Nontraditional Notions of Polynomial Ordering with Computational Applications

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Introduction and Motivation

Why consider nontraditional notions of degree?

Suppose we approximate some function f, analytic in the hypercube $H_d := [-1,1]^d$. Then consider

$$\inf_{\text{degree}(p)\leq n} \|f - p\|_{H_d}$$

In [T17], it is shown that for analytic functions with an analytic continuation to a "particular" set around the hypercube.

$$\begin{cases} O(\rho^{-n/\sqrt{d}} + \epsilon), & \text{for traditional degree} \\ O(\rho^{-n} + \epsilon), & \text{for two selected nontraditional degrees} \end{cases}$$

for any $\epsilon > 0$, where $\rho > 0$ depends on the "particular set".

Notation

- We begin with a compact set $K \subset \mathbb{C}^d$ for positive $d \in \mathbb{Z}$.
- Then for positive $k \in \mathbb{Z}$, define the polynomial spaces $\mathcal{P}_\Sigma(k)$ as all polynomials of standard degree less than or equal to k.
- Let M_k be the dimension of $\mathcal{P}_\Sigma(k)$.
- For positive $s \in \mathbb{Z}$, let $\alpha(s)$ be an enumeration of the multi-exponents, so that $e_s = z^{\alpha(s)}$ for $s = 1, \ldots, M_k$ forms a basis for $P_\Sigma(k)$.

Vandermonde Matrix and Determinant

Then for a given k and sequence $(z_i)_{i=1}^{s} \subset K$, we can form the s by s Vandermonde matrix VDM.

$$[\mathrm{VDM}(z_1, z_2, \ldots, z_s)]_{i,j} = z_i^{\alpha(j)}$$

And its determinant

$$V(z_1, \ldots, z_s) = |\det(\mathrm{VDM}(z_1, \ldots, z_s))|$$

We will focus on maximizing $V(z_1, \ldots, z_s)$ over sets of s points in K.

Vandermonde Matrix and Determinant

Note: In one dimension, the determinant of the Vandermonde is just the product of the distances between each pair of points.

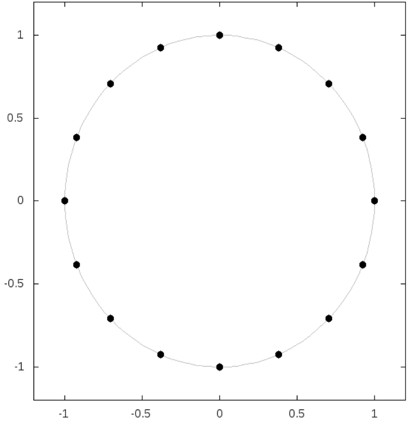
$$V(z_1, \ldots, z_s) = \prod_{i=1}^{s} \prod_{j>i}^{s} |z_i - z_j|$$

Fekete Points

- For any s, there exist set(s) of points $(\zeta_i)_{i=1}^s \subset K$ that maximize V. We call these *Fekete Points*.
- We define the measure μ^s on K via $\mu^s(z) = \dfrac{1}{s}\sum_{i=1}^{s} \delta_{\zeta_i}(z)$.
- These measures converge weak-*: $\mu^s \rightharpoonup \mu_K$, where μ_K is the potential theoretic *equilibrium measure*

Fekete Points: Example

If K is the unit complex disk, $\{z : |z| \leq 1\}$, then the sth order Fekete points are the roots of unity of order s.



Fekete Points: Asymptotics

Let $L_k = \sum_{i=1}^{k} i(M_i - M_{i-1})$.

and *we want to* define $\tau(K) = \lim_{k\to\infty} V(\zeta_1^{(k)}, \ldots, \zeta_{M_k}^{(k)})^{1/L_K}$.

In one dimension, it straightforward to prove that $V(\zeta_1, \ldots, \zeta_{M_k})^{1/L_k}$ is decreasing in k, and has a limit as $k \to \infty$ which we call the *transfinite diameter* [R95]

In multiple dimensions, this is *much more difficult*, but was done in [Z75].

We say that an array $\left(z_i^{(k)}\right)_{i=1}^{M_k}$ for each k is *Asymptotically Fekete* if

$$V(z_1^{(k)}, \ldots, z_{M_k}^{(k)})^{1/L_k} \to \tau(K) \qquad \text{as } k \to \infty$$

A Preview of Orderings

Definition

Let $\prec$ denote the grevlex ordering on $\mathbb{Z}^d$, [M19], where $\alpha \prec \beta$ if

- $|\alpha| < |\beta|$; or
- $|\alpha| = |\beta|$, and there exists $k \in \{1, \ldots, d\}$ such that $\alpha_j = \beta_j$ for all $j < k$, and $\alpha_k < \beta_k$

Ex: For $d = 2$, this gives

$$1, x, y, x^2, xy, y^2, x^3, x^2y, \ldots$$

We enumerate the monomials using this ordering, so $e_1(z), e_2(z), \ldots$ are a polynomial basis.

Monomial Classes and Tchebyshev Constants

We then, following [Z75] in [BBCL92], define the following monomial class,

Definition

We define the sth monomial class,

$$\mathcal{M}(s) := \left\{ p : p(z) = e_s(z) + \sum_{j=1}^{s-1} c_j e_j(z) : c_j \in \mathbb{C} \right\}$$

Definition

We define the discrete Tchebyshev constant

$$T_s := \inf \{ \|p\|_K : p \in M(s) \}^{1/\deg(e_s)}$$

And in preparation, we define the set
$D = \{x_1, \ldots, x_d \in \mathbb{R}_+ : \sum x_i = 1\}$.

Zaharjuta Conclusion

Definition

For $\theta \in \text{int}(C)$, the directional Chebyshev constant is the function

$$T(K,\theta) := \limsup_{s\to\infty,\, \frac{\alpha(s)}{\deg(e_s)}\to\theta} T_s$$

And this gives us the formula for the transfinite diameter

$$\log(\tau(K)) = \lim_{k\to\infty} \log\left(\prod_{\deg(e_s)=k} T_s\right)^{1/(M_k - M_{k-1})}$$
$$= \frac{1}{\text{meas}(D)} \int_D \ln T(K,\theta)\, d\theta$$

Zaharjuta Conclusion

Lastly, from [Z75] and [BBCL92], we note the connection between the maximum Vandermonde matrix determinant, $V(\zeta_1, \ldots, \zeta_s)$, and these averages. For $k \geq 1$,

$$\left(\prod_{s=1}^{M_k} T_s^{\deg(s)} \right) \leq V(\zeta_1, \ldots, \zeta_{M_k}) \leq M_k! \left(\prod_{s=1}^{M_k} T_s^{\deg(s)} \right)$$

Leja Points

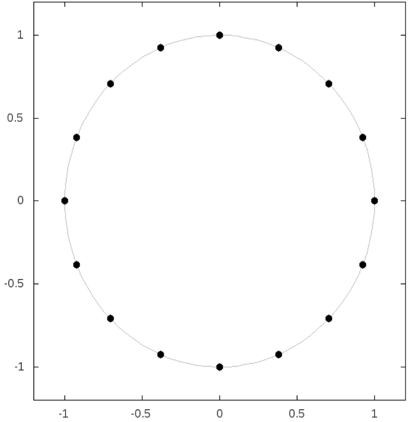
Fekete sequences are very difficult to find, so for given k we define a *Leja Sequence* $(\ell_i)_{i=1}^s \subset K$ as a sequence given by the following procedure.

- Choose $\ell_1 \in K$ at a maximum of $z^{\alpha(1)}$.
- Assuming $\ell_1, \dots, \ell_{s-1}$ have been chosen,
- For each subsequent point ℓ_s, choose a maximum of $\ell \to V(\ell_1, \dots, \ell_{s-1}, \ell)$.

We know that Leja points are asymptotically Fekete, in both the one and several variable cases.

Leja Points: Example

If K is the unit complex disk, $\{z : |z| \leq 1\}$, then the 2^nth order Leja points are the roots of unity of order 2^n.



Leja Points in $\mathbb{C}^d$: Proof of Asymptotically Fekete

From [BBCL92],

- Let $\mathcal{L}_s = V(\ell_1, \ldots, \ell_s)$. Then $\mathcal{L}_s \leq V(\zeta_1, \ldots, \zeta_s)$ is clear.
- $\dfrac{V(\ell_1, \ldots, \ell_{s-1}, \ell)}{V(\ell_1, \ldots, \ell_{s-1})} = p_s(\ell)$.
- $p_s(\ell)$ is monic, with $e_s(\ell)$ being the monic term.
- Further, $\mathcal{L}_s / \mathcal{L}_{s-1} = \|p_s\|_K \geq T_s^{\deg(e_s)}$.
- So $\mathcal{L}_{M_k} = \dfrac{\mathcal{L}_{M_k}}{\mathcal{L}_{M_k-1}} \dfrac{\mathcal{L}_{M_k-1}}{\mathcal{L}_{M_k-2}} \cdots \dfrac{\mathcal{L}_1}{1} = \|p_s\|_K \geq \prod_{s=1}^{M_k} T_s^{\deg(e_s)}$.
- Taking L_kth roots, we achieve $\mathcal{L}_s \geq V_s / M_k!$.

Thus, in $\mathbb{C}^d$, Leja point sequences are asymptotically Fekete.

Polynomial Spaces associated with Convex Bodies

So far, we have used $\mathcal{P}_\Sigma(k)$ to denote our polynomial space. We let

$$\Sigma := \left\{ x_1, \ldots, x_d : x_i \geq 0, \sum_{i=1}^{d} x_i \leq 1 \right\}$$

More generally, let $C \in \mathbb{R}_+^d$ be a convex body containing $\frac{1}{n}\Sigma$ for some positive n.

C Polynomial Spaces

Then we define the polynomial space $\mathcal{P}_C(k)$ as

$$\mathcal{P}_C(k) := \left\{ p(z) = \sum_{J \in C \cap \mathbb{Z}_+^d} c_J z^J \right\}$$

(Which encompasses our use of $\mathcal{P}_\Sigma(k)$ thus far.)

With these new polynomials comes the question of C-degree, which we answer

$$\deg_C(p) = \min_{p \in \mathcal{P}_C(k)} (k)$$

Grevlex Ordering

Definition

Let $\prec$ denote the grevlex ordering on $\mathbb{Z}^d$, [M19], where $\alpha \prec \beta$ if

- $|\alpha| < |\beta|$; or
- $|\alpha| = |\beta|$, and there exists $k \in \{1, \ldots, d\}$ such that $\alpha_j = \beta_j$ for all $j < k$, and $\alpha_k < \beta_k$

Ex: For $d = 2$, this gives

$$1, x, y, x^2, xy, y^2, x^3, x^2y, \ldots$$

We note that this definition pays no attention to the gradiation given by the convex body C.

Grevlex Ordering

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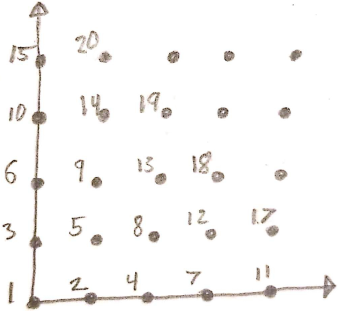


Nested and Additive

We now deal with the question of ordering within these monomial classes. Two properties are desireable for an ordering $<$.

Definition (Additivity)

For any α, β, δ such that $\alpha < \beta$, we have

$$\alpha + \delta < \beta + \delta$$

Definition (Nested)

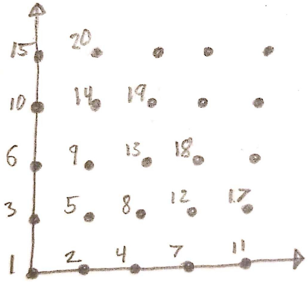
For $k_1 < k_2$, both in $\mathbb{Z}_+$, let $\alpha \in \mathcal{P}_C(k_1), \beta \in \mathcal{P}_C(k_2) \setminus \mathcal{P}_C(k_1)$. Then

$$\alpha < \beta$$

We often say that an order "respects the C-degree", if it is nested.

Grevlex Ordering Not Usually Nested

We consider the grevlex ordering again.



We see quickly that the only ordering this respects is the canonical one, associated with $C = \Sigma$.
Ex: Consider $C = [0, 1] \times [0, 1]$.

Modified Grevlex Ordering

Definition

For a given convex body C, let $\prec_C$ denote the modified grevlex ordering on $\mathbb{Z}^d$, where $\alpha \prec \beta$ if

- $\deg_C(\alpha) < \deg_C(\beta)$; or
- $\deg_C(\alpha) = \deg_C(\beta)$, and $\alpha \prec \beta$.

Ex: For $d = 2$ and $C = [0,1] \times [0,1]$, this provides

$$1, x, y, xy,$$

$$x^2, y^2, x^2y, xy^2, x^2y^2,$$

$$x^3, y^3, x^3y, xy^3, x^3y^2, x^2y^3, x^3y^3, \ldots$$

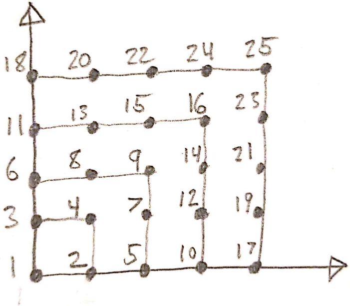
Modified Grevlex Ordering

Definition

For a given convex body C, let $\prec_C$ denote the modified grevlex ordering on $\mathbb{Z}^d$, where $\alpha \prec \beta$ if

- $\deg_C(\alpha) < \deg_C(\beta)$; or
- $\deg_C(\alpha) = \deg_C(\beta)$, and $z^\alpha \prec z^\beta$.

Ex: For $d = 2$ and $C = [0,1] \times [0,1]$, this provides



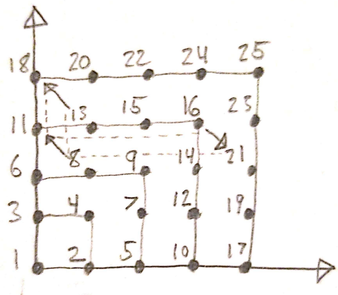
Modified Grevlex Ordering (Not Additive!)

Definition

For a given convex body C, let $\prec_C$ denote the modified grevlex ordering on $\mathbb{Z}^d$, where $\alpha \prec \beta$ if

- $\deg_C(\alpha) < \deg_C(\beta)$; or
- $\deg_C(\alpha) = \deg_C(\beta)$, and $z^\alpha \prec z^\beta$.

Ex: For $d = 2$ and $C = [0,1] \times [0,1]$, this provides



Standard Spaces

- For $C = \Sigma$, both the grevlex and modified grevlex orderings result in the standard polynomial ordering, and this ordering satisfies *both* the additive and nesting properties.
- For C an irregular simplex, we can construct an ordering which is both additive and nested, but it is not the modified or standard grevlex ordering.
- For C not a simplex of any kind, *there is no ordering* which is both nested and additive.

Continuous Grevlex Ordering

We let the fractional C-degree, $\mathrm{rdeg}_C(z^\alpha) = inf\{r \in \mathbb{R} : \alpha \in rC\}$.

Definition

For a convex body C, let $\lhd_C$ denote the continuous grevlex ordering on $\mathbb{Z}^d$, where $\alpha \lhd_C \beta$ if

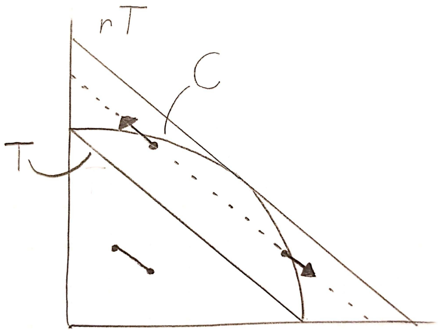
- $\mathrm{rdeg}_C(\alpha) < \mathrm{rdeg}_C(\beta)$; or
- $\mathrm{rdeg}_C(\alpha) = \mathrm{rdeg}_C(\beta)$, and $\alpha \prec \beta$.

We make two remarks

- For any convex body C, $\lhd_C$ is nested.
- For a simplex T of any kind, $\lhd_T$ is *both* nested and additive.

Result: No A.N. Ordering if C not a Simplex

- We construct $T \subset C \subset rT$, $r > 1$.
- Consider the scaled lattice, $\frac{1}{k}\mathbb{Z}_+^d$.
- For large k, we can find two pairs of points that are abitrarily close to realizing this line segment's passing in and out of C
- Then we can find a contradiction through additivity



Making Do

When C is not a simplex of any kind, we must make do:

- The grevlex ordering is always additive: necessary for Zaharjuta Theory
- The modified grevlex ordering is always nested: necessary for Leja Point Construction

Following [M19], we use the grevlex ordering to develop the main asymptotics, and compare to get results for the modified grevlex ordering.

Generalized Zaharjuta: Conclusion

Through detailed work in [M19, Thm. 4.6], we have.

Theorem

Let V_k be the maximum determinant of the matrix $V(z_1, \ldots, z_{M_k})$ for the M_k points $z_1, \ldots, z_{M_k} \in K$.

$$\frac{1}{M_k!} V_k \leq \prod_{\alpha \in kC} T_k^{\prec}(\alpha)^k \leq V_k$$

The same inequality holds for $T_k^{\prec c}(\alpha)$.

Generalized Zaharjuta: Conclusion

We also have a recreation of the transfinite diameter formula, but now integrating across the entire convex body C.

$$\tau(K)^c = \exp\left(\frac{1}{vol_N(C)}\int_{\mathrm{int}(C)} \log T_{\prec}(K,\theta)\, d\theta\right)$$

C-Leja and C-Fekete Points

Now we use the ordering $\prec_C$ to give a reordered basis of polynomials that respects $\deg_C$. We again refer to these as $e_s(z) = z^{\alpha}(s)$.

Then as before, for a given k and sequence $(z_i)_{i=1}^{M_k} \subset K$, we can form the M_k by M_k matrix VDM.

$$\mathrm{VDM}_C(z_1, z_2, \ldots, z_{M_k})_{i,j} = z_i^{\alpha(j)}$$

And its determinant

$$V(z_1, \ldots, z_{M_k})_C = |\det(\mathrm{VDM}(z_1, \ldots, z_{M_k}))|$$

C-Fekete and C-Leja Sequences

For any $s \geq 1$, there exist set(s) of points $(\zeta_i)_{i=1}^{s} \subset K$ that maximize V. We call these *C-Fekete Points*.

For given k we define a *C-Leja Sequence* $(\ell_i)_{i=1}^{s} \subset K$ as a sequence given by the following procedure.

- Choose $\ell_1 \in K$ at a maximum of $z^{\alpha(1)}$.
- Assuming $\ell_1, \ldots, \ell_{s-1}$ have been chosen,
- For each subsequent point ℓ_s, choose a maximum of $\ell \to V(\ell_1, \ldots, \ell_{s-1}, \ell)$.

Asymptotically Fekete Sequences

We let $\tau(K)^c = \lim_{k\to\infty} V(\zeta_1^{(k)}, \ldots, \zeta_{M_k}^{(k)}))$

We say that an array $\left(z_i^{(k)}\right)_{i=1}^{M_k}$ is *Asymptotically Fekete* if

$$V(z_1^{(k)}, \ldots, z_{M_k}^{(k)})^{1/L_k} \to \tau(K)^c$$

Weakly Admissable Meshes

We approximate our compact set K by an array of points, $A_k \subset K$, which "approximates" K in the following way. [CL08]

Definition

A weakly admissable mesh is a sequence of finite sets $A_k \subset K$, and sequence of constants C_k, which satisfy the following conditions:

- For any $p \in \mathcal{P}_C(k)$,

$$\|p\|_K \leq C_k \|p\|_{A_k}$$

- $\lim_{k\to\infty} (\#A_k)^{1/k} = \lim_{k\to\infty} (C_k)^{1/k} = 1$

A Brief Return to Uniform Estimates

Given, for any $p \in \mathcal{P}_C(k)$,

$$\|p\|_K \leq C_k \|p\|_{A_k}$$

Let k be some degree, then for $p \in \mathcal{P}_C(k)$ minimizing $\|f - p\|_{A_k}$
From [CL08], we have the following:

$$\|f - p\|_K \leq \left(1 + C_k \left(1 + \sqrt{\#A_k}\right)\right) \mathrm{dist}_K(f, \mathcal{P}_C(k))$$

Discrete Leja

To generate n standard Leja points, given the first s points, we choose the next point to maximize this function of ℓ:

$$\ell \hookrightarrow V(\ell_1, \ldots, \ell_s, \ell) = |\det(\mathrm{VDM}(z_1, \ldots, z_s))|$$

over all $\ell \in K$.

To generate a Discrete Leja Sequence [BMS10], we take k large enough so $p_1, \ldots, p_n \in \mathcal{P}_C(k)$, and given s points, maximize the function above for $\ell \in A_k$.

Discrete Leja Example

To perform this procedure, we form the non-square Vandermonde matrix, of dimensions A_k by k. (We transpose, so we are choosing rows.)

Goal: Permute rows to find the 3 by 3 submatrix of maximal determinant.

$$\begin{array}{c|c} & \begin{matrix} 1 & z & z^2 \end{matrix} \\ \hline \begin{matrix} -1 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{matrix} & \left[\begin{matrix} 1 & -1 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & -1 & 1 \end{matrix}\right] \end{array} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -\frac{1}{2} & \frac{1}{4} \\ 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{4} \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & \frac{1}{2} & -\frac{3}{4} \\ 0 & 1 & -1 \\ 0 & \frac{3}{2} & \frac{3}{4} \\ 0 & 2 & 0 \end{bmatrix}$$

Discrete Leja Example

After eliminating the first column, we select to move the $[0\ \ 2\ \ 0]$ row to be the next pivot row.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & \frac{1}{2} & -\frac{3}{4} \\ 0 & 1 & -1 \\ 0 & \frac{3}{2} & \frac{3}{4} \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & \frac{3}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} & -\frac{3}{4} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & \frac{3}{4} \\ 0 & 0 & -\frac{3}{4} \end{bmatrix}$$

Since we only care about maximizing the absolute value of the determinant, we are done.
We want a more automated way to do this, which is LU factorization with pivoting.

Discrete Leja in Matlab

This process is very easily automated. We adapt [BL09] and [BMS10].

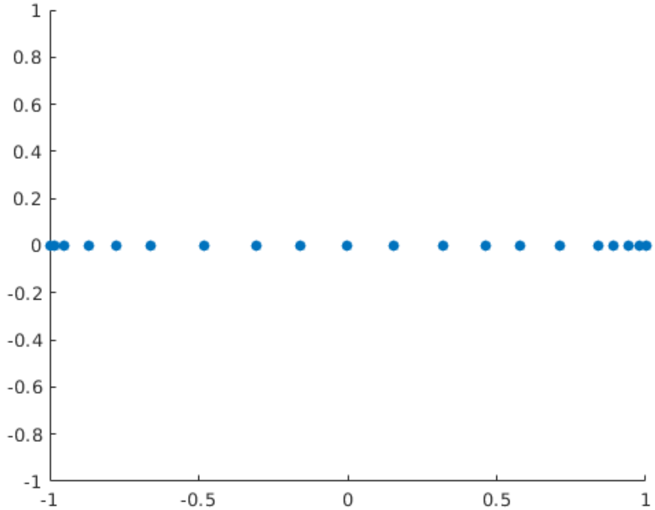
```
n = 20;
m = 1000;
x = linspace(-1,1,m);
V = gallery('chebvand', n, x)';

%LU decomposition
[L, U, sigma] = lu(V, 'vector');

%Extra points
ind = sigma(1:n);

display('chosen points')
zeta = x(ind)
```

Discrete Leja in Matlab



Discrete Leja in Matlab: Comparison

From [BL09].

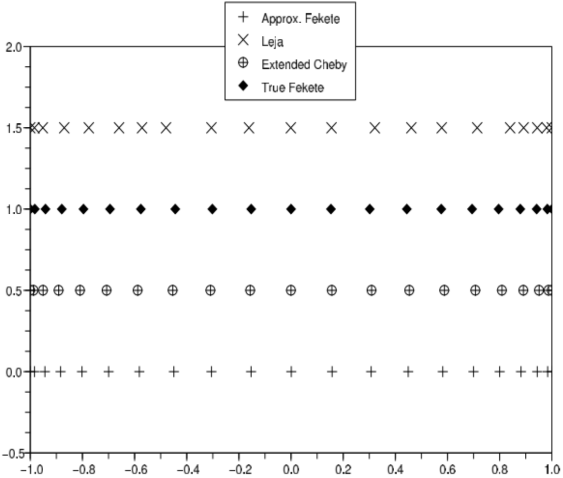


FIG. 1.1. *Plot of various point sets for* $n = 21$

Asymptotics and Convex Body Generalization

We first make a theoretical comment: the Discrete Leja point scheme described above *does* generate an Asymptotically Fekete array, as long as the underlying mesh is in fact a weakly admissable mesh.

Second, we note that to generalize this process to the convex body case–that is, to find Discrete C-Leja points–all that is needed is to reorder the polynomials using an ordering that respects the C-degree. We have also proven that these arrays are asymptotically fekete.

Extremal Function

From the work of Zaharjuta, we can define the extremal function as

$$V_K(z) := \sup\left\{\left(\frac{1}{\deg(p)}\right)\log|p(z)| : \|p\|_K \leq 1\right\}$$

Remark: In one variable, this is the potential generated by the equilibrium measure, up to an additive constant.

Extremal: Lev and Bayraktar, Menuja

Siciak proved that the sequence of functions $\Phi_k(z)$, defined

$$\log(\Phi_k(z)) := \sup\left\{\frac{1}{k}\log|p(z)| : p \in \mathcal{P}_C(k), \|p\|_K \leq 1\right\}$$

Converges locally uniformly in $\mathbb{C}^n$ to $V_K(z)$

Most of convex generalization is completed: current work by T. Bayraktar, N. Levenberg, M. Perera, and SH.

With this proven, we can move forward with generalizing Thm. 1 from Pluripotential Numerics, (F. Piazzon), which gives several ways to approximate the extremal function using various kernels.

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Thank you all for attending!