambda) = \int_0^\lambda \left(1 - \frac{x}{\lambda}\right)^\alpha dN(x), \qquad \lambda > 0. \tag{4}$$

[a] There exists an alternative convention according to which the Cauchy Integral of $f(t)$ is defined as $\int_0^\infty (\zeta + t)^{-1} f(t)\, dt$.

$\exists$ Non-negative Constants

Theorem 1 (Theorem 3) *Let function $N(\lambda)$ defined on $\mathbb{R}_+$ be nondecreasing, equal zero near $\lambda = 0$, and satisfy condition (6). For any $\alpha > 0$ and any integer $q = 2, 3, \ldots$ there exist nonnegative constants $C_0, \ldots, C_{q-2}$, depending on q and α, such that for any $\lambda > 0$*

$$\begin{aligned} &\left| N^\alpha(\lambda_0) - \frac{\alpha\, B(q,\alpha)}{2\pi i} \int_\Gamma S_q(\zeta)\,(\zeta - \lambda_0)^{q-1} \left(1 - \frac{\zeta}{\lambda_0}\right)^\alpha d\zeta \right| \\ &\qquad \le \sum_{m=0}^{q-2} C_m \left(\frac{\eta_0}{\lambda_0}\right)^\alpha \cdot \eta_0^{q-1-m} \left| \int_\Gamma S_q(\zeta)\,(\zeta - \lambda_0)^m\, d\zeta \right|, \end{aligned}$$

where

$$B(q, \alpha) = \frac{\Gamma(q)\Gamma(\alpha)}{\Gamma(q + \alpha)}.$$

Its proof mimics that of Theorem 2 by substituting $T_{q,q-1}(\mu)$ with

$$T_{q,q-1+\alpha}(\mu) = \int_{-1}^1 \frac{\tau^{q-1+\alpha}}{(\mu - i\tau)^q}\, d\tau.$$

This theorem is connected to Theorem 2, as Theorem 1 to the inequality.

Epilogue

An estimated Riesz means of the distribution function from its Cauchy integral and another with power growth for which the ordinary Cauchy Integral is non-existent, wherein a power of the Cauchy kernel is utilized to yield the generalized Cauchy integral. A plethora of more applications in and beyond spectral theory are expected to realized.

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